

Dedicated to Prof. Suthep Suantai on the occasion of his 60th anniversary

On the Additive (s_1, s_2) –Functional Inequality and Its Stability in Banach Spaces

Wuttichai Suriyacharoen and Wutiphol Sintunavarat*

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University
Rangsit Center, Pathum Thani 12120, Thailand
e-mail : wutiphol@mathstat.sci.tu.ac.th (W. Sintunavarat)

Abstract The additive (s_1, s_2) –functional inequality of the form

$$\|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\|$$

is solved without any regularity assumptions, where s_1, s_2 are fixed nonzero complex numbers satisfying $|s_1| + |s_2| < 1$. An analysis of its stability is also proved in complex Banach spaces. The solution of the proposed functional inequality is related to the additive mapping.

MSC: 30D05; 39B52; 47H10

Keywords: functional equation; functional inequality; stability

Submission date: 12.04.2020 / Acceptance date: 23.06.2020

1. INTRODUCTION

In the article [1], Park introduced and solved the following functional inequalities:

$$\|f(x + y) - f(x) - f(y)\| \leq \left\| \rho \left(2f\left(\frac{x + y}{2}\right) - f(x) - f(y) \right) \right\|, \quad (1.1)$$

$$\left\| 2f\left(\frac{x + y}{2}\right) - f(x) - f(y) \right\| \leq \|\rho(f(x + y) - f(x) - f(y))\| \quad (1.2)$$

*Corresponding author.

over 2-divisible Abelian groups, where ρ is a fixed non-Archimedean number with $|\rho| < 1$. He also treated, in 2015 [2], the following functional inequalities:

$$\left\| f\left(\sum_{j=1}^k x_j\right) - \sum_{j=1}^k f(x_j) \right\| \leq \left\| \rho \left(kf\left(\frac{1}{k}\sum_{j=1}^k x_j\right) - \sum_{j=1}^k f(x_j) \right) \right\|, \quad (1.3)$$

$$\left\| kf\left(\frac{1}{k}\sum_{j=1}^k x_j\right) - \sum_{j=1}^k f(x_j) \right\| \leq \left\| \rho \left(f\left(\sum_{j=1}^k x_j\right) - \sum_{j=1}^k f(x_j) \right) \right\|, \quad (1.4)$$

which are the extensions of (1.1) and (1.2), respectively, over k -divisible Abelian group, where $k \geq 2$ is a fixed integer and ρ is a fixed nonzero complex number with $|\rho| < 1$. The stabilities of (1.1), (1.2), (1.3), and (1.4) are also investigated over complex Banach spaces. The solutions of these inequalities are related to the additive mapping. We shall refer collectively to functional inequalities like (1.1), (1.2), (1.3), or (1.4) as *additive ρ -functional inequalities*.

Recently, the following additive s -functional inequality:

$$\|f(x+y) - f(x) - f(y)\| \leq \|s(f(x-y) - f(x) - f(-y))\| \quad (1.5)$$

was treated in 2019 by Park et al. [3] over the complex vector spaces, where $s \in \mathbb{C} \setminus \{0\}$ is a fixed complex number with $|s| < 1$, as well as its stability. He also solved, in 2019 [4], the following functional inequality:

$$\|f(x+y) - f(x) - f(y)\| \leq \|\rho_1(f(x+y) + f(x-y) - 2f(x))\| + \left\| \rho_2 \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \quad (1.6)$$

over complex normed spaces underlying the condition that $f(0) = 0$, where ρ_1, ρ_2 are fixed complex numbers satisfying $\sqrt{2}|\rho_1| + |\rho_2| < 1$; its stability is also proved there.

In the present work, we introduce and solve the following functional inequality:

$$\begin{aligned} & \|f(x+2y+3z) - f(x+z) - 2f(y+z)\| \\ & \leq \|s_1(f(x+2y-3z) - f(x-z) - 2f(y-z))\| \\ & + \left\| s_2 \left(f(x+z) + 2f(y+z) - 2f\left(\frac{x+2y+3z}{2}\right) \right) \right\| \end{aligned} \quad (1.7)$$

over complex vector spaces without any regularity assumptions on the unknown function, where s_1, s_2 are fixed nonzero complex numbers subjecting to the condition that $|s_1| + |s_2| < 1$. This inequality is closely related to the additive functional equation. We shall refer to the functional inequality like (1.7) as *additive (s_1, s_2) -functional inequality*.

Furthermore, an analysis of the stability of (1.7) is here also carried out in complex Banach space. Our investigation consists of first using the fixed point method, and second of using the direct method.

The following result is needed in our main results, which is the well-known fundamental theorem in fixed point theory; see also [5]:

Theorem 1.1 ([6]). *Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n , or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty \quad \forall n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.

Throughout this paper, denote generically by X a complex normed space, and denote by Y a complex Banach space. Let s_1, s_2 be fixed nonzero complex numbers with $|s_1| + |s_2| < 1$.

2. ADDITIVE (s_1, s_2) -FUNCTIONAL INEQUALITY

In this section, the functional equation

$$f(x + 2y + 3z) = f(x + z) + 2f(y + z)$$

and the functional inequality

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ & \quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| \end{aligned}$$

are solved without any regularity conditions. Both of above equation and inequality are closely related to the additive functional equation. The results so obtained are needed in the next parts.

The equations and inequalities of our results involve the use of *additive Cauchy equation* ([7, Chapter 1]), which is the functional equation of the form

$$f(x + y) = f(x) + f(y).$$

The solution of this equation is called an *additive mapping*.

Lemma 2.1. *Let X and Y be real or complex vector spaces. Then a mapping $f : X \rightarrow Y$ satisfies the following functional equation*

$$f(x + 2y + 3z) = f(x + z) + 2f(y + z) \tag{2.1}$$

for all $x, y, z \in X$ if and only if $f : X \rightarrow Y$ is an additive.

Proof. Assume that f satisfies (2.1). Putting $x = y = z = 0$ into (2.1), we get $f(0) = 0$. Next, taking $x = z = 0$ in (2.1) we have $f(2y) = 2f(y)$, and so

$$f\left(\frac{y}{2}\right) = \frac{1}{2}f(y).$$

Replacing y by $y/2$ and setting $z = 0$ in (2.1), we see that

$$f(x + y) = f(x) + 2f\left(\frac{y}{2}\right) = f(x) + f(y),$$

showing that f is an additive.

The converse is obviously holds. ■

Lemma 2.2. *Let X and Y be real or complex vector spaces. Then a mapping $f : X \rightarrow Y$ satisfies the following functional inequality*

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ & \quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f \left(\frac{x + 2y + 3z}{2} \right) \right) \right\| \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$ if and only if $f : X \rightarrow Y$ is an additive.

Proof. Putting $x = y = z = 0$ in (2.2), we have

$$\|f(0)\| |2 - 2|s_1| - |s_2| | \leq 0.$$

If $|2 - 2|s_1| - |s_2| | = 0$, then $|s_2| = 2 - 2|s_1|$ and so $1 > |s_1| + |s_2| = 2 - |s_1|$ giving that $|s_1| > 1$. This contradicts the condition of s_1, s_2 , and so we must have $f(0) = 0$.

Next, taking $x = z = 0$ into (2.2), we have

$$\|f(2y) - 2f(y)\| \leq \|s_1 (f(2y) - 2f(y))\|$$

for all $y \in X$, and so

$$f \left(\frac{y}{2} \right) = \frac{1}{2} f(y) \tag{2.3}$$

for all $y \in X$ since $|s_1| < 1$. Using (2.2) twice and (2.3), we see that

$$\begin{aligned} & (1 - |s_2|)^2 \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1^2 (f(x + 2y + 3z) - f(x + z) - 2f(y + z))\| \end{aligned}$$

for all $x, y, z \in X$. This implies, since $|s_1| + |s_2| < 1$, that

$$f(x + 2y + 3z) = f(x + z) + 2f(y + z)$$

for all $x, y, z \in X$. The result follows from Lemma 2.1.

The converse is obviously holds. ■

3. STABILITY RESULTS USING FIXED POINT METHOD

Throughout this section, denoted by $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$ the set of positive real numbers. Stability results of the functional inequality (1.7) are here investigated by using fixed point method. We begin with:

Theorem 3.1. *Let $\phi : X^3 \rightarrow [0, \infty)$ be a fixed function satisfying*

$$\phi \left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right) \leq \frac{L}{2} \phi(x, y, z) \tag{3.1}$$

for some $L < 1$ and all $x, y, z \in X$. If $f : X \rightarrow Y$ satisfies the following inequality

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ & \quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f \left(\frac{x + 2y + 3z}{2} \right) \right) \right\| + \phi(x, y, z) \end{aligned} \tag{3.2}$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{L}{2(1-L)(1-|s_1|)} \right) \phi(0, x, 0) \tag{3.3}$$

for all $x \in X$.

Proof. First note that $\phi(0, 0, 0) = 0$, by taking $x = y = z = 0$ in (3.1). Putting $x = y = z = 0$ in (3.2) and using the previous result, we then also get that $\|f(0)\| 2-2|s_1|-|s_2\| \leq 0$ and thus $f(0) = 0$.

Next, setting $x = z = 0$ into (3.2) we get

$$(1 - |s_1|) \|f(2y) - 2f(y)\| \leq \phi(0, y, 0) \tag{3.4}$$

for all $y \in X$. Constructing the set

$$G := \{g : X \rightarrow Y : g(0) = 0\}$$

with the generalized metric

$$d(g, h) := \inf\{\mu \in \mathbb{R}_{>0} : \|g(x) - h(x)\| \leq \mu \cdot \phi(0, x, 0) \quad \forall x \in X\},$$

where $\inf \emptyset = +\infty$. Then it is easily shows that the pair (G, d) is a complete generalized metric space; see also [8]. Define a linear mapping $J : G \rightarrow G$ by

$$Jg(x) = 2g\left(\frac{x}{2}\right) \tag{3.5}$$

for all $x \in X$ and for each $g \in G$. For given $g, h \in G$ with $d(g, h) = \varepsilon$, we have

$$\|g(x) - h(x)\| \leq \varepsilon \phi(0, x, 0)$$

and so

$$\|Jg(x) - Jh(x)\| = \left\| 2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right) \right\| \leq 2\varepsilon \phi\left(0, \frac{x}{2}, 0\right) \leq \varepsilon L \phi(0, x, 0)$$

yielding that $d(Jg, Jh) \leq \varepsilon L$, and we arrived at

$$d(Jg, Jh) \leq Ld(g, h) \tag{3.6}$$

for all $g, h \in G$. This means that J is a contraction mapping with the contractive constant $L < 1$. The inequalities (3.1) and (3.4) give

$$\|f(x) - Jf(x)\| = \left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \left(\frac{1}{1-|s_1|} \right) \phi\left(0, \frac{x}{2}, 0\right) \leq \left(\frac{L}{2(1-|s_1|)} \right) \phi(0, x, 0),$$

and so $d(f, Jf) \leq \frac{L}{2(1-|s_1|)}$. Theorem 1.1 is now applicable and yields that there exists a mapping $A : X \rightarrow Y$ satisfying the following assertions:

- (i) A is a fixed point of J , i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{3.7}$$

for all $x \in X$. Moreover, the mapping A is a unique fixed point in the set

$$\hat{G} := \{g \in S : d(f, g) < \infty\}. \tag{3.8}$$

This means that A is a unique mapping, which is defined by (3.7), satisfying

$$\|f(x) - A(x)\| \leq \mu \phi(0, x, 0)$$

for some $\mu \in \mathbb{R}_{>0}$ and all $x \in X$;

(ii) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This gives

$$A(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) \tag{3.9}$$

for all $x \in X$;

(iii) $d(f, A) \leq \left(\frac{1}{1-L}\right) d(f, Jf)$. This implies that

$$\|f(x) - A(x)\| \leq \left(\frac{L}{2(1-L)(1-|s_1|)}\right) \phi(0, x, 0) \tag{3.10}$$

for all $x \in X$, as desired.

The remaining is to check that a mapping A is an additive. Indeed, note from (3.1), (3.2), (3.3), and (3.9) that

$$\begin{aligned} & \|A(x + 2y + 3z) - A(x + z) - 2Af(y + z)\| \\ &= \lim_{n \rightarrow \infty} 2^n \left\| f\left(\frac{x + 2y + 3z}{2^n}\right) - f\left(\frac{x + z}{2^n}\right) - 2f\left(\frac{y + z}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^n |s_1| \left\| f\left(\frac{x + 2y - 3z}{2^n}\right) - f\left(\frac{x - z}{2^n}\right) - 2f\left(\frac{y - z}{2^n}\right) \right\| \\ &\quad + \lim_{n \rightarrow \infty} 2^n |s_2| \left\| f\left(\frac{x + z}{2^n}\right) + 2f\left(\frac{y + z}{2^n}\right) - 2f\left(\frac{x + 2y + 3z}{2^n \cdot 2}\right) \right\| \\ &\quad + \lim_{n \rightarrow \infty} 2^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\ &\leq \|s_1 (A(x + 2y - 3z) - A(x - z) - 2A(y - z))\| \\ &\quad + \left\| s_2 \left(A(x + z) + 2A(y + z) - 2A\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| \end{aligned}$$

for all $x, y, z \in X$, and the result follows from Lemma 2.2. This proves Theorem 3.1. ■

When the function ϕ satisfies the condition similar to (3.1), we get the following result:

Theorem 3.2. *Let $\phi : X^3 \rightarrow [0, \infty)$ be a fixed function satisfying*

$$\phi(x, y, z) \leq 2L\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{3.11}$$

for some $L < 1$ and all $x, y, z \in X$. If $f : X \rightarrow Y$ satisfies the following inequality

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ &\leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ &\quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| + \phi(x, y, z) \end{aligned} \tag{3.12}$$

for all $x, y, z \in X$ subjecting to the condition that $f(0) = 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2(1-L)(1-|s_1|)}\right) \phi(0, x, 0) \tag{3.13}$$

for all $x \in X$.

Proof. Constructing a complete generalized metric space (G, d) as in the proof of Theorem 3.1, and define a linear mapping $\hat{J} : G \rightarrow G$ by

$$\hat{J}g(x) = \frac{1}{2}g(2x) \tag{3.14}$$

for all $x \in X$ and for each $g \in G$. The inequality (3.4) then gives

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \left(\frac{1}{2(1 - |s_1|)} \right) \phi(0, x, 0) \tag{3.15}$$

for all $x \in X$. Using (3.14) and (3.15) with the same arguments as in the proof of the former, the result follows. ■

Immediate from Theorems 3.1 and 3.2 are the following, which illustrate some examples of functions ϕ satisfying (3.1) and (3.11), respectively.

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying the following inequality*

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ & \quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| \\ & \quad + \theta (\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{3.16}$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{\theta}{(2^r - 2)(1 - |s_1|)} \right) \|x\|^r \tag{3.17}$$

for all $x \in X$.

Proof. Taking $\phi(x, y, z) := \theta (\|x\|^r + \|y\|^r + \|z\|^r)$ and choosing $L := 2^{1-r}$ in Theorem 3.1, the result follows. ■

Corollary 3.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying the following inequality*

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ & \quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| \\ & \quad + \theta (\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{3.18}$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{\theta}{(2 - 2^r)(1 - |s_1|)} \right) \|x\|^r \tag{3.19}$$

for all $x \in X$.

Proof. Taking $\phi(x, y, z) := \theta (\|x\|^r + \|y\|^r + \|z\|^r)$ and choosing $L := 2^{r-1}$ in Theorem 3.2, the result follows. ■

4. STABILITY RESULTS USING DIRECT METHOD

Regarding the (s_1, s_2) -functional inequality, its stability is here proved by using direct method.

Theorem 4.1. *Let $\phi : X^3 \rightarrow [0, \infty)$ be a fixed function satisfying*

$$\Phi(x, y, z) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{4.1}$$

for all $x, y, z \in X$. If $f : X \rightarrow Y$ satisfies the following inequality

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ & \quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| \\ & \quad + \phi(x, y, z) \end{aligned} \tag{4.2}$$

for all $x, y, z \in X$ subjecting to the condition that $f(0) = 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2(1 - |s_1|)}\right) \Phi(0, x, 0) \tag{4.3}$$

for all $x \in X$.

Proof. Taking $x = z = 0$ in (4.2), we have

$$(1 - |s_1|) \|f(2y) - 2f(y)\| \leq \phi(0, y, 0),$$

and so

$$\left\| f(y) - 2f\left(\frac{y}{2}\right) \right\| \leq \left(\frac{1}{1 - |s_1|}\right) \phi\left(0, \frac{y}{2}, 0\right) \tag{4.4}$$

for all $y \in X$. By (4.4) and triangle inequality, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, we get

$$\begin{aligned} \left\| 2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| & \leq \sum_{i=n}^{m-1} \left\| 2^i f\left(\frac{x}{2^i}\right) - 2^{i+1} f\left(\frac{x}{2^{i+1}}\right) \right\| \\ & \leq \sum_{i=n}^{m-1} \left(\frac{2^i}{1 - |s_1|}\right) \phi\left(0, \frac{x}{2^{i+1}}, 0\right) \\ & = \left(\frac{1}{2(1 - |s_1|)}\right) \sum_{i=n}^{m-1} 2^{i+1} \phi\left(0, \frac{x}{2^{i+1}}, 0\right) \end{aligned} \tag{4.5}$$

$$= S_{m-1} - S_{n-1}, \tag{4.6}$$

where

$$S_k := \left(\frac{1}{2(1 - |s_1|)}\right) \sum_{i=1}^k 2^{i+1} \phi\left(0, \frac{x}{2^{i+1}}, 0\right).$$

From (4.1), there exists $s \in \mathbb{R}_{>0}$ such that $S_k \rightarrow s$ as $k \rightarrow \infty$. Taking limit as $n, m \rightarrow \infty$ in (4.6), we have

$$\left\| 2^m f\left(\frac{x}{2^m}\right) - 2^n f\left(\frac{x}{2^n}\right) \right\| \rightarrow 0$$

showing that the sequence $(2^n f(\frac{x}{2^n}))$ is a Cauchy sequence in Y . By the completeness of Y , one can define a mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) \tag{4.7}$$

for all $x \in X$. Putting $n = 0$ and also taking limit as $m \rightarrow \infty$ in (4.5), and using (4.7) we get

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2(1 - |s_1|)}\right) \Phi(0, x, 0) \tag{4.8}$$

for all $x \in X$, as desired. By using (4.1), (4.2), and (4.7) with the same arguments as in the proof of Theorem 3.1, it is easily shows that a mapping A is an additive.

It remains now to verify that a mapping A is unique. Assume that there exists another additive mapping $\hat{A} : X \rightarrow Y$ satisfying the inequality (4.3). For each $x \in X$, we obtain

$$\begin{aligned} \|A(x) - \hat{A}(x)\| &= \left\|2^t A\left(\frac{x}{2^t}\right) - 2^t \hat{A}\left(\frac{x}{2^t}\right)\right\| \\ &\leq \left\|2^t A\left(\frac{x}{2^t}\right) - 2^t f\left(\frac{x}{2^t}\right)\right\| + \left\|2^t f\left(\frac{x}{2^t}\right) - 2^t \hat{A}\left(\frac{x}{2^t}\right)\right\| \\ &\leq \left(\frac{2^t}{1 - |s_1|}\right) \Phi\left(0, \frac{x}{2^t}, 0\right) \\ &= \left(\frac{1}{1 - |s_1|}\right) \sum_{j=1}^{\infty} 2^{t+j} \phi\left(0, \frac{x}{2^{t+j}}, 0\right) < \infty. \end{aligned}$$

This prove the uniqueness of A , and the proof of Theorem 4.1 now completes. ■

The same conclusion still valid when the function ϕ in Theorem 4.1 satisfies the following condition similar to (4.1).

Theorem 4.2. *Let $\phi : X^3 \rightarrow [0, \infty)$ be a fixed function satisfying*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \phi\left(2^j x, 2^j y, 2^j z\right) < \infty \tag{4.9}$$

for all $x, y, z \in X$. If $f : X \rightarrow Y$ satisfies the following inequality

$$\begin{aligned} &\|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ &\leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ &\quad + \left\|s_2\left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right)\right)\right\| \\ &\quad + \phi(x, y, z) \end{aligned} \tag{4.10}$$

for all $x, y, z \in X$ subjecting to the condition that $f(0) = 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2(1 - |s_1|)}\right) \Phi(0, x, 0) \tag{4.11}$$

for all $x \in X$.

Proof. Taking $x = z = 0$ in (4.10), we have

$$(1 - |s_1|) \|f(2y) - 2f(y)\| \leq \phi(0, y, 0),$$

and so

$$\left\| f(y) - \frac{1}{2}f(2y) \right\| \leq \left(\frac{1}{2(1 - |s_1|)} \right) \phi(0, y, 0) \tag{4.12}$$

for all $y \in X$. Using the same arguments as in the proof of Theorem 4.2, one can see that

$$\begin{aligned} \left\| \frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x) \right\| &\leq \sum_{i=n}^{m-1} \left\| \frac{1}{2^i}f(2^i x) - \frac{1}{2^{i+1}}f(2^{i+1} x) \right\| \\ &\leq \left(\frac{1}{2(1 - |s_1|)} \right) \sum_{i=n}^{m-1} \frac{1}{2^i} \phi(0, 2^i x, 0) \end{aligned} \tag{4.13}$$

for all $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, yielding that the sequence $(\frac{1}{2^n}f(2^n x))$ is Cauchy in Y . By the completeness of Y , we can define a mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n}f(2^n x) \tag{4.14}$$

for all $x \in X$. Putting $n = 0$ and also taking limit as $m \rightarrow \infty$ in (4.13), and using (4.14) we get

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2(1 - |s_1|)} \right) \Phi(0, x, 0) \tag{4.15}$$

for all $x \in X$, which is the desired assertion. The rest of the proof is similar to that of the former. ■

As in the proof of Corollaries 3.3 and 3.4, one can choose the functions ϕ with the real numbers r, θ for being the functions satisfying the conditions (4.1) and (4.9), respectively, and the following results follow immediately.

Corollary 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying the following inequality*

$$\begin{aligned} &\|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ &\leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ &\quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| \\ &\quad + \theta (\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{4.16}$$

for all $x, y, z \in X$ subjecting to the condition that $f(0) = 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{\theta}{(2^r - 2)(1 - |s_1|)} \right) \|x\|^r \tag{4.17}$$

for all $x \in X$.

Corollary 4.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying the following inequality*

$$\begin{aligned} & \|f(x + 2y + 3z) - f(x + z) - 2f(y + z)\| \\ & \leq \|s_1(f(x + 2y - 3z) - f(x - z) - 2f(y - z))\| \\ & \quad + \left\| s_2 \left(f(x + z) + 2f(y + z) - 2f\left(\frac{x + 2y + 3z}{2}\right) \right) \right\| \\ & \quad + \theta (\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \quad (4.18)$$

for all $x, y, z \in X$ subjecting to the condition that $f(0) = 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{\theta}{(2 - 2^r)(1 - |s_1|)} \right) \|x\|^r \quad (4.19)$$

for all $x \in X$.

FUNDING

The second author would like to thank the Thailand Research Fund and Office of the Higher Education Commission under grant no. MRG6180283 for financial support during the preparation of this manuscript.

AUTHORS' CONTRIBUTIONS

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

REFERENCES

- [1] C. Park, Additive ρ -functional inequalities and equations, *J. Math. Inequal.* 9 (2015) 17–26.
- [2] C. Park, Additive ρ -functional inequalities in non-Archimedean normed spaces, *J. Math. Inequal.* 9 (2015) 397–407.
- [3] C. Park, J.R. Lee, X. Zhang, Additive s -functional inequality and hom-derivations in Banach algebras, *J. Fixed Point Theory Appl.* (2019) 21:18.
- [4] C. Park, The stability of an additive (ρ_1, ρ_2) -functional inequality in Banach spaces, *J. Math. Inequal.* 1 (2019) 95–104.
- [5] W.A.J. Luxemburg, On the convergence of successive approximations in the theory of ordinary differential equations. II, *Koninkl. Nederl. Akademie van Wetenschappen, Amsterdam, Proc. Ser. A* 5 (61), and *Indag. Math.* 5 (20) (1958) 540–546.
- [6] J.B. Diaz, B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metrix space, *Bull. Am. Math. Soc.* 74 (1968) 305–309.
- [7] P.I. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, Dordrecht, 2009.
- [8] D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J. Math. Anal. Appl.* 343 (2008) 567–572.