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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Modified Explicit Self-Adaptive Two-Step Extragradient Method for Equilibrium Programming in a Real Hilbert Space

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Abstract The primary objective of this study is to present a new self-adaptive method to solve an equilibrium problem involving pseudomonotone bifunction in real Hilbert spaces. This method could be viewed as an improvement of the paper title Extragradient algorithms extended to equilibrium problem by Tran et al. [D.Q. Tran, M.L. Dung, V.H. Nguyen, Extragradient algorithms extended to equilibrium problems, Optim. 57 (2008) 749–776]. A weak convergence theorem for the generated sequence has been proven and implemented to solve variational inequality problems. We have used different numerical examples to illustrate our well-established convergence results.

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1. Introduction

Let C to be a nonempty closed, convex subset of $\mathbb{E}$ and $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$, for all $x \in C$. The *equilibrium problem* for the bifunction f on C is defined as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \ \forall y \in C. \tag{EP}$$

Equilibrium problem (EP) contains many mathematical problems as a specific case, i.e. the variational inequality problems (shortly, VIP), complementarity problems, minimization problems, the fixed point problems, Nash equilibrium of noncooperative games, saddle point problems and vector minimization problem (see e.g., [1, 2]). The explicit formula of an equilibrium problem was established in 1992 by Muu and Oettli [3] and has been further developed by Blum and Oettli [1]. The equilibrium problem also known as Ky Fan inequality problem, Fan [4] established conditions on a bifunction for the existence of a solution of an equilibrium problem. Several authors have achieved and generalized many results with regard to the existence of an equilibrium problem solution and particular form of an equilibrium problem (e.g., see [5–9] and the references therein).

On the other hand, iterative methods are useful tools for determining the approximate solution of an equilibrium problem. A large number of methods were established to deal different types of equilibrium problems for both finite and infinite dimensional spaces (for details, see [10–29]). More precisely, Quoc et al. [30] provided an iterative sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in C \\ y_n = \arg\min\{\lambda f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\} \\ x_{n+1} = \arg\min\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\} \end{cases}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ with c_1, c_2 are Lipschitz constants of a bifunction. The extra-gradient method was introduced by Quoc et al. [30] for dealing with the equilibrium problem involving pseudomonotone bifunction in real Hilbert spaces. Quoc proved a weak convergence results for iterative sequence generated based upon the hypotheses of Lipschitz type-continuity and monotonicity. It is essential to consider two minimization problems on a closed convex set in each iteration of the method and there is a reasonable fixed step-size in each minimization problems.

In this paper, we consider Quoc et al. [30] extragradient method and provide its improvement by providing the explicit step-size formula for both minimization problems. We introduce a self-adaptive extragradient method for dealing with a pseudomonotone equilibrium problem in real Hilbert spaces. A weak convergence theorem for our suggested method is proved by choosing an adequate explicit step-size formula. Some application to the variational inequality problem is presented and also some numerical examples in finite and infinite dimensional spaces are considered for supporting the practicality of our established results.

The rest of the paper was organized as follows: Section 2 presents a number of definitions and key results to be used in this paper. Section 3 includes our first method involving pseudomonotone bifunction and produces a weak convergence result. Section 4 presents an application of our results in the variational inequality problems. Section 5 sets out the numerical experimental work to show the scientific performance of the proposed algorithms.

2. Preliminaries

We require some relevant lemmas, definitions and other notions that will be needed throughout the convergence analysis. The notion $\langle .,. \rangle$ and $\|.\|$ represents the inner product and norm on the Hilbert space. We write down $x_n \rightharpoonup x^*$ mention the sequence $\{x_n\}$ weakly converges to x^*. In addition, $EP(f, C)$ means the solution set of an equilibrium problem over C and $VI(G, C)$ denotes the the solution set of an variational inequality problem over C with x^* an arbitrary element of $EP(f, C)$ or $VI(G, C)$.

Let $g : C \to \mathbb{R}$ is a convex function and *subdifferential of* g at $x \in C$ is define as follows:

$$\partial g(x) = \{z \in \mathbb{E} : g(y) - g(x) \geq \langle z, y - x\rangle, \ \forall y \in C\}.$$

The *normal cone of* C at $x \in C$ is defined by

$$N_C(x) = \{z \in \mathbb{E} : \langle z, y - x\rangle \leq 0, \ \forall y \in C\}.$$

We can see the different notions of the bifunction monotonicity (see [1, 31] for more details).

Definition 2.1. A bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ on C for $\gamma > 0$ is said to be:

(i) *strongly monotone* if

$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2, \ \forall x, y \in C;$$

(ii) *monotone* if

$$f(x, y) + f(y, x) \leq 0, \ \forall \ x, y \in C;$$

(iii) *strongly pseudomonotone* if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\gamma\|x - y\|^2, \ \forall x, y \in C;$$

(iv) *pseudomonotone* if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \ \forall x, y \in C;$$

(v) satisfying the *Lipschitz-type condition* on C if there exist two real numbers $c_1, c_2 > 0$ such that

$$f(x, z) \leq f(x, y) + f(y, z) + c_1\|x - y\|^2 + c_2\|y - z\|^2, \ \forall x, y, z \in C.$$

Remark 2.2. We shall have the following implications from the above notions.

$$\textit{strongly monotone} \Longrightarrow \textit{monotone} \Longrightarrow \textit{pseudomonotone}$$

$$\textit{strongly monotone} \Longrightarrow \textit{strongly pseudomonotone} \Longrightarrow \textit{pseudomonotone}$$

Definition 2.3. [32] The *metric projection* $P_C(x)$ of x onto a closed, convex subset C of $\mathbb{E}$ is define as follows:

$$P_C(x) = \arg\min_{y \in C}\{\|y - x\|\}.$$

Lemma 2.4. [33] *Let* $P_C : \mathbb{E} \to C$ *be the metric projection from* $\mathbb{E}$ *onto* C*. Then*

(i) *For all* $x \in C$ *and* $y \in \mathbb{E}$

$$\|x - P_C(y)\|^2 + \|P_C(y) - y\|^2 \leq \|x - y\|^2.$$

(ii) $z = P_C(x)$ *if and only if*

$$\langle x - z, y - z \rangle \leq 0, \ \forall y \in C.$$

Lemma 2.5. [34] *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{E}$ and $g : C \to \mathbb{R}$ be a convex, subdifferentiable and lower semicontinuous function on C. Moreover, $x \in C$ is a minimizer of a function g if and only if $0 \in \partial g(x) + N_C(x)$ where $\partial g(x)$ and $N_C(x)$ denotes the subdifferential of g at x and the normal cone of C at x respectively.*

Lemma 2.6. [35] *Let $\{x_n\}$ be a sequence in $\mathbb{E}$ and $C \subset \mathbb{E}$ such that the following assumptions are true:*

(i) *For each $x \in C$ the $\lim_{n\to\infty} \|x_n - x\|$ exists;*

(ii) *Every sequentially weak cluster point of $\{x_n\}$ lies in C;*

Thus $\{x_n\}$ converges weakly to a point in C.

Lemma 2.7. [36] *For every $a, b \in \mathbb{E}$ and $\xi \in \mathbb{R}$ the the following relation is true:*

$$\|\xi a + (1-\xi)b\|^2 = \xi\|a\|^2 + (1-\xi)\|b\|^2 - \xi(1-\xi)\|a-b\|^2.$$

Lemma 2.8. [37] *Assume $\{a_n\}, \{b_n\}$ are two sequences in $\mathbb{R}$ in such a way that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Take $\varrho, \sigma \in (0,1)$ and $\mu \in (0, \sigma)$. Then, there is a sequence ζ_n in a manner that $\zeta_n a_n \leq \mu b_n$ and $\zeta_n \in (\varrho\mu, \sigma)$.*

The Lipschitz-type condition with above result gives the following inequality.

Corollary 2.9. *Assume that bifunction f satisfy a Lipschitz-type condition on C through two positive constants c_1 and c_2. Let $\varrho \in (0,1)$, $\sigma < \min\left\{1, \frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and $\mu \in (0, \sigma)$. Then, there is a real number ζ such that*

$$\zeta\big(f(x,z) - f(x,y) - c_1\|x-y\|^2 - c_2\|y-z\|^2\big) \leq \mu f(y,z)$$

and $\varrho\mu < \zeta < \sigma$ where $x, y, z \in C$.

Assumption 1. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfies the following conditions:*

f_1. *$f(x,x) = 0, \forall x \in C$ and f is pseudomontone on feasible set C.*

f_2. *f satisfy the Lipschitz-type condition on $\mathbb{E}$ with constants c_1 and c_2.*

f_3. *$\limsup_{n\to\infty} f(x_n, y) \leq f(u^*, y)$ for every $y \in C$ and $\{x_n\} \subset C$ satisfy $x_n \rightharpoonup u^*$.*

f_4. *$f(x, .)$ is convex and subdifferentiable on C for every $x \in C$.*

3. An Algorithm and Its Convergence Analysis

In this section, we discuss our first method and set a weak convergence theorem for our proposed method. The illustration of the method is given below.

Algorithm 1 (A self-adaptive two-step extragradient method for PEP)

Initialization: Choose $x_0 \in \mathbb{E}$, $\varrho \in (0,1)$, $\sigma < \min\left\{1, \frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\mu \in (0,\sigma)$ and $\lambda_0 > 0$.
Iterative steps: Given x_n and λ_n are known for $n \geq 0$.
Step 1: Compute

$$y_n = \arg\min_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\}.$$

If $x_n = y_n$; STOP. Otherwise, go to next step.
Step 2: Compute $z_n = \arg\min_{y\in C}\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}$ and

$$x_{n+1} = (1-\beta_n)x_n + \beta_n z_n.$$

The step-size sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu f(y_n, z_n)}{f(x_n, z_n) - f(x_n, y_n) - c_1\|x_n - y_n\|^2 - c_2\|z_n - y_n\|^2 + 1}\right\}. \tag{3.1}$$

Set $n := n+1$ and return back to **Iterative steps**.

Remark 3.1. By Corollary 2.9 and λ_{n+1} in eq. (3.1) is well-defined and

$$\lambda_{n+1}\Big[f(x_n, z_n) - f(x_n, y_n) - c_1\|x_n - y_n\|^2 - c_2\|y_n - z_n\|^2\Big] \leq \mu f(y_n, z_n). \tag{3.2}$$

Lemma 3.2. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfying the Assumptions 1. Thus, for each $x^* \in EP(f, C) \neq \emptyset$, we have*

$$\begin{aligned}\|z_n - x^*\|^2 \leq\ & \|x_n - x^*\|^2 - (1-\lambda_{n+1})\|z_n - x_n\|^2 \\ & - \lambda_{n+1}(1 - 2c_1\lambda_n)\|x_n - y_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|z_n - y_n\|^2.\end{aligned}$$

Proof. By Lemma 2.5 and definition of z_n gives that

$$0 \in \partial_2\Big\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\Big\}(z_n) + N_C(z_n).$$

There is $\omega \in \partial_2 f(y_n, z_n)$ and $\overline{\omega} \in N_C(z_n)$ such that $\lambda_n\omega + z_n - x_n + \overline{\omega} = 0$. The above implies that

$$\langle x_n - z_n, y - z_n\rangle = \lambda_n\langle \omega, y - z_n\rangle + \langle \overline{\omega}, y - z_n\rangle,\ \forall y \in C.$$

By $\overline{\omega} \in N_C(z_n)$ and $\langle \overline{\omega}, y - z_n\rangle \leq 0$ for all $y \in C$. This gives that

$$\lambda_n\langle \omega, y - z_n\rangle \geq \langle x_n - z_n, y - z_n\rangle,\ \forall y \in C. \tag{3.3}$$

By $\omega \in \partial_2 f(y_n, z_n)$ we obtain

$$f(y_n, y) - f(y_n, z_n) \geq \langle \omega, y - z_n\rangle,\ \forall y \in C. \tag{3.4}$$

Combining (3.3) and (3.4) gives that

$$\lambda_n f(y_n, y) - \lambda_n f(y_n, z_n) \geq \langle x_n - z_n, y - z_n\rangle,\ \forall y \in C. \tag{3.5}$$

By substituting $y = x^*$ into the expression (3.5) we get

$$\lambda_n f(y_n, x^*) - \lambda_n f(y_n, z_n) \geq \langle x_n - z_n, x^* - z_n\rangle,\ \forall y \in C. \tag{3.6}$$

Since $x^* \in EP(f, C)$ implies that $f(x^*, y_n) \geq 0$ and due to the pseudomonotonicity of a bifunction f implies that $f(y_n, x^*) \leq 0$. From eq. (3.6) we obtain

$$\langle x_n - z_n, z_n - x^* \rangle \geq \lambda_n f(y_n, z_n). \tag{3.7}$$

The expression (3.2) implies that

$$f(y_n, z_n) \geq \lambda_{n+1}\big(f(x_n, z_n) - f(x_n, y_n) - c_1\|x_n - y_n\|^2 - c_2\|y_n - z_n\|^2\big). \tag{3.8}$$

Combining (3.7) and (3.8) provides that

$$\begin{aligned}\langle x_n - z_n, z_n - x^* \rangle \geq \lambda_{n+1}\Big[&\lambda_n\{f(x_n, z_n) - f(x_n, y_n)\} \\ &- c_1\lambda_n\|x_n - y_n\|^2 - c_2\lambda_n\|z_n - y_n\|^2\Big].\end{aligned} \tag{3.9}$$

Similarly by the definition of y_n obtain the following

$$\lambda_n\{f(x_n, y) - f(x_n, y_n)\} \geq \langle x_n - y_n, y - y_n \rangle. \tag{3.10}$$

Substituting $y = z_n$ into above inequality

$$\lambda_n\{f(x_n, z_n) - f(x_n, y_n)\} \geq \langle x_n - y_n, z_n - y_n \rangle. \tag{3.11}$$

By combining the expression (3.9) and (3.11) implies that

$$\begin{aligned}2\langle x_n - z_n, z_n - x^* \rangle \geq \lambda_{n+1}\Big[&2\langle x_n - y_n, z_n - y_n \rangle \\ &- 2c_1\lambda_n\|x_n - y_n\|^2 - 2c_2\lambda_n\|z_n - y_n\|^2\Big].\end{aligned} \tag{3.12}$$

We have the following facts:

$$2\langle x_n - z_n, z_n - x^* \rangle = \|x_n - x^*\|^2 - \|z_n - x_n\|^2 - \|z_n - x^*\|^2.$$
$$2\langle x_n - y_n, z_n - y_n \rangle = \|x_n - y_n\|^2 + \|z_n - y_n\|^2 - \|x_n - z_n\|^2.$$

From the above last two inequalities and expression (3.12) we obtain

$$\begin{aligned}\|z_n - x^*\|^2 \leq\; &\|x_n - x^*\|^2 - (1 - \lambda_{n+1})\|z_n - x_n\|^2 \\ &- \lambda_{n+1}(1 - 2c_1\lambda_n)\|x_n - y_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|z_n - y_n\|^2.\end{aligned}$$

■

Theorem 3.3. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfying the Assumptions 1. Thus, for each $x^* \in EP(f, C) \neq \emptyset$, the sequence $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 1 converges weakly to x^*. Moreover, $\lim_{n\to\infty} P_{EP(f,C)}(x_n) = x^*$.*

Proof. By the definition of x_{n+1} and Lemma 2.7 gives that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n)x_n + \beta_n z_n - x^*\|^2 \\ &= \|(1 - \beta_n)(x_n - x^*) + \beta_n(z_n - x^*)\|^2 \\ &= (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 - \beta_n(1 - \beta_n)\|x_n - z_n\|^2 \\ &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2.\end{aligned} \tag{3.13}$$

By combining the expressions (3.13) and Lemma 3.2 such that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq\; &\|x_n - x^*\|^2 - \beta_n(1 - \lambda_{n+1})\|z_n - x_n\|^2 \\ &- \beta_n\lambda_{n+1}(1 - 2c_1\lambda_n)\|x_n - y_n\|^2 \\ &- \beta_n\lambda_{n+1}(1 - 2c_2\lambda_n)\|z_n - y_n\|^2.\end{aligned} \tag{3.14}$$

From Corollary 2.9 we have $\varrho\mu < \lambda_n < \sigma$ for each $n > 0$. The above implies that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{3.15}$$

This implies that the sequence $\{\|x_n - x^*\|\}$ is non-increasing. Then, there exists the finite limit of $\lim_{n\to\infty} \|x_n - x^*\|$ and also implies that the sequence $\{x_n\}$ is bounded. By using the expression (3.14) we have

$$\begin{aligned} 0 &\leq \beta_n(1 - \lambda_{n+1})\|z_n - x_n\|^2 + \beta_n\lambda_{n+1}(1 - 2c_1\lambda_n)\|x_n - y_n\|^2 \\ &\quad + \beta_n\lambda_{n+1}(1 - 2c_2\lambda_n)\|z_n - y_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.16}$$

Since $0 < \beta_n \leq 1$ and $\varrho\mu < \lambda_n < \sigma$ for each $n > 0$, implies that

$$\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.17}$$

We conclude that the sequences $\{x_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded and for every $x^* \in EP(f, C)$, the $\lim_{n\to\infty} \|x_n - x^*\|^2$ exists. Now further, we show that for every sequential weak cluster point of the sequence $\{x_n\}$ is in $EP(f, C)$. Assume that z is a weak cluster point of $\{x_n\}$ i.e. there exists a subsequence denoted by $\{x_{n_k}\}$ of $\{x_n\}$ weakly converging to z. Then $\{y_{n_k}\}$ and $\{z_{n_k}\}$ also weakly converges to z and $z \in C$. Let us show that $z \in EP(f, C)$. By expression (3.5), (3.8) and (3.11) we have

$$\begin{aligned} \lambda_n f(y_{n_k}, y) &\geq \lambda_n f(y_{n_k}, z_{n_k}) + \langle x_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\ &\geq \lambda_n\lambda_{n+1} f(x_{n_k}, z_{n_k}) - \lambda_n\lambda_{n+1} f(x_{n_k}, y_{n_k}) - c_1\lambda_n\lambda_{n+1}\|x_{n_k} - y_{n_k}\|^2 \\ &\quad - c_2\lambda_n\lambda_{n+1}\|y_{n_k} - z_{n_k}\|^2 + \langle x_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\ &\geq \lambda_{n+1}\langle x_{n_k} - y_{n_k}, z_{n_k} - y_{n_k}\rangle - c_1\lambda_n\lambda_{n+1}\|x_{n_k} - y_{n_k}\|^2 \\ &\quad - c_2\lambda_n\lambda_{n+1}\|y_{n_k} - z_{n_k}\|^2 + \langle x_{n_k} - z_{n_k}, y - z_{n_k}\rangle \end{aligned} \tag{3.18}$$

where y is an any element in C. It follows from (3.17) and the boundness of $\{x_n\}$ that the right-hand side of the last inequality tends to zero. Using $\lambda_n > 0$, condition (f_3) and $y_{n_k} \rightharpoonup z$ implies that

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(z, y), \quad \forall\, y \in C. \tag{3.19}$$

We have $f(z, y) \geq 0$ for all $y \in C$ shows that $z \in EP(f, C)$. Thus, Lemma 2.6 provides that $x_n \rightharpoonup x^*$, $y_n \rightharpoonup x^*$ and $z_n \rightharpoonup x^*$.

Define $t_n := P_{EP(f,C)}(x_n)$ for all $n \in \mathbb{N}$. Since $x^* \in EP(f, C)$, we have

$$\|t_n\| \leq \|t_n - x_n\| + \|x_n\| \leq \|x^* - x_n\| + \|x_n\|. \tag{3.20}$$

Thus, $\{t_n\}$ is bounded. By expression (3.15) we deduce that

$$\|x_{n+1} - t_{n+1}\|^2 \leq \|x_{n+1} - t_n\|^2 \leq \|x_n - t_n\|^2, \ \forall\, n \geq 1. \tag{3.21}$$

From (3.20) and (3.21) imply the existence of the $\lim_{n\to\infty} \|x_n - t_n\|$. By using the expression (3.15) for all $m > n \geq 1$, we have

$$\|t_n - x_m\|^2 \leq \|t_n - x_{m-1}\|^2 \leq \cdots \leq \|t_n - x_n\|^2. \tag{3.22}$$

Next, we show that $\{t_n\}$ is a Cauchy sequence. Let take $t_m, t_n \in EP(f, C)$, for $m > n \geq 1$ and Lemma 2.4(i) with (3.22) gives that

$$\|t_n - t_m\|^2 \leq \|t_n - x_m\|^2 - \|t_m - x_m\|^2 \leq \|t_n - x_n\|^2 - \|t_m - x_m\|^2. \tag{3.23}$$

The existence of $\lim_{n\to\infty} \|t_n - x_n\|$ imply that the $\lim_{m,n\to\infty} \|t_n - t_m\| = 0$ for all $m > n$. Consequently $\{t_n\}$ is a Cauchy sequence. Since $EP(f, C)$ is closed implies that $\{t_n\}$ converges strongly to $t^* \in EP(f, C)$. Now we prove that $t^* = x^*$. It follows from Lemma 2.4(ii) and $x^*, t^* \in EP(f, C)$ we have

$$\langle x_n - t_n, x^* - t_n\rangle \leq 0. \tag{3.24}$$

Since $t_n \to t^*$ and $x_n \rightharpoonup x^*$ we have

$$\langle x^* - t^*, x^* - t^*\rangle \leq 0$$

which implies that $x^* = t^* = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$. Further $\|x_n - y_n\| \to 0$ implies that $\lim_{n\to\infty} P_{EP(f,C)}(y_n) = x^*$. ∎

4. Application to Variational Inequality Problems

We consider the application of our results discussed in above mentioned sections to solve a problem of variational inequality involving pseudomonotone with Lipschitz-type continuous operator. The variational inequality problem is expressed in the following manner:

$$\text{Find } x^* \in C \text{ such that } \big\langle G(x^*), y - x^*\big\rangle \geq 0, \ \forall y \in C.$$

An operator $G : \mathbb{E} \to \mathbb{E}$ is said to be

i. L-Lipschitz continuous on C if $\|G(x) - G(y)\| \leq L\|x - y\|, \ \forall x, y \in C$.
ii. pseudomonotone on C if $\big\langle G(x), y - x\big\rangle \geq 0 \Rightarrow \big\langle G(y), x - y\big\rangle \leq 0, \ \forall x, y \in C$.

Note: If bifunction $f(x, y) := \big\langle G(x), y - x\big\rangle$ for all $x, y \in C$, thus the equilibrium problem translate into the above variational inequality problem with $L = 2c_1 = 2c_2$. By definitions of y_n in the Algorithm 1, we have

$$\begin{aligned}
y_n &= \arg\min_{y\in C}\Big\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\Big\} \\
&= \arg\min_{y\in C}\Big\{\lambda_n\langle G(x_n), y - x_n\rangle + \frac{1}{2}\|x_n - y\|^2\Big\} \\
&= \arg\min_{y\in C}\Big\{\lambda_n\langle G(x_n), y - x_n\rangle + \frac{1}{2}\|x_n - y\|^2 + \frac{\lambda_n^2}{2}\|G(x_n)\|^2 - \frac{\lambda_n^2}{2}\|G(x_n)\|^2\Big\} \\
&= \arg\min_{y\in C}\Big\{\frac{1}{2}\|y - (x_n - \lambda_n G(x_n)\|^2\Big\} - \frac{\lambda_n^2}{2}\|G(x_n)\|^2 \\
&= P_C(x_n - \lambda_n G(x_n)).
\end{aligned} \tag{4.1}$$

Similarly the value of z_n in Algorithm 1 convert into

$$z_n = P_C(x_n - \lambda_n G(y_n)).$$

Assumption 2. *We assume that G satisfying the following assumptions:*

G_1. *G is pseudomonotone on C and $VI(G, C)$ is nonempty;*
G_2. *G is L-Lipschitz continuous on C through positive constant $L > 0$;*
G_3. $\limsup_{n\to\infty}\langle G(x_n), y - x_n\rangle \leq \langle G(u^*), y - u^*\rangle$ *for every $y \in C$ and $\{x_n\} \subset C$ satisfying $x_n \rightharpoonup u^*$.*

We have reduced the following results from our main results that are applicable to solve variational inequality problems.

Corollary 4.1. *Assume that $G : C \to \mathbb{E}$ is satisfying Assumption 2. Let the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generating in the following way:*

(i) Choose $x_0 \in \mathbb{E}$, $\varrho \in (0,1)$, $\sigma < \min\{1, \frac{1}{L}\}$, $\mu \in (0,\sigma)$ and $\lambda_0 > 0$. Compute

$$\begin{cases} y_n = P_C\big(x_n - \lambda_n G(x_n)\big), \\ z_n = P_C\big(x_n - \lambda_n G(y_n)\big), \\ x_{n+1} = (1-\beta_n)x_n + \beta_n z_n, \end{cases}$$

where step-size λ_{n+1} updated as follows:

$$\lambda_{n+1} = \min\Big\{\sigma, \frac{\mu\langle G(y_n), z_n - y_n\rangle}{\langle G(x_n), z_n - y_n\rangle - \frac{L}{2}\|x_n - y_n\|^2 - \frac{L}{2}\|z_n - y_n\|^2 + 1}\Big\}$$

Then the sequence $\{x_n\}$, $\{z_n\}$ and $\{y_n\}$ weakly converges to the solution x^* of $VI(G,C)$.

5. Computational Experiment

Some numerical results will be produced in this part to demonstrate the performance of our proposed methods. The MATLAB codes run in MATLAB version 9.5 (R2018b) on a PC Intel(R) Core(TM)i5-6200 CPU @ 2.30GHz 2.40GHz, RAM 8.00 GB. During all these examples y-axes represent for the value of $D_n = \|x_{n+1} - x_n\|$ while the x-axis indicates to the number of iterations or the execution time (in seconds).

Example 5.1. Assume that a function $f : C \times C \to \mathbb{E}$ is

$$f(x,y) = \langle Ax + By + c, y - x\rangle$$

where $c \in \mathbb{R}^n$ and A, B are matrices of order n such that B is symmetric positive semidefinite and $B - A$ is symmetric negative definite with Lipschitz constants $c_1 = c_2 = \frac{1}{2}\|A - B\|$ (see [30]). Two matrices A, B are randomly generated and entries of a vector c entries randomly belongs to $[-1, 1]$. The feasible set $C \subset \mathbb{R}^n$ is closed and convex and write as

$$C := \{x \in \mathbb{R}^n : -5 \le x_i \le 5\}.$$

The numerical results are shown in Figures 1-6 and Table 1 with $x_0 = (1, \cdots, 1)$, $\lambda = \frac{1}{2.1c_1}$, $\sigma = \frac{1}{2.2c_1}$, $\mu = \frac{1}{2.3c_1}$, $\lambda_0 = \frac{1}{2c_1}$ and $TOL = 10^{-4}$.

Table 1. Numerical results for Example 5.1.

	Quoc Algorithm [30]		Algorithm 1	
n	No. of Iter.	CPU(s) time	No. of Iter.	CPU(s) time
5	41	0.6449	8	0.0786
10	35	0.2643	12	0.0883
20	42	0.3483	13	0.1073

Two matrices are randomly generated E and F with entries from $[-2, 2]$. The matrix $B = E^T E$, $S = F^T F$ and $A = S + B$.

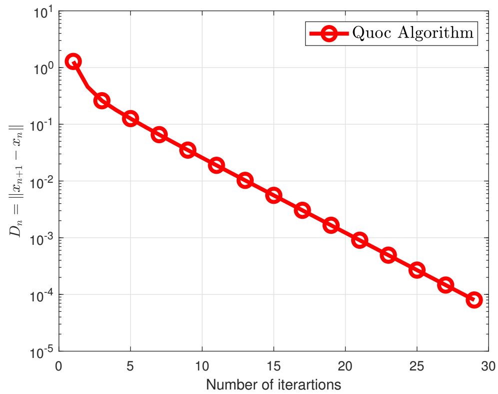

FIGURE 1. Example 5.1 when $n = 5$.

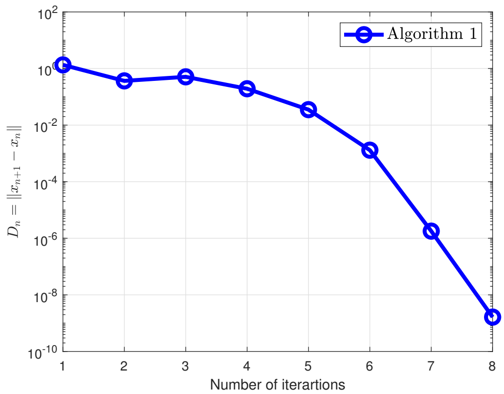

FIGURE 2. Example 5.1 when $n = 5$.

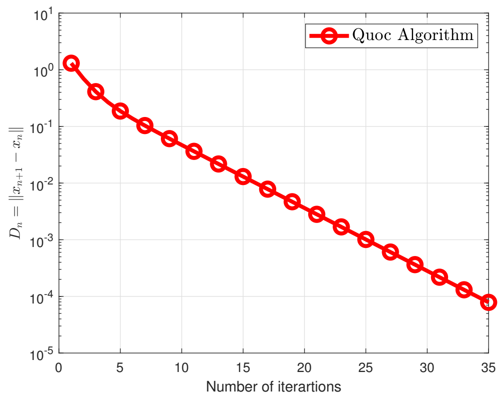

FIGURE 3. Example 5.1 when $n = 10$.

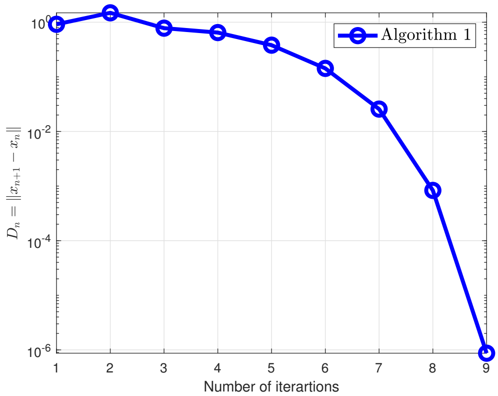

FIGURE 4. Example 5.1 when $n = 10$.

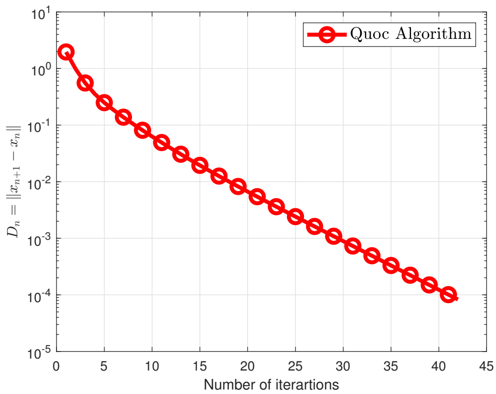

FIGURE 5. Example 5.1 when $n = 20$.

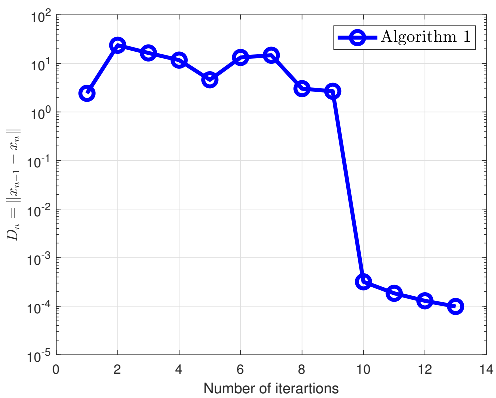

FIGURE 6. Example 5.1 when $n = 20$.

Example 5.2. Suppose $G : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$G(x) = \begin{pmatrix} 0.5x_1x_2 - 2x_2 - 10^7 \\ -4x_1 - 0.1x_2^2 - 10^7 \end{pmatrix}$$

and $C = \{x \in \mathbb{R}^2 : (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\}$. It is not hard to check that G is Lipschitz continuous with $L = 5$ and pseudomonotone. Now we use Corollary 4.1 and step-size $\lambda = 10^{-8}$, $\lambda_0 = 0.1$, $\sigma = \frac{1}{1.1L}$ and $\mu = \frac{1}{1.2L}$. The experimental results are shown in Table 2 and Figures 7-10.

TABLE 2. Numerical results for Example 5.2.

	Quoc Algorithm [30]		Algorithm 1	
$x_0 = y_0$	No. of Iter.	CPU(s) time	No. of Iter.	CPU(s) time
$(1.5, 1.7)$	15	0.6449	4	0.2107
$(2.0, 3.0)$	16	0.7224	9	0.6880
$(1.0, 2.0)$	17	0.6681	5	0.3384
$(2.7, 2.6)$	14	0.6439	8	0.6027

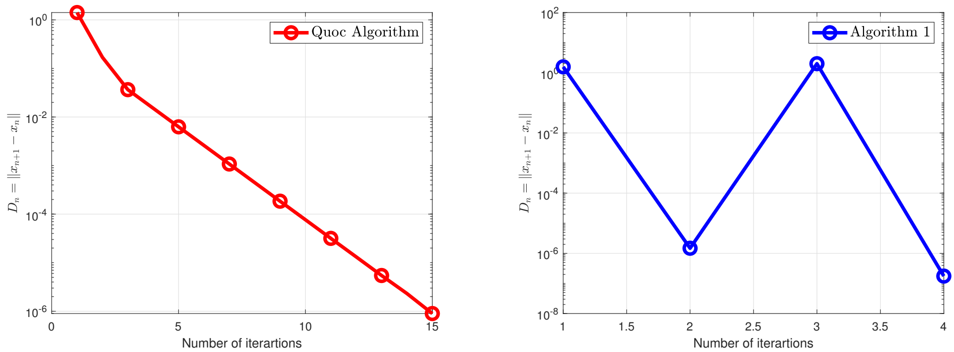

FIGURE 7. Example 5.2 for $x_0 = (1.5, 1.7)$.

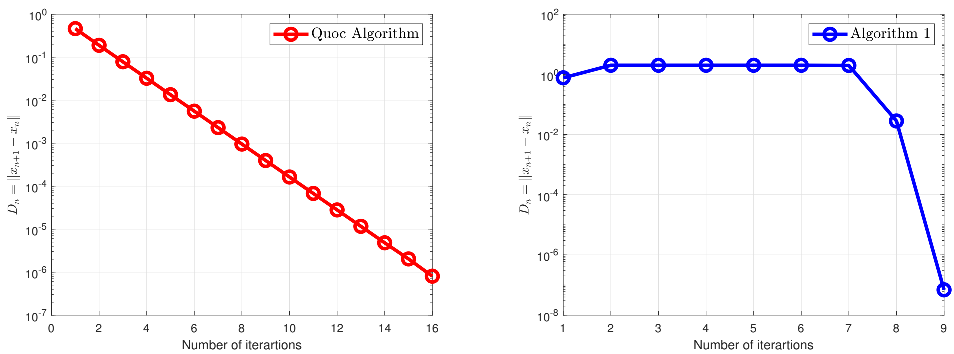

FIGURE 8. Example 5.2 for $x_0 = (2.0, 3.0)$.

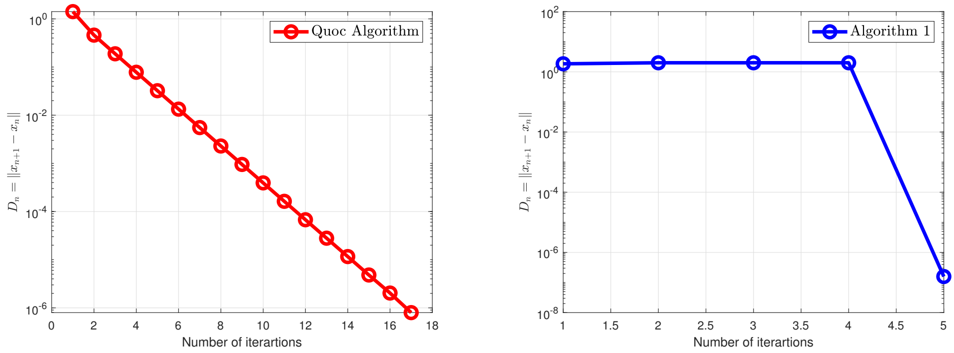

FIGURE 9. Example 5.2 for $x_0 = (1.0, 2.0)$.

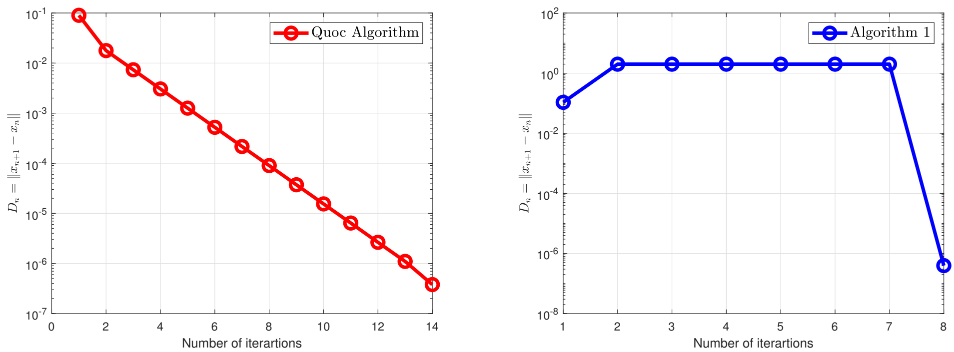

FIGURE 10. Example 5.2 for $x_0 = (2.7, 2.6)$.

6. Conclusion

Two new methods are introduced in this study to deal with pseudomonotone equilibrium problems and variational inequality problems. These methods are a two-step proximal method that gives a weak converging iterative sequence. Numerical results have been accounted for to demonstrate the computational behaviour of methods related to other methods.

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Disclosure Statement

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