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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Numerical Reckoning Fixed Points for Nonexpansive Mappings via a Faster Iteration Process and Its Application to Constrained Minimization Problems, Split Feasibility Problems and Image Deblurring Problems

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Abstract In this paper, we introduce a new faster iteration scheme and establish convergence results for approximation of fixed points of nonexpansive mappings in the framework of Banach spaces. Further, we show that our iteration process is faster than a number of existing iteration processes. We support our analytic proof by numerical examples in which we approximate the fixed point by a computer using Matlab program. Furthermore, we apply our results to find solutions of constrained minimization problems, split feasibility problems and image deblurring problems. Our results are the extension, improvement and generalization of many known results in the literature of iterations in fixed point theory.

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weak and strong convergence; nonexpansive mapping

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1. Introduction

Once the existence of a fixed point of some mapping is established, then to find the value of that fixed point is not an easy task, that is why we use iterative processes for computing them. By time, many iterative processes have been developed and it is impossible to cover them all.

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Numerical reckoning fixed points for nonlinear operators is nowadays an active research direction of nonlinear analysis. This because they found applications to variational inequalities, equilibrium problems, computer simulation, image encoding and much more. Classical iterations such as Picard, Mann and Ishikawa represent pioneers research work in this regard; please, see Mann [1] and Ishikawa [2]. Nowadays, this research direction is developed by Agarwal et al. [3] and Noor [4]. Speed of convergence play important role for an iteration process to be preferred on another iteration process. In [5], Rhoades mentioned that the Mann iteration process for decreasing function converges faster than the Ishikawa iteration process and for increasing function the Ishikawa iteration process is better than the Mann iteration process. Also the Mann iteration process appears to be independent of the initial guess (see also [6]). In [3], the authors claimed that Agarwal iteration process converge at a rate same as that of the Picard iteration process and faster than the Mann iteration process for contraction mappings.

Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $F(T) := \{x : Tx = x\}$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$. For arbitrary chosen $x_1 \in C$, construct a sequence $\{x_n\}$, where x_n is defined iteratively for each positive integer $n \geq 1$ by:

$$x_{n+1} = Tx_n, \tag{1}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \tag{2}$$

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n. \end{cases} \tag{3}$$

The sequences $\{x_n\}$ generated by (1), (2) and (3) are called Picard, Mann [1] and Ishikawa [7] iteration sequences respectively.

In 1955, Krasnoselskii [8] showed that the Picard iteration scheme (1) for a nonexpansive mapping T may fail to converge to fixed point of T even if T has a unique fixed point, but the Mann sequence (2) for $\alpha_n = \frac{1}{2}, \forall n \geq 1$ converges strongly to the fixed point of T.

Mann and Ishikawa iteration methods have been studied by several authors for approximation fixed points of nonexpansive mappings, see, e.g., [2, 9–19].

In 2000, Noor [4] defined the following iterative scheme, by $x_1 \in C$ and

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \end{cases} \tag{4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1).

Recently, Agarwal et al. [3] introduced the following iteration process. For arbitrary chosen $x_1 \in C$ construct a sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in (0,1). They showed that this process converges at a rate that is the same as that of the Picard iteration (1) and faster than the Mann iteration (2) for contractions mapping.

Motivated by the previous ones, we introduce a new faster iteration process for numerical reckoning fixed points of nonexpansive mappings, where the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$ and

$$\begin{cases} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n, \end{cases} \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in (0,1).

The purpose of this paper is to prove convergence results for nonexpansive mappings using the iteration (6). We also prove that the iteration (6) converges faster than Picard, Mann, Ishikawa, Noor and Agarwal et al. iteration processes for contractive mappings in the sense of Berinde [20]. We also present numerical examples to compare the convergence of (6) with Picard, Mann, Ishikawa, Noor and Agarwal et al. iterations. Moreover, we apply our results to find solutions of constrained minimization problems, split feasibility problems and image deblurring problems.

2. Preliminaries

Let E be a Banach space and $S_E = \{x \in E : \|x\| = 1\}$ unit sphere on E. For all $\lambda \in (0,1)$, and $x, y \in S_E$ with $x \neq y$, if $\|(1-\lambda)x + \lambda y\| < 1$, then E is called strictly convex. If E is a strictly convex Banach space and $\|x\| = \|y\| = \|\alpha x + (1-\alpha)y\|$ for $x, y \in E$ and $\alpha \in (0,1)$, then $x = y$.

The space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{7}$$

exists for each x and y in S_E. In this case, the norm of E is called Gateaux differentiable. For all $y \in S_E$, if the limit (7) is attained uniformly for $x \in S_E$, then the norm is said to be uniformly Gateaux differentiable or Frechet differentiable.

We call the space E satisfies the Opial's condition [21] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

A mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

The following definitions about the rate of convergence are due to Berinde [20].

Definition 2.1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$, then $\{a_n\}$ converges faster than $\{b_n\}$.

Definition 2.2. Suppose that for two fixed-point iteration processes $\{x_n\}$ and $\{u_n\}$, both converging to the same fixed point p, the error estimates

$$\|x_n - p\| \leq a_n \quad \text{for all } n \geq 1,$$
$$\|u_n - p\| \leq b_n \quad \text{for all } n \geq 1$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{x_n\}$ *converges faster than* $\{u_n\}$ to p.

We state the following lemmas to be used later on.

Lemma 2.3 ([22]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and T a nonexpansive mapping on C. Then, $I - T$ is demiclosed at zero.*

Lemma 2.4 ([23]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5 ([3]). *Let E be a reflexive Banach space satisfying the Opial's condition, C a nonempty convex subset of E, and $T : C \to X$ an operator such that $I - T$ demiclosed at zero and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

3. Rate of Convergence

In this section, we show that the iteration process (6) converges faster than the iteration of Picard (1). To support our understanding using MATLAB software, we provide two numerical examples.

Theorem 3.1. *Let C be a nonempty closed convex subset of a norm space E. Let T be a contraction with a contraction factor $k \in (0, 1)$ and fixed point p. Let $\{u_n\}$ be defined by the iteration process (1) and $\{x_n\}$ by (6), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\epsilon, 1 - \epsilon]$ for all $n \in \mathbb{N}$ and for some ϵ in (0, 1). Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our process (6) converges faster than (1).*

Proof. Using (6), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1 - \gamma_n)x_n + \gamma_n Tx_n - p\| \\
&= \|(1 - \gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| \\
&\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\
&\leq (1 - \gamma_n)\|x_n - p\| + k\gamma_n\|x_n - p\| \\
&= (1 - \gamma_n + k\gamma_n)\|x_n - p\| \\
&= 1 - (\gamma_n - k\gamma_n)\|x_n - p\| \\
&= (1 - (1 - k)\gamma_n)\|x_n - p\|,
\end{aligned}$$

and so

$$\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n)Tx_n + \beta_n Tz_n - p\| \\
&= \|(1 - \beta_n)(Tx_n - p) + \beta_n(Tz_n - p)\| \\
&\leq (1 - \beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\
&\leq k(1 - \beta_n)\|x_n - p\| + k\beta_n\|z_n - p\| \\
&= k(1 - \beta_n)\|x_n - p\| + k\beta_n((1 - (1 - k)\gamma_n)\|x_n - p\|)
\end{aligned}$$

$$
\begin{aligned}
&= k(1-\beta_n)\|x_n - p\| + (k\beta_n - (1-k)k\beta_n\gamma_n)\|x_n - p\| \\
&= k((1-\beta_n) + \beta_n - (1-k)\beta_n\gamma_n)\|x_n - p\| \\
&= k(1-(1-k)\beta_n\gamma_n)\|x_n - p\|.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\| \\
&= \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \\
&\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\| \\
&\le k(1-\alpha_n)\|x_n - p\| + k\alpha_n\|y_n - p\| \\
&= k(1-\alpha_n)\|x_n - p\| + k\alpha_n(k(1-(1-k)\beta_n\gamma_n)\|x_n - p\|) \\
&= k(1-\alpha_n)\|x_n - p\| + k^2\alpha_n(1-(1-k)\beta_n\gamma_n)\|x_n - p\| \\
&= k(1-\alpha_n) + k^2\alpha_n(1-(1-k))\beta_n\gamma_n\|x_n - p]\| \\
&= k((1-\alpha_n) + k\alpha_n(1-(1-k)\beta_n\gamma_n))\|x_n - p\| \\
&= k((1-\alpha_n) + k(\alpha_n - (1-k)\alpha_n\beta_n\gamma_n))\|x_n - p\| \\
&< k((1-\alpha_n) + (\alpha_n - (1-k)\alpha_n\beta_n\gamma_n))\|x_n - p\| \\
&= k(1-(1-k)\alpha_n\beta_n\gamma_n)\|x_n - p\|.
\end{aligned}
$$

Repetition of above processes gives the following inequalities

$$
\begin{cases}
\|x_{n+1} - p\| & \le k(1-(1-k)\alpha_n\beta_n\gamma_n)\|x_n - p\|, \\
\|x_n - p\| & \le k(1-(1-k)\alpha_{n-1}\beta_{n-1}\gamma_{n-1})\|x_{n-1} - p\|, \\
\|x_{n-1} - p\| & \le k(1-(1-k)\alpha_{n-2}\beta_{n-2}\gamma_{n-2})\|x_{n-2} - p\|, \\
 & \vdots \\
\|x_2 - p\| & \le k(1-(1-k)\alpha_1\beta_1\gamma_1)\|x_1 - p\|, \\
\|x_1 - p\| & \le k(1-(1-k)\alpha_0\beta_0\gamma_0)\|x_0 - p\|.
\end{cases}
$$

From above inequalities, we derive

$$\|x_{n+1} - p\| \le \|x_0 - p\| k^{n+1} \prod_{j=0}^{n} (1-(1-k)\alpha_j\beta_j\gamma_j).$$

It follows that

$$\|x_{n+1} - p\| \le \|x_0 - p\| k^{n+1} (1-(1-k)\alpha\beta\gamma)^{n+1}$$

for all $n \in \mathbb{N}$ and for some $\alpha, \beta, \gamma > 0$ such that $\alpha \le \alpha_n < 1, \beta \le \beta_n < 1$ and $\gamma \le \gamma_n < 1$. By the definition of the Picard iteration process, we have

$$\|u_{n+1} - p\| \le k\|u_n - p\|$$

for all $n \in \mathbb{N}$. Note that $\|u_{n+1} - p\| \le \|u_0 - p\| k^{n+1}$.

Let

$$a_n = k^{n+1}\|u_0 - p\|$$

and

$$b_n = k^{n+1}(1-(1-k)\alpha\beta\gamma)^{n+1}\|x_0 - p\|.$$

Then

$$\begin{aligned}\frac{b_n}{a_n} &= \frac{k^{n+1}(1-(1-k)\alpha\beta\gamma)^{n+1}\|x_0 - p\|}{k^{n+1}\|u_0 - p\|}\\ &= \frac{(1-(1-k)\alpha\beta\gamma)^{n+1}\|x_0 - p\|}{\|u_0 - p\|}.\end{aligned}$$

Define $\theta_n = (1-(1-k)\alpha\beta\gamma)^{n+1}$. Therefore, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} &= \frac{(1-(1-k)\alpha\beta\gamma)^{n+2}}{(1-(1-k)\alpha\beta\gamma)^{n+1}}\\ &= 1-(1-k)\alpha\beta\gamma\\ &< 1.\end{aligned}$$

It thus follows from well-known ratio test that $\sum_{n=0}^{\infty}\theta_n < \infty$. Hence, we have $\lim_{n\to\infty}\theta_n = 0$ which implies that

$$\lim_{n\to\infty}\frac{b_n}{a_n} = 0.$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. ■

Now, we present an example which shows that our iteration process (6) converges at a rate faster than Agarwal et al. iteration process (5), Mann iteration process (2), Ishikawa iteration process (3), Noor iteration process (4) and Picard iteration process (1).

Example 3.2. Let $E = \mathbb{R}$ and $C = [1, 50]$. Let $T : C \to C$ be a mapping, which is defined by

$$T(x) = \sqrt{x^2 - 8x + 40}$$

for all $x \in C$. Choose $\alpha_n = 0.85, \beta_n = 0.65, \gamma_n = 0.45$, with the initial value $x_1 = 40$. The corresponding our iteration process, Agarwal et al. iteration process, Noor iteration process, Ishikawa iteration process, Mann iteration process and Picard iteration processes are respectively given in Table 1.

All sequences converges to $x^* = 5$. Comparison shows that our iteration process (6) requires least number of iterations among all the iterations mentioned below.

Step	Picard	Mann	Ishikawa	Noor	Agarwal	New iteration
1	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000
2	36.3318042492	36.8820336118	34.8751575132	33.9816211055	34.3249281505	32.3527454021
3	32.7008496221	33.7905308732	29.8335259837	28.0887816012	28.7529148550	24.9217277329
4	29.1159538575	30.7306375124	24.9067432334	22.3811620460	23.3289757744	17.8627609012
5	25.5892777970	27.7090706072	20.1467307646	16.9736024952	18.1321892967	11.5463790857
6	22.1381326176	24.7347891266	15.6449263114	12.0962209155	13.3147454600	6.9395376930
7	18.7880774656	21.8200359935	11.5741197024	8.2289280979	9.1939307941	5.2225332988
8	15.5784221001	18.9820007784	8.2638548016	6.0182077910	6.3717274607	5.0138389524
9	12.5721859009	16.2455313784	6.1736938982	5.2517005165	5.2434387591	5.0007808271
10	9.8733161157	13.6475866165	5.3185408455	5.0576355955	5.0298139084	5.0007808271
11	7.6482574613	11.2442765494	5.0768890301	5.0129587850	5.0033662656	5.0000024535
12	6.1081734180	9.1201110370	5.0179832209	5.0029016212	5.0003761718	5.0000001375
13	5.3333287129	7.3913650188	5.0041744485	5.0006491038	5.0000419870	5.0000000077
14	5.0771808572	6.1732610225	5.0009673150	5.0001451769	5.0000046858	5.0000000004
15	5.0160062399	5.4814708358	5.0002240577	5.0000324684	5.0000005229	5.0000000000
16	5.0032258274	5.1725897008	5.0000518932	5.0000072614	5.0000000584	5.0000000000
17	5.0006461643	5.0576419946	5.0000120186	5.0000016240	5.0000000065	5.0000000000
18	5.0001292729	5.0187159301	5.0000027835	5.0000003632	5.0000000007	5.0000000000
19	5.0000258562	5.0060176595	5.0000006447	5.0000000812	5.0000000001	5.0000000000
20	5.0000051713	5.0019286052	5.0000001493	5.0000000182	5.0000000000	5.0000000000
21	5.0000010343	5.0006174572	5.0000000346	5.0000000041	5.0000000000	5.0000000000
22	5.0000002069	5.0001976174	5.0000000080	5.0000000009	5.0000000000	5.0000000000
23	5.0000000414	5.0000632408	5.0000000019	5.0000000002	5.0000000000	5.0000000000
24	5.0000000083	5.0000202374	5.0000000004	5.0000000000	5.0000000000	5.0000000000
25	5.0000000017	5.0000064760	5.0000000001	5.0000000000	5.0000000000	5.0000000000
26	5.0000000003	5.0000020723	5.0000000000	5.0000000000	5.0000000000	5.0000000000
27	5.0000000001	5.0000006631	5.0000000000	5.0000000000	5.0000000000	5.0000000000
28	5.0000000000	5.0000002122	5.0000000000	5.0000000000	5.0000000000	5.0000000000
29	5.0000000000	5.0000000679	5.0000000000	5.0000000000	5.0000000000	5.0000000000
30	5.0000000000	5.0000000217	5.0000000000	5.0000000000	5.0000000000	5.0000000000
31	5.0000000000	5.0000000070	5.0000000000	5.0000000000	5.0000000000	5.0000000000
32	5.0000000000	5.0000000022	5.0000000000	5.0000000000	5.0000000000	5.0000000000
33	5.0000000000	5.0000000007	5.0000000000	5.0000000000	5.0000000000	5.0000000000
34	5.0000000000	5.0000000002	5.0000000000	5.0000000000	5.0000000000	5.0000000000
35	5.0000000000	5.0000000001	5.0000000000	5.0000000000	5.0000000000	5.0000000000
36	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000

TABLE 1. Comparative results.

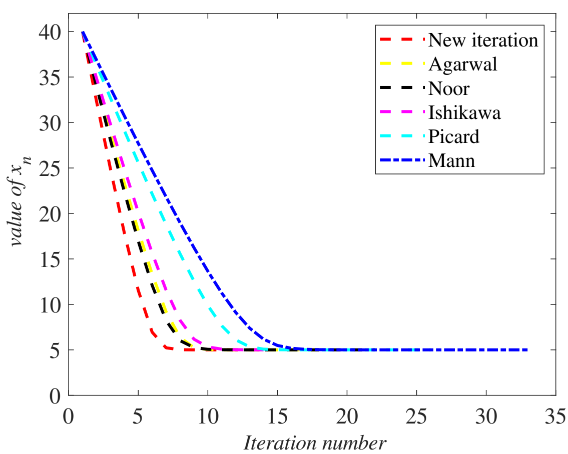

FIGURE 1. Convergence behavior of the Mann, the Picard, the Ishikawa, the Noor, the Argarwal and new iterations for the function given in Example 3.2.

Example 3.3. Let $E = \mathbb{R}$ and $C = [1, 50]$. Let $T : C \to C$ be a mapping, which is defined by

$$T(x) = \sqrt{x^2 - 9x + 54}$$

for all $x \in C$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{3}{4}$, with the initial value $x_1 = 30$. The corresponding our iteration process, Agarwal et al. iteration process, Noor iteration process, Ishikawa iteration process, Mann iteration process and Picard iteration processes are respectively given in Table 2.

Step	Picard	Mann	Ishikawa	Noor	Agarwal	New iteration
1	30.0000000000	30.0000000000	30.0000000000	30.0000000000	30.0000000000	30.0000000000
2	26.1533936612	27.1150452459	25.0119824036	23.4891033190	24.0503308189	31.8347714371
3	22.4191761010	24.2907437151	20.2547559071	17.4668190633	18.4372719353	14.5055413771
4	18.8373796516	21.5420343135	15.8509087868	12.3265857284	13.3938203603	8.9907882699
5	15.4696624163	18.8892775011	12.0133051549	8.7275766163	9.3725555853	6.5301998399
6	12.4130372403	16.3606498049	9.0688620373	6.9585711603	6.9939357160	6.0555659291
7	9.8166266286	13.9954171304	7.2820400289	6.3102146269	6.1862068754	6.0050669583
8	7.8750567432	11.8475686983	6.4668031480	6.0979255677	6.0283693653	6.0045473127
9	6.7187058292	9.9869851099	6.1600652383	6.0306808428	6.0041338820	6.0000407497
10	6.2187342406	8.4900396666	6.0537250393	6.0095903071	6.0095981884	6.0000407497
11	6.0583865336	7.4083030742	6.0179028366	6.0029956076	6.0000864719	6.0000032715
12	6.0148623083	6.7246651786	6.0059514305	6.0009354914	6.0000124982	6.0000000293
13	6.0037328233	6.3468134658	6.0019768478	6.0002921200	6.0000018064	6.0000000026
14	6.0009342942	6.1586728531	6.0006564620	6.0000912177	6.0000002611	6.0000000002
15	6.0002336418	6.0708846663	6.0002179755	6.0000284834	6.0000000377	6.0000000000
16	6.0000584147	6.0313055772	6.0000723757	6.0000088941	6.0000000055	6.0000000000
17	6.0000146039	6.0137535390	6.0000240311	6.0000027772	6.0000000008	6.0000000000
18	6.0000036510	6.0060282506	6.0000079791	6.0000008672	6.0000000001	6.0000000000
19	6.0000009128	6.0026394884	6.0000026493	6.0000002708	6.0000000000	6.0000000000
20	6.0000002282	6.0011551843	6.0000008797	6.0000000846	6.0000000000	6.0000000000
21	6.0000000570	6.0005054713	6.0000002921	6.0000000264	6.0000000000	6.0000000000
22	6.0000000143	6.0002211587	6.0000000970	6.0000000082	6.0000000000	6.0000000000
23	6.0000000036	6.0000967598	6.0000000322	6.0000000026	6.0000000000	6.0000000000
24	6.0000000009	6.0000423330	6.0000000107	6.0000000008	6.0000000000	6.0000000000
25	6.0000000002	6.0000185208	6.0000000035	6.0000000003	6.0000000000	6.0000000000
26	6.0000000001	6.0000081029	6.0000000012	6.0000000001	6.0000000000	6.0000000000
27	6.0000000000	6.0000025450	6.0000000004	6.0000000000	6.0000000000	6.0000000000
28	6.0000000000	6.0000015509	6.0000000001	6.0000000000	6.0000000000	6.0000000000
29	6.0000000000	6.0000006785	6.0000000000	6.0000000000	6.0000000000	6.0000000000
30	6.0000000000	6.0000002969	6.0000000000	6.0000000000	6.0000000000	6.0000000000
31	6.0000000000	6.0000001299	6.0000000000	6.0000000000	6.0000000000	6.0000000000
32	6.0000000000	6.0000000586	6.0000000000	6.0000000000	6.0000000000	6.0000000000
33	6.0000000000	6.0000000249	6.0000000000	6.0000000000	6.0000000000	6.0000000000
34	6.0000000000	6.0000000109	6.0000000000	6.0000000000	6.0000000000	6.0000000000
35	6.0000000000	6.0000000048	6.0000000000	6.0000000000	6.0000000000	6.0000000000
36	6.0000000000	6.0000000021	6.0000000000	6.0000000000	6.0000000000	6.0000000000
37	6.0000000000	6.0000000009	6.0000000000	6.0000000000	6.0000000000	6.0000000000
38	6.0000000000	6.0000000004	6.0000000000	6.0000000000	6.0000000000	6.0000000000
39	6.0000000000	6.0000000002	6.0000000000	6.0000000000	6.0000000000	6.0000000000
40	6.0000000000	6.0000000001	6.0000000000	6.0000000000	6.0000000000	6.0000000000
41	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000

TABLE 2. Comparative results.

All sequences converges to $x^* = 6$. Comparison shows that our iteration process (6) requires least number of iterations among all the iterations mentioned above.

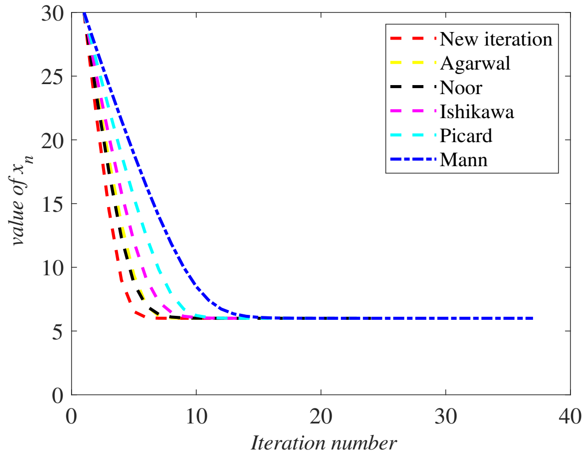

FIGURE 2. Convergence behavior of the Mann, the Picard, the Ishikawa, the Noor, the Argarwal and new iterations for the function given in Example 3.3.

4. Convergence Theorems

In this section, we give some convergence theorems using our iteration process (6); please, see Table 1, Table 2, Figure 1 and Figure 2. Before proving the main theorems, we have the following lemmas.

Example 4.1. Let C be a nonempty closed convex subset of a normed linear space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence defined by (6) and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.

Proof. Let $p \in F(T)$. From (6), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\
&= \|(1-\gamma_n)(x_n - p) + \gamma_n(T x_n - p)\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|T x_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\
&= \|x_n - p\| \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)T x_n + \beta_n T z_n - p\| \\
&= \|(1-\beta_n)(T x_n - p) + \beta_n(T z_n - p)\| \\
&\leq (1-\beta_n)\|T x_n - p\| + \beta_n\|T z_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&= \|x_n - p\|. \qquad (9)
\end{aligned}$$

By using (8) and (9), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)Tx_n+\alpha_n Ty_n-p\|\\ &= \|(1-\alpha_n)(Tx_n-p)+\alpha_n(Ty_n-p)\|\\ &\le (1-\alpha_n)\|Tx_n-p\|+\alpha_n\|Ty_n-p\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|x_n-p\|\\ &= \|x_n-p\|.\end{aligned}$$

This implies that $\{\|x_n-p\|\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n\to\infty}\|x_n-p\|$ exists, as required. ∎

Lemma 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence given by (6) and $F(T)\neq\emptyset$. Then* $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. By Lemma 4.1, we see that $\lim_{n\to\infty}\|x_n-p\|$ exists. Assume that $\lim_{n\to\infty}\|x_n-p\|=c$. Using (8) and (9), we have

$$\limsup_{n\to\infty}\|y_n-p\|\le c \tag{10}$$

and

$$\limsup_{n\to\infty}\|z_n-p\|\le c. \tag{11}$$

Since T is nonexpensive mapping, we have

$$\|Tx_n-p\|\le\|x_n-p\|,\ \|Ty_n-p\|\le\|y_n-p\| \text{ and } \|Tz_n-p\|\le\|z_n-p\|.$$

Taking lim sup on both sides, using (10) and (11), we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\|Tx_n-p\|\le c,\\ &\limsup_{n\to\infty}\|Ty_n-p\|\le c\end{aligned}$$

and

$$\limsup_{n\to\infty}\|Tz_n-p\|\le c. \tag{12}$$

Since

$$\begin{aligned}c &= \lim_{n\to\infty}\|x_{n+1}-p\|\\ &= \lim_{n\to\infty}\|(1-\alpha_n)(Tx_n-p)+\alpha_n(Ty_n-p)\|,\end{aligned}$$

by using (12) and Lemma (2.4), we have

$$\lim_{n\to\infty}\|Tx_n-Ty_n\|=0. \tag{13}$$

Now

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)Tx_n+\alpha_nTy_n-p\| \\ &= \|(1-\alpha_n)(Tx_n-p)+\alpha_n(Ty_n-p)\| \\ &= \|(Tx_n-p)-\alpha_n(Tx_n-p)+\alpha_n(Ty_n-p)\| \\ &= \|(Tx_n-p)+\alpha_n((Ty_n-p)-(Tx_n-p))\| \\ &= \|(Tx_n-p)+\alpha_n(Ty_n-p-Tx_n+p)\| \\ &= \|(Tx_n-p)+\alpha_n(Ty_n-Tx_n)\| \\ &\leq \|Tx_n-p\|+\alpha_n\|Ty_n-Tx_n\|.\end{aligned}$$

Using (13), we have

$$c \leq \liminf_{n\to\infty} \|Tx_n-p\|. \tag{14}$$

It follows from (12) and (14) that

$$\lim_{n\to\infty} \|Tx_n-p\| = c. \tag{15}$$

On the other hand, we have

$$\begin{aligned}\|Tx_n-p\| &= \|Tx_n-Ty_n+Ty_n-p\| \\ &\leq \|Tx_n-Ty_n\|+\|Ty_n-p\| \\ &\leq \|Tx_n-Ty_n\|+\|y_n-p\|,\end{aligned}$$

and this yields that

$$c \leq \liminf_{n\to\infty} \|y_n-p\|. \tag{16}$$

Form (10) and (16), we get

$$\lim_{n\to\infty} \|y_n-p\| = c.$$

Since

$$c = \lim_{n\to\infty} \|y_n-p\| = \lim_{n\to\infty} \|(1-\beta_n)(Tx_n-p)+\beta_n(Tz_n-p)\|. \tag{17}$$

From (12) and (17), by using Lemma 2.4 we obtain

$$\lim_{n\to\infty} \|Tz_n-Tx_n\| = 0. \tag{18}$$

In addition,

$$\begin{aligned}\|Tx_n-p\| &\leq \|Tx_n-Tz_n+Tz_n-p\| \\ &\leq \|Tx_n-Tz_n\|+\|Tz_n-p\| \\ &\leq \|Tx_n-Tz_n\|+\|z_n-p\|.\end{aligned} \tag{19}$$

Using (15), (18) and (19), we have

$$c \leq \liminf_{n\to\infty} \|z_n-p\|. \tag{20}$$

By (11) and (20), we obtain $\lim_{n\to\infty} \|z_n-p\| = c$. Thus

$$\begin{aligned}c &= \lim_{n\to\infty} \|z_n-p\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)(x_n-p)+\gamma_n(Tx_n-p)\|,\end{aligned}$$

gives by Lemma 2.4 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. ∎

By using Lemma 2.3, Lemma 2.5, Lemma 4.1 and Lemma 4.2, we will establish the following theorems.

Theorem 4.3. *Let E be a real uniformly convex Banach space which satisfies the Opial's condition, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by iteration process (6). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in F(T)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By $\lim_{n\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 2.3, therefore we obtain $Tu = u$. Again in the same manner, we can prove that $v \in F(T)$. Next, we prove the uniqueness. From Lemma 4.1 the limits $\lim_{n\to\infty} \|x_n - v\|$ exists. For this suppose that $u \neq v$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| < \lim_{n_i\to\infty} \|x_{n_i} - v\| = \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{n_j\to\infty} \|x_{n_j} - v\| < \lim_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is a contradiction, so $u = v$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. ∎

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (6) and $F(T) \neq \emptyset$. Then $\{x_n\}$ converges to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.*

Proof. Necessity is obvious. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 4.2, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(T)$, therefore $\lim_{n\to\infty} d(x_n, F(T))$ exists. But by hypothesis, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, therefore we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We will show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for given $\epsilon > 0$, there exists n_0 in $\mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, F(T)) < \frac{\epsilon}{2}.$$

Particularly, $\inf\{\|x_{n_0} - p\| : p \in F(T)\} < \frac{\epsilon}{2}$. Hence, there exist $p^* \in F(T)$ such that $\|x_{n_0} - p^*\| < \frac{\epsilon}{2}$. Now, for $m, n \geq n_0$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \leq 2\|x_{n_0} - p^*\| < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed in the Banach space E, so that there exists a point p in C such that $\lim_{n\to\infty} x_n = p$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(p, F(T)) = 0$. Since F is closed, $p \in F(T)$. This completes the proof. ∎

A mapping $T : C \to C$, where C is a subset of a normed space E, is said to satisfy Condition (A) [14] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in (0, 1)$, such that $\|x - Tx\| \geq f(d(x, F(T))$, for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

It is to be noted that Condition (A) is weaker than compactness of the domain C.

Applying Theorem 4.4, we obtain a strong convergence of the process (6) under Condition (A) as follows:

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (6) and $F(T) \neq \emptyset$. Let T satisfy Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We proved in Lemma 4.2 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{21}$$

From Condition (A) and (21), we get

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

i.e., $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Therefore, by Theorem 4.4, the sequence $\{x_n\}$ converges strongly to a point of $F(T)$. The proof is completed. ∎

5. Application to Constrained Minimization Problems, Split Feasibility Problems and Image Deblurring Problems

This section is devoted to some applications. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and $T : C \to H$ a nonlinear operator. T is said to be:
(i) monotone if $\langle Tx - Ty, x - y \rangle \geq 0$ for all $x, y \in C$,
(ii) λ-strongly monotone if there exists a constant $\lambda > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \lambda \|x - y\|^2$$

for all $x, y \in C$,
(iii) v-inverse strongly monotone (v-ism) if there exists a constant $v > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq v \|Tx - Ty\|^2$$

for all $x, y \in C$.

Construction of fixed points of nonexpansive operators is an important subject in the theory of nonexpansive operators and has applications in a number of applied areas such as image recovery and signal processing (see, [24–26]). For instance, split feasibility problem of C and T (denoted by $SFP(C, T)$) is to find a point

$$x \text{ in } C \text{ such that } Tx \in Q, \tag{22}$$

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of another Hilbert space H_2 and $T : H_1 \to H_2$ is a bounded linear operator. The $SFP(C,T)$ is said to be consistent if (22) has a solution. It is easy to see that $SFP(C,T)$ is consistent if and only if the following fixed point problem has a solution:

$$\text{find } x \in C \text{ such that } x = P_C(I - \gamma T^*(I - P_Q)T)x, \tag{23}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively; $\gamma > 0$, and T^* is the adjoint of T. Note that for sufficient small $\gamma > 0$, the operator $P_C(I-\gamma T^*(I-P_Q)T)$ in (23) is nonexpansive.

Application to constrained minimization problems. Let C be a closed convex subset of a Hilbert space H, P_C the metric projection of H onto C and $T : C \to H$ a v-ism where $v > 0$ is a constant. It is well known that $P_C(I - \mu T)$ is nonexpansive operator provided that $\mu \in (0, 2v)$.

The algorithms for signal and image processing are often iterative constrained optimization processes designed to minimize a convex differentiable function T over a closed convex set C in H. It is well known that every L-Lipschitzian operator is $2/L$-ism.

Therefore, we have the following result which generates the sequence of vectors in the constrained or feasible set C which converges weakly to the optimal solution which minimizes T.

Theorem 5.1. *Let C be a closed convex subset of a Hilbert space H and T a convex and differentiable function on an open set D containing the set C. Assume that $\bigtriangledown T$ is an L-Lipschitz operator on D, $\mu \in (0, 2/L)$ and minimizers of T relative to the set C exist. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} &= (1-\alpha_n)P_C(I-\mu \bigtriangledown T)x_n + \alpha_n P_C(I-\mu \bigtriangledown T)y_n, \\ y_n &= (1-\beta_n)P_C(I-\mu \bigtriangledown T)x_n + \beta_n P_C(I-\mu \bigtriangledown T)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu \bigtriangledown T)x_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0, 1)$. Then $\{x_n\}$ converges weakly to a minimizer of T.

Application to image deblurring problems. Let us consider the linear system: find $x \in C$ such that

$$Ax = b,$$

where $A : H \to H$ is bounded linear operator and $b \in H$ is fixed. An algorithm in Theorem 5.1 can be applied directly to solve

$$\min_x \|b - Ax\|_2, \tag{24}$$

by setting

$$T = \frac{1}{2}\|b - Ax\|_2^2.$$

Theorem 5.2. *Let H be a real Hilbert space. Let $A : H \to H$ be a bounded linear operator and $b \in H$. Let $\{x_n\}$ be a sequence generated by $C_1 = H$, $x_0 \in H$ and*

$$\begin{cases} x_{n+1} &= (1-\alpha_n)\left(x_n - \mu A^T(Ax_n - b)\right) + \alpha_n\left(y_n - \mu A^T(Ay_n - b)\right), \\ y_n &= (1-\beta_n)\left(x_n - \mu A^T(Ax_n - b)\right) + \beta_n\left(z_n - \mu A^T(Az_n - b)\right), \\ z_n &= (1-\gamma_n)x_n + \gamma_n\left(x_n - \mu A^T(Ax_n - b)\right), \ n \in \mathbb{N}, \end{cases}$$

where $\mu \subset \left(0, \frac{2}{||A||_2^2}\right)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0,1)$. Then $\{x_n\}$ converges weakly to its solution.

The algorithm in Theorem 5.2 (implemented algorithm) can be used in solving image restoration problem

$$b = Ax + \upsilon.$$

Here b is the observed blurred and noisy image (degraded image), υ is an unknown gaussian noise, and A is a blurring matrix. The blurring matrix A is often ill-conditioned. The aim is to compute an approximation of the original image x. In the most case, the blur generally has much more significant effect than the noise, and thus, the emphasis is on removing the blur. Therefore, the original image x can be approximated by solving an equation (24). We call this kind of problem solving as image deblurring problem. An implemented algorithm is proposed in solving the image deblurring problem. Two kinds of image deblurring consists of Gaussian blur and motion blur are used to test the implemented algorithm.

The original grey and its degraded images on Figure 3 from a Gaussian blur of size 9×9, $\sigma = 4$ and the motion blur with len $= 21$, $\theta = 11$ respectively are used to test an implemented algorithm.

FIGURE 3. Data used in numerical experiments. The true grey image of size 336×252 and two types of its degraded image.

The parameters $\alpha_n, \beta_n, \gamma_n$ and μ on an implemented algorithm in solving the image deblurring problem is set as

$$\alpha_n = \frac{n}{n+1}, \beta_n = \frac{n}{\sqrt{4n^3+2}}, \gamma_n = \frac{n+1}{5n+3}, \mu = 1/\left(||A||_1||A||_\infty\right).$$

These parameters is called as the default choice of set parameter. The quality improvements of the reconstructed grey images sized 336×252 being used implemented algorithm are illustrated in Figure 4 and Figure 5.

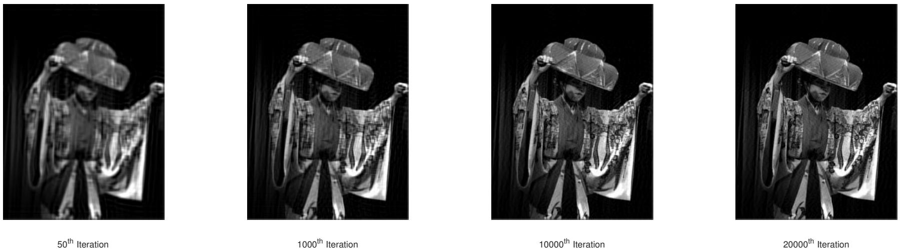

FIGURE 4. The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the Gaussian blurred image on Figure 3.

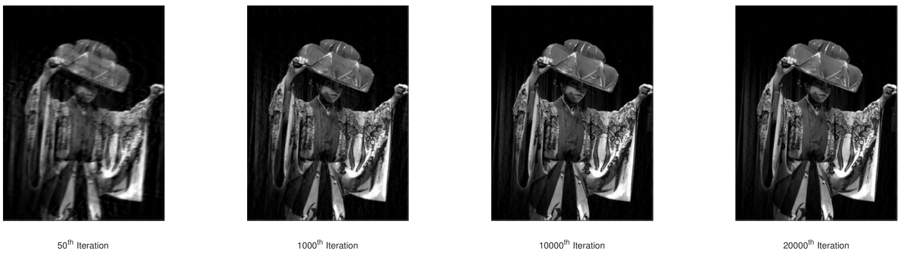

FIGURE 5. The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the motion blurred image on Figure 3.

Next, we are also apply our algorithm in solving the color deblurring problems. The following RGB images illustrate example of blurring adjustment. The three independent deblurring problem consists of red green and blue deblurring channel which are solved with the default parameter.

FIGURE 6. Data used in numerical experiments. The true RGB image of size $336 \times 252 \times 3$, Degraded image with Gaussian blur of size 9×9 and $\sigma = 4$ and the motion blur image with len $= 21$ and $\theta = 11$.

The quality improvements of the reconstructed RGB images sized $336 \times 252 \times 3$ being used the implemented algorithm are also illustrated on Figure 7 and Figure 8.

FIGURE 7. The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the Gaussian blurred image on Figure 6.

FIGURE 8. The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the Motion blurred image on Figure 6.

It can be seen that the restored images on Figures 4, 5, 7, 8 are clearly evident and we also essentially have the original image when the number of iteration is sufficient.

Application to split feasibility problems. Recall that a mapping T in a Hilbert space H is said to be averaged if T can be written as $(1-\alpha)I+\alpha S$, where $\alpha \in (0,1)$ and S is a nonexpansive map on H. Set

$$q(x) := \frac{1}{2}\|(T-P_Q T)x\|, x \in C.$$

Consider the minimization problem

$$\text{find} \min_{x\in C} q(x).$$

By [27], the gradient of q is $\bigtriangledown q = T^*(I-P_Q)T$, where T^* is the adjoint of T. Since $I-P_Q$ is nonexpansive, it follows that $\bigtriangledown q$ is L-Lipschitzian with $L = \|T\|^2$. Therefore, $\bigtriangledown q$ is $1/L$ism and for any $0<\mu<2/L, I-\mu \bigtriangledown q$ is averaged. Therefore, the composition $P_C(I-\mu \bigtriangledown q)$ is also averaged. Set $T := P_C(I-\mu \bigtriangledown q)$. Note that the solution set of $SFP(C,T)$ is $F(T)$.

We now present an iterative process that can be used to find solutions of $SFP(C,T)$.

Theorem 5.3. *Assume that $SFP(C,T)$ is consistent. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n\in\mathbb{N}$ and for some δ in $(0,1)$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} &= (1-\alpha_n)P_C(I-\mu \bigtriangledown q)x_n + \alpha_n P_C(I-\mu \bigtriangledown q)y_n, \\ y_n &= (1-\beta_n)P_C(I-\mu \bigtriangledown q)x_n + \beta_n P_C(I-\mu \bigtriangledown q)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu \bigtriangledown q)x_n,\ n\in\mathbb{N}, \end{cases}$$

where $0<\mu<2/\|T\|^2$. Then $\{x_n\}$ converges weakly to a solution of $SFP(C,T)$.

Proof. Since $T := P_C(I-\mu \bigtriangledown q)$ is nonexpansive, the result follows from Theorem 4.3. ∎

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