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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Stability of the General Mixed Additive and Quadratic Functional Equation in Quasi Banach Spaces

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Abstract In this paper, we prove the generalized Hyers-Ulam-Rassias stability of the following general mixed additive and quadratic functional equation

$$f(\lambda x+y)+f(\lambda x-y)=f(x+y)+f(x-y)+(\lambda-1)[(\lambda+2)f(x)+\lambda f(-x)]$$

where $\lambda \in \mathbb{N}$ and $\lambda \neq 1$ in quasi Banach spaces. Moreover, we use contractive subadditive and expansively superadditive function to prove stability of the general mixed additive and quadratic functional equation in quasi Banach spaces.

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1. Introduction

The stability problem of functional equations was initiated by Ulam [1] in 1940 arising from concerning the stability of group homomorphisms. These question form is the object of the stability theory. In 1941, Hyers [2] provided a first affirmative partial answer to Ulam's problem for the case of approximately additive mapping in Banach spaces. In 1978, Rassias [3] gave a generalization of Hyers's theorem for linear mapping by considering an unbounded Cauchy difference. A generalization of Rassias's result was developed by Găvruţa [4] in 1994 by replacing the unbounded Cauchy difference by a general control function. For more information on that subject and further references we refer to a survey paper [5] and to a recent monograph on Ulam stability [6]. One way to develop the

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stability of functional equations is to replace the class of Banach spaces by quasi-Banach spaces.

Definition 1.1. ([7]) Let X be a vector space over the field $\mathbb{K}$, $\kappa \geqslant 1$ and $\|\cdot\| : X \to \mathbb{R}_+$ be a function such that for all $x, y \in X$ and all $a \in \mathbb{K}$,
(1) $\|x\| = 0$ if and only if $x = 0$.
(2) $\|ax\| = |a| \, \|x\|$.
(3) $\|x + y\| \leq \kappa(\|x\| + \|y\|)$.
Then, $\|\cdot\|$ is called a quasi-normed on X. The smallest possible κ is called the modulus of concavity and $(X, \|\cdot\|, \kappa))$ is called a quasi-normed space. For a quasi-normed space $(X, \|\cdot\|, \kappa)$, without loss of generality we can assume κ is the modulus of concavity.
$(X, \|\cdot\|, \kappa)$ is called a p-normed space if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \tag{1.1}$$

for some $0 < p \leq 1$ and for all $x, y \in X$.

Definition 1.2. ([8]) Let $(X, \|\cdot\|, \kappa)$ be a quasi-normed space.
(1) The sequence $\{x_n\}$ is called convergent sequence if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$, denoted by $\lim_{n\to\infty} x_n = x$.
(2) The sequence $\{x_n\}$ is called Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.
(3) $(X, \|\cdot\|, \kappa)$ is called quasi-Banach space if each Cauchy sequence is a convergent sequence.
(4) $(X, \|\cdot\|, \kappa)$ is p-Banach space if it is p-normed space and quasi-Banach space.

The first difference between a quasi-norm and a norm is that the modulus of concavity of a quasi-norm is greater than or equal to 1, while that of a norm is equal to 1. The quasi-norm is not continuous in general, while a norm is always continuous. For example a noncontinuous quasi-norm, see [[9], example 3]. However, every p-norm is continuous quasi-norm. Moreover,by Aoki-Rolewicz Theorem [[7], Theorem 5], each quasi-norm is equivalent to some-p-norm.

The stability results of functional equation in quasi-Banach spaces was first studied by Najati and Eskandani [10] and Najati and Mogahimi [11]. They establish the general solution of the functional equation

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 2f(2x) + 2f(x)$$

and investigate the Hyers-Ulam-Rassias stability of this equation in quasi-Banach spaces. Then many authors have been interested in this topic; see more [12],[13] and the references therein.

By the Aoki-Rolewicz Theorem, each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

However, quantities relevant to the stability of functional equations are not preserve even by equivalent norms in general. Moreover, the inequality (1.1), which may be seen to have the modulus of concavity equal to 1, and the continuity of p-norms were used in many proofs such as in proving the inequalities (3.17) and (3.20) in the proof of [[11], Theorem 3.2], in proving the inequalities (3.32) and (3.35) in the proof of [[14], Theorem 3.2].

Inspired by the above facts, Dung and Hang [13] were interested in studying the stability of functional equations in quasi-Banach spaces where the quasi-norm is not assumed to be a p-norm, and thus, the modulus of concavity is greater than 1 and the quasi-norm is not continuous in general. To overcome the modulus of concavity greater than 1 and the discontinuity of quasi-norms, They used the squeeze inequality presented in an explicit revision of Aoki-Rolewicz, Theorem [[9], Theorem 1]. As illustrations, They proved an extension of the main result of [11] in p-Banach spaces to quasi-Banach spaces with better approximation; see more [15] and [16]. The technique may be used to prove extensions of other results on the stability of functional equations in p-Banach spaces to quasi-Banach spaces.

Theorem 1.3. ([9]) Let $(Y, \|\cdot\|, \kappa_y)$ be a quasi-normed space, $p = \log_{2\kappa_Y} 2$, and

$$|||x|||_Y = \inf\left\{\left(\sum_{i=1}^{n}\|x_i\|_Y^p\right)^{\frac{1}{p}} : x = \sum_{i=1}^{n} x_i, x_i \in X, n \geqslant 1\right\}$$

for all $x \in Y$. Then, $|||\cdot|||_Y$ is a quasi-norm on Y satisfying

$$|||x+y|||_Y^p \leq |||x|||_Y^p + |||y|||_Y^p \tag{1.2}$$

and

$$\frac{1}{2\kappa_Y}\|x\|_Y \leq |||x|||_Y \leq \|x\|_Y \tag{1.3}$$

for all $x, y \in Y$. In particular, the quasi-norm $|||\cdot|||_Y$ is p-norm, and if $\|\cdot\|_Y$ is a norm then $p = 1$ and $|||\cdot|||_Y = \|\cdot\|_Y$.

In 2011, Eskandani et al. [17] established the general solution and investigate the generalized Hyers-Ulam stability of the following mixed additive and quadratic functional equation

$$\begin{aligned}&f(\lambda x+y)+f(\lambda x-y)\\&=f(x+y)+f(x-y)+(\lambda-1)[(\lambda+2)f(x)+\lambda f(-x)]\end{aligned}$$

where $\lambda \in \mathbb{N}$, $\lambda \neq 1$ in quasi-β-normed spaces, where the quasi-norm is assumed to be p-norm.

In this paper, we use the technique proof of results of [13] to prove the generalized Hyers-Ulam-Rassais stability of the general mixed additive and quadratic functional equation

$$f(\lambda x+y)+f(\lambda x-y)=f(x+y)+f(x-y)+(\lambda-1)[(\lambda+2)f(x)+\lambda f(-x)] \tag{1.4}$$

where $\lambda \in \mathbb{N}$ and $\lambda \neq 1$ in quasi Banach spaces such that the quasi-norm is not assumed to be p-norm.

2. The Stability of the Functional Equation (1.4) in Quasi Banach Spaces

In this section, we prove the Hyer-Ulam-Rassias stability of the general mixed additive and quadratic functional equation in quasi-Banach spaces. Thoughout of this section, let X be a real vector space and $(Y, \|\cdot\|_Y, k_Y)$ be a real quasi-Banach spaces. Let f be a mapping of X into Y. Conveniently, we define

$$\begin{aligned}D_\lambda f(x,y) =& f(\lambda x+y)+f(\lambda x-y)-f(x+y)-f(x-y)\\&-(\lambda-1)[(\lambda+2)f(x)+\lambda f(-x)]\end{aligned}$$

for all $x, y \in X$, $\lambda \in \mathbb{N}$ and $\lambda \neq 1$.

Theorem 2.1. *Suppose that*

(1) $\varphi : X \times X \to [0, \infty)$ *is a function satisfying*

$$\lim_{n\to\infty} \lambda^{2n}\varphi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0 \tag{2.1}$$

and

$$\sum_{i=1}^{\infty} \lambda^{2ip}\varphi^p\left(\frac{x}{\lambda^i}, 0\right) < \infty \tag{2.2}$$

for all $x, y \in X$ *with* $p = \log_{2\kappa_Y} 2$.

(2) $f : X \to Y$ *is an even function such that* $f(0) = 0$ *and satisfying*

$$\|D_\lambda f(x, y)\| \leq \varphi(x, y) \tag{2.3}$$

for all $x, y \in X$.

Then there exists a unique general mixed additive and quadratic functional equation $Q : X \to Y$ *satisfying*

$$\|f(x) - Q(x)\|_Y \leq \frac{\kappa_Y}{\lambda^2}\left(\tilde{\psi}(x)\right)^{\frac{1}{p}} \tag{2.4}$$

where

$$\tilde{\psi}(x) = \sum_{i=1}^{\infty} \lambda^{2ip}\varphi^p\left(\frac{x}{\lambda^i}, 0\right)$$

for all $x \in X$.

Proof. Let $x \in X$. Since f is an even function, we get $f(x) = f(-x)$. Replacing y by 0 in (2.3), we have

$$\begin{aligned}
\varphi(x, 0) &\geq \|f(\lambda x) + f(\lambda x) - f(x) - f(x) - (\lambda - 1)[(\lambda + 2)f(x) + \lambda f(-x)]\|_Y \\
&= \left\|2f(\lambda x) - 2\lambda^2 f(x)\right\|_Y \\
&= 2\left\|f(\lambda x) - \lambda^2 f(x)\right\|_Y
\end{aligned} \tag{2.5}$$

for all $x \in X$. Replacing x by $\frac{x}{\lambda^{n+1}}$ and multiplying λ^{2n} on the both sides in (2.5) with non-negetive integers n, we get

$$\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2(n+1)} f\left(\frac{x}{\lambda^{n+1}}\right)\right\|_Y \leq \frac{\lambda^{2n}}{2}\varphi\left(\frac{x}{\lambda^{n+1}}, 0\right)$$

for all $x \in X$ and all non-negative integers n. By using Theorem 1.3, for any $m, n \in \mathbb{N}$ with $m < n$, we have

$$\begin{aligned}
&\frac{1}{2}\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|_Y^p \\
&\leq \left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|\right|_Y^p \\
&= \left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right) + \lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|\right|_Y^p \\
&\leq \left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right)\right\|\right|_Y^p + \left|\left\|\lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|\right|_Y^p \\
&\leq \left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2(m+2)} f\left(\frac{x}{\lambda^{m+2}}\right)\right\|\right|_Y^p + \left|\left\|\lambda^{2(m+2)} f\left(\frac{x}{\lambda^{m+2}}\right) - \lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right)\right\|\right|_Y^p \\
&\quad + \left|\left\|\lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|\right|_Y^p \\
&\quad \vdots \\
&\leq \left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2(n-1)} f\left(\frac{x}{\lambda^{n-1}}\right)\right\|\right|_Y^p + \left|\left\|\lambda^{2(n-1)} f\left(\frac{x}{\lambda^{n-1}}\right) - \lambda^{2(n-2)} f\left(\frac{x}{\lambda^{n-2}}\right)\right\|\right|_Y^p \\
&\quad + \cdots + \left|\left\|\lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|\right|_Y^p \\
&\leq \left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2(n-1)} f\left(\frac{x}{\lambda^{n-1}}\right)\right\|_Y^p + \left\|\lambda^{2(n-1)} f\left(\frac{x}{\lambda^{n-1}}\right) - \lambda^{2(n-2)} f\left(\frac{x}{\lambda^{n-2}}\right)\right\|_Y^p \\
&\quad + \cdots + \left\|\lambda^{2(m+1)} f\left(\frac{x}{\lambda^{m+1}}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|_Y^p \\
&\leq \frac{\lambda^{2(n-1)p}}{2^p}\varphi^p\left(\frac{x}{\lambda^n}, 0\right) + \frac{\lambda^{2(n-2)p}}{2^p}\varphi^p\left(\frac{x}{\lambda^{n-1}}, 0\right) + \ldots + \frac{\lambda^{2mp}}{2^p}\varphi^p\left(\frac{x}{\lambda^{m+1}}, 0\right) \\
&= \frac{1}{2^p}\sum_{i=m}^{n-1} \lambda^{2ip}\varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right) \\
&\leq \frac{1}{2^p}\sum_{i=m}^{\infty} \lambda^{2ip}\varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right) \qquad (2.6)
\end{aligned}$$

for all $x \in X$. Letting $n, m \to \infty$ in (2.6) and using (2.2), we get

$$\lim_{n\to\infty}\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|_Y = 0$$

for all $x \in X$. This show that the sequence $\left\{\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)\right\}$ is Cauchy in $(Y, \|\cdot\|_Y, \kappa_Y)$ for all $x \in X$. Since $(Y, \|\cdot\|_Y, \kappa_Y)$ is a real quasi-Banach space, the sequence $\left\{\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)\right\}$ is convergent for all $x \in X$. So, we can define the function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \lambda^{2n} f\left(\frac{x}{\lambda^n}\right) \qquad (2.7)$$

for all $x \in X$. Let $m = 0$ in (2.6), we get

$$\left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right\|\right|_Y^p \leq \frac{1}{2^p}\sum_{i=0}^{n-1} \lambda^{2ip}\varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right) \qquad (2.8)$$

for all $x \in X$. By using the continuity of $|||\cdot|||_Y$ and the inequality (2.8), we have

$$\begin{aligned}
|||Q(x)-f(x)|||_Y^p &= \left|\left|\left|\lim_{n\to\infty} \lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right|\right|\right|_Y^p \\
&= \left|\left|\left|\lim_{n\to\infty} \left(\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right)\right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \left|\left|\left|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right|\right|\right|_Y^p \\
&\leq \frac{1}{2^p} \sum_{i=0}^{\infty} \lambda^{2ip} \varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right) \\
&= \frac{1}{2^p\lambda^{2p}} \sum_{i=1}^{\infty} \lambda^{2ip} \varphi^p\left(\frac{x}{\lambda^{i}}, 0\right) \\
&= \frac{1}{2^p\lambda^{2p}} \tilde{\psi}(x)
\end{aligned} \tag{2.9}$$

for all $x \in X$. It follows from (2.9) that we get

$$\|Q(x)-f(x)\|_Y \leq 2\kappa_Y\, |||Q(x)-f(x)|||_Y \leq \frac{2\kappa_y}{2\lambda^2} \tilde{\psi}(x)^{\frac{1}{p}} = \frac{\kappa_Y}{\lambda^2} \tilde{\psi}(x)^{\frac{1}{p}}$$

Thus, (2.4) holds for all $x \in X$. Next, we will prove that Q is mixed general additive quadratic. By using the continuity of $|||\cdot|||$, we have

$$\begin{aligned}
&|||Q(\lambda x+y)+Q(\lambda x-y)-Q(x+y)-Q(x-y)-(\lambda-1)[(\lambda+2)Q(x)+\lambda Q(-x)]|||_Y \\
&= \left|\left|\left|\lim_{n\to\infty} \lambda^{2n} f\left(\frac{\lambda x+y}{\lambda^n}\right) + \lim_{n\to\infty} \lambda^{2n} f\left(\frac{\lambda x-y}{\lambda^n}\right) - \lim_{n\to\infty} \lambda^{2n} f\left(\frac{x+y}{\lambda^n}\right)\right.\right.\right. \\
&\quad \left.\left.\left. - \lim_{n\to\infty} \lambda^{2n} f\left(\frac{x-y}{\lambda^n}\right) -(\lambda-1)\left[(\lambda+2)\lim_{n\to\infty} \lambda^{2n} f\left(\frac{x}{\lambda^n}\right) + \lambda \lim_{n\to\infty} \lambda^{2n} f\left(-\frac{x}{\lambda^n}\right)\right]\right|\right|\right|_Y \\
&= \lim_{n\to\infty} \left|\left|\left|\lambda^{2n} f\left(\frac{\lambda x+y}{\lambda^n}\right) + \lambda^{2n} f\left(\frac{\lambda x-y}{\lambda^n}\right) - \lambda^{2n} f\left(\frac{x+y}{\lambda^n}\right)\right.\right.\right. \\
&\quad \left.\left.\left. - \lambda^{2n} f\left(\frac{x-y}{\lambda^n}\right) -(\lambda-1)\left[(\lambda+2)\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) + \lambda\cdot\lambda^{2n} f\left(-\frac{x}{\lambda^n}\right)\right]\right|\right|\right|_Y \\
&\leq \lim_{n\to\infty} \lambda^{2n} \varphi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) \\
&= 0
\end{aligned}$$

for all $x, y \in X$. This implies that

$$Q(\lambda x+y)+Q(\lambda x-y) = Q(x+y)+Q(x-y)+(\lambda-1)[(\lambda+2)Q(x)+\lambda Q(-x)] \tag{2.10}$$

for all $x, y \in X$. Since f is an even function and $f(0) = 0$, from (2.7), we get that Q is also even and $Q(0) = 0$. Since Q satisfy (2.10), Q is quadratic.

Finally, we prove the uniqueness of Q. Let $T : X \to Y$ be another general mixed additive quadratic function satisfy

$$\|f(x)-T(x)\|_Y \leq \frac{\kappa_Y}{\lambda^2} \left(\tilde{\psi}(x)\right)^{\frac{1}{p}}$$

for all $x \in X$. Since T is mixed general additive quadratic, we have

$$T(\lambda x + y) + T(\lambda x - y) = T(x+y) + T(x-y) + (\lambda - 1)[(\lambda+2)T(x) + \lambda T(-x)] \tag{2.11}$$

for all $x \in X$. Setting $y = 0$ in (2.11), we obtain that

$$T(\lambda x) = \lambda^2 T(x) \tag{2.12}$$

for all $x \in X$. Replacing x by $\frac{x}{\lambda}$ in (2.12), we get

$$T(x) = \lambda^2 T\left(\frac{x}{\lambda}\right) \tag{2.13}$$

for all $x \in X$. Replacing x by $\frac{x}{\lambda}$ in (2.13), we get

$$T\left(\frac{x}{\lambda}\right) = \lambda^2 T\left(\frac{x}{\lambda^2}\right) \tag{2.14}$$

for all $x \in X$. From (2.13) and (2.14) for all $n \in \mathbb{N}$, we have

$$T(x) = \lambda^2 T(x) = \lambda^2 \left(\lambda^2 T\left(\frac{x}{\lambda^2}\right)\right) = (\lambda^2)^2 T\left(\frac{x}{\lambda^2}\right)$$

for all $x \in X$. By induction process, we have

$$T(x) = \lambda^{2n} T\left(\frac{x}{\lambda^n}\right)$$

for all $x \in X$ and all $n \in \mathbb{N}$. By using the continuity of $|||\cdot|||_Y$ and Theorem 1.3, we have

$$\begin{aligned}
|||Q(x) - T(x)|||_Y^p &= \left|\left|\left|\lim_{n\to\infty} \lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2n} T\left(\frac{x}{\lambda^n}\right)\right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \left|\left|\left|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2n} T\left(\frac{x}{\lambda^n}\right)\right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \lambda^{2np} \left|\left|\left|f\left(\frac{x}{\lambda^n}\right) - T\left(\frac{x}{\lambda^n}\right)\right|\right|\right|_Y^p \\
&\leq \lim_{n\to\infty} \lambda^{2np} \left\|f\left(\frac{x}{\lambda^n}\right) - T\left(\frac{x}{\lambda^n}\right)\right\|_Y^p \\
&\leq \lim_{n\to\infty} \lambda^{2np} \frac{\kappa_Y}{\lambda^2} \left(\tilde{\psi}(\frac{x}{\lambda^n})\right) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \lambda^{2np} \tilde{\psi}(\frac{x}{\lambda^n}) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \sum_{i=1}^{\infty} \lambda^{2(n+i)p} \varphi^p \left(\frac{x}{\lambda^{n+i}}, 0\right) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \sum_{i=n+1}^{\infty} \lambda^{2ip} \varphi^p \left(\frac{x}{\lambda^{i}}, 0\right) \\
&= 0
\end{aligned}$$

for all $x \in X$. This implies that

$$|||Q(x) - T(x)|||_Y^p = 0.$$

Then $Q(x) = T(x)$ for all $x \in X$. So Q is unique. ∎

Theorem 2.2. *Suppose that*

(1) *X is a real quasi-normed space and $(Y, \|\cdot\|_Y, k_Y)$ is a real quasi-Banach space.*
(2) *$\varphi : X \times X \to [0, \infty)$ is a function satisfying*

$$\lim_{n\to\infty} \frac{1}{\lambda^{2n}} \varphi\left(\lambda^n x, \lambda^n y\right) = 0 \tag{2.15}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{\lambda^{2ip}} \varphi^p\left(\lambda^i x, 0\right) < \infty \tag{2.16}$$

for all $x \in X$ with $p = \log_{2\kappa_Y} 2$.
(3) *$f : X \to Y$ is an even function such that $f(0) = 0$ and satisfying*

$$\|D_\lambda f(x, y)\| \le \varphi(x, y) \tag{2.17}$$

for all $x, y \in X$.

Then there exists a unique general mixed additive and quadratic functional equation $G : X \to Y$ satisfying

$$\|f(x) - G(x)\|_Y \le \frac{\kappa_Y}{\lambda^2} \left(\tilde{\psi}(x)\right)^{\frac{1}{p}} \tag{2.18}$$

where

$$\tilde{\psi}(x) = \sum_{i=1}^{\infty} \frac{1}{\lambda^{2ip}} \varphi^p\left(\lambda^i x, 0\right)$$

for all $x \in X$.

Proof. By the same argument of Theorem 2.1, we also have

$$2\left\|f(\lambda x) - \lambda^2 f(x)\right\|_Y \le \varphi(x, 0) \tag{2.19}$$

for all $x \in X$. Replacing x by $\lambda^n x$ and Multiply $\frac{1}{\lambda^{2(n+1)}}$ on the both sides in (2.19), we have

$$\left\|\frac{1}{\lambda^{2(n+1)}} f(\lambda^{n+1} x) - \frac{1}{\lambda^{2n}} f(\lambda^n x)\right\|_Y \le \frac{1}{2\lambda^{2(n+1)}} \varphi(\lambda^n x, 0) \tag{2.20}$$

for all $x \in X$ and all non-negative integers n. By using Theorem 1.3 and the equality (2.20), for any $m, n \in \mathbb{N}$ with $m < n$, we have

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{1}{\lambda^{2n}}f(\lambda^n x) - \frac{1}{\lambda^{2m}}f(\lambda^m x)\right\|_Y^p \\
&\leq \left|\left|\left|\frac{1}{\lambda^{2n}}f(\lambda^n x) - \frac{1}{\lambda^{2m}}f(\lambda^m x)\right|\right|\right|_Y^p \\
&= \left|\left|\left|\frac{1}{\lambda^{2n}}f(\lambda^n x) - \frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x) + \frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x) - \frac{1}{\lambda^{2m}}f(\lambda^m x)\right|\right|\right|_Y^p \\
&\leq \left|\left|\left|\frac{1}{\lambda^{2n}}f(\lambda^n x) - \frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x)\right|\right|\right|_Y^p + \left|\left|\left|\frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x) - \frac{1}{\lambda^{2m}}f(\lambda^m x)\right|\right|\right|_Y^p \\
&\leq \left|\left|\left|\frac{1}{\lambda^{2n}}f(\lambda^n x) - \frac{1}{\lambda^{2(m+2)}}f(\lambda^{m+2}x)\right|\right|\right|_Y^p + \left|\left|\left|\frac{1}{\lambda^{2(m+2)}}f(\lambda^{n+2}x) - \frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x)\right|\right|\right|_Y^p \\
&\quad + \left|\left|\left|\frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x) - \frac{1}{\lambda^{2m}}f(\lambda^m x)\right|\right|\right|_Y^p \\
&\ \ \vdots \\
&\leq \left|\left|\left|\frac{1}{\lambda^{2n}}f(\lambda^n x) - \frac{1}{\lambda^{2(n-1)}}f(\lambda^{n-1}x)\right|\right|\right|_Y^p + \left|\left|\left|\frac{1}{\lambda^{2(n-1)}}f(\lambda^(n-1)x) - \frac{1}{\lambda^{2(n-2)}}f(\lambda^{n-2}x)\right|\right|\right|_Y^p \\
&\quad + \ldots + \left|\left|\left|\frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x) - \frac{1}{\lambda^{2m}}f(\lambda^m x)\right|\right|\right|_Y^p \\
&\leq \left\|\frac{1}{\lambda^{2n}}f(\lambda^n x) - \frac{1}{\lambda^{2(n-1)}}f(\lambda^{n-1}x)\right\|_Y^p + \left\|\frac{1}{\lambda^{2(n-1)}}f(\lambda^(n-1)x) - \frac{1}{\lambda^{2(n-2)}}f(\lambda^{n-2}x)\right\|_Y^p \\
&\quad + \ldots + \left\|\frac{1}{\lambda^{2(m+1)}}f(\lambda^{m+1}x) - \frac{1}{\lambda^{2m}}f(\lambda^m x)\right\|_Y^p \\
&\leq \frac{1}{2^p\lambda^{2np}}\varphi(\lambda^{n-1}x, 0) + \frac{1}{2^p\lambda^{2(n-1)p}}\varphi(\lambda^{n-2}x, 0) + \ldots + \frac{1}{2^p\lambda^{2(m+1)p}}\varphi(\lambda^{m+1}x, 0) \\
&= \frac{1}{2^p}\sum_{i=m}^{n-1}\frac{1}{\lambda^{2(i+1)p}}\varphi^p(\lambda^i x, 0)
\end{aligned} \tag{2.21}$$

for all $x \in X$. Letting $n, m \to \infty$ in (2.21) and using (2.16), we get

$$\lim_{n\to\infty}\left\|\frac{1}{\lambda^{2n}x}f(\lambda^{2n}x) - \frac{1}{\lambda^{2m}x}f(\lambda^{2m}x)\right\|_Y = 0$$

for all $x \in X$. This show that the sequence $\left\{\frac{1}{\lambda^{2n}x}f(\lambda^{2n}x)\right\}$ is Cauchy in $(Y, \|\cdot\|_Y, \kappa_Y)$ for all $x \in X$. Since $(Y, \|\cdot\|_Y, \kappa_Y)$ is a real quasi-Banach space, the sequence $\left\{\frac{1}{\lambda^{2n}x}f(\lambda^{2n}x)\right\}$ is convergent for all $x \in X$. So we can define the function $H : X \to Y$ by

$$G(x) = \lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(\lambda^n x) \tag{2.22}$$

for all $x \in X$. Let $m = 0$ in (2.21), we get

$$\left|\left|\left|\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right) - f(x)\right|\right|\right|_Y^p \leq \frac{1}{2^p}\sum_{i=0}^{n-1}\frac{1}{\lambda^{2(i+1)p}}\varphi^p\left(\lambda^i x, 0\right) \tag{2.23}$$

for all $x \in X$. By using the continuity of $|||\cdot|||_Y$ and the inequality (2.23), we have

$$\begin{aligned}|||G(x)-f(x)|||_Y^p &= \left|\left|\left|\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(\lambda^n x)-f(x)\right|\right|\right|_Y^p\\ &= \left|\left|\left|\lim_{n\to\infty}\left(\frac{1}{\lambda^{2n}}f(\lambda^n x)-f(x)\right)\right|\right|\right|_Y^p\\ &= \lim_{n\to\infty}\left|\left|\left|\frac{1}{\lambda^{2n}}f(\lambda^n x)-f(x)\right|\right|\right|_Y^p\\ &\leq \frac{1}{2^p}\sum_{i=0}^{\infty}\frac{1}{\lambda^{2(i+1)p}}\varphi^p(\lambda^i x,0)\\ &= \frac{1}{2^p\lambda^{2p}}\tilde{\psi}(x) \end{aligned}\tag{2.24}$$

for all $x \in X$. From Theorem 1.3 and the inequality (2.24), we have

$$\|G(x)-f(x)\|_Y \leq 2\kappa_Y\,|||G(x)-f(x)||| \leq \frac{\kappa_Y}{\lambda^2}\tilde{\psi}(x)^{\frac{1}{p}}$$

for all $x \in X$. Thus, (2.18) holds for all $x \in X$. Next, we will prove that G is mixed general additive and quadratic function. By using the continuity of $|||\cdot|||$, we have

$$\begin{aligned}&|||G(\lambda x+y)+G(\lambda x-y)-G(x+y)-G(x-y)-(\lambda-1)[(\lambda+2)G(x)+\lambda G(-x)]|||_Y\\ &= \left|\left|\left|\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(\lambda^n(\lambda x+y))+\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(\lambda^n(\lambda x-y))-\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(\lambda^n(x+y))\right.\right.\right.\\ &\quad\left.\left.\left.-\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(\lambda^n(x-y))-(\lambda-1)\left[(\lambda+2)\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(\lambda^n x)+\lambda\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f(-\lambda^n x)\right]\right|\right|\right|_Y\\ &= \lim_{n\to\infty}\left|\left|\left|\frac{1}{\lambda^{2n}}f(\lambda^n(\lambda x+y))+\frac{1}{\lambda^{2n}}f(\lambda^n(\lambda x-y))-\frac{1}{\lambda^{2n}}f(\lambda^n(x+y))\right.\right.\right.\\ &\quad\left.\left.\left.-\frac{1}{\lambda^{2n}}f(\lambda^n(x-y))-(\lambda-1)\left[(\lambda+2)\frac{1}{\lambda^{2n}}f(\lambda^n x)+\lambda\cdot\frac{1}{\lambda^{2n}}f(-\lambda^n x)\right]\right|\right|\right|_Y\\ &\leq \lim_{n\to\infty}\lambda^{2n}\varphi\left(\frac{x}{\lambda^n},\frac{y}{\lambda^n}\right)\\ &= 0 \end{aligned}\tag{2.25}$$

for all $x, y \in X$. This implies that

$$G(\lambda x+y)+G(\lambda x-y)=G(x+y)+G(x-y)+(\lambda-1)[(\lambda+2)G(x)+\lambda G(-x)]\tag{2.26}$$

for all $x, y \in X$. Since f is an even function and $f(0)=0$, from (2.22) we get that G is also even and $G(0)=0$. Since G satisfy (2.26), G is quadratic.

Finally, we prove the uniqueness of G. Let $H : X \to Y$ be another general mixed additive and quadratic function satisfy

$$\|f(x)-H(x)\|_Y \leq \frac{\kappa_Y}{\lambda^2}\left(\tilde{\psi}(x)\right)^{\frac{1}{p}}$$

for all $x \in X$. Since T is mixed general additive quadratic, we have

$$\begin{aligned}&H(\lambda x+y)+H(\lambda x-y)\\ &=H(x+y)+H(x-y)+(\lambda-1)[(\lambda+2)H(x)+\lambda H(-x)]\end{aligned}\tag{2.27}$$

for all $x, y \in X$. Setting $y = 0$ in (2.27), we obtain that $H(\lambda x) = \lambda^2 H(x)$, that is,

$$H(x) = \frac{1}{\lambda^2} H(\lambda x) \tag{2.28}$$

for all $x \in X$. Replacing x by λx in (2.28), we get $H(\lambda x) = \frac{1}{\lambda^2} H(\lambda^2 x)$, that is,

$$H(x) = \frac{1}{\lambda^2} H(\lambda x) = \frac{1}{\lambda^2}\left(\frac{1}{\lambda^2} H(\lambda^2 x)\right) = \frac{1}{(\lambda^2)^2} H(\lambda^2 x)$$

for all $x \in X$. Continuing this process, we have

$$H(x) = \frac{1}{\lambda^{2n}} H(\lambda^n x) \tag{2.29}$$

for all $x \in X$ and all $n \in \mathbb{N}$. By using the continuity of $|||\cdot|||_Y$, Theorem 1.3 and the inequality (2.29), we have

$$\begin{aligned}
|||H(x) - G(x)|||_Y^p &= \left|\left|\left|\lim_{n\to\infty} \frac{1}{\lambda^{2n}} f(\lambda^n x) - \frac{1}{\lambda^{2n}} H(\lambda^n x)\right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \left|\left|\left|\frac{1}{\lambda^{2n}} f(\lambda^n x) - \frac{1}{\lambda^{2n}} H(\lambda^n x)\right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \frac{1}{\lambda^{2np}} \left|\left|\left|\frac{1}{\lambda^{2n}} f(\lambda^n x) - \frac{1}{\lambda^{2n}} H(\lambda^n x)\right|\right|\right|_Y^p \\
&\leq \lim_{n\to\infty} \frac{1}{\lambda^{2np}} \left\|\frac{1}{\lambda^{2n}} f(\lambda^n x) - \frac{1}{\lambda^{2n}} H(\lambda^n x)\right\|_Y^p \\
&\leq \frac{\kappa_Y^p}{\lambda^{2np}\lambda^{2p}} \tilde{\psi}(\lambda^n x) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \frac{1}{\lambda^{2np}} \tilde{\psi}(\lambda^n x) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \frac{1}{\lambda^{2np}} \sum_{i=1}^{\infty} \frac{1}{\lambda^{2ip}} \varphi^p(\lambda^{n+i} x, 0) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{1}{\lambda^{2(n+i)p}} \varphi^p(\lambda^{n+i} x, 0) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \sum_{i=n+1}^{\infty} \frac{1}{\lambda^{2ip}} \varphi^p(\lambda^{i} x, 0) \\
&\leq \frac{\kappa_Y^p}{\lambda^{2p}} \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{1}{\lambda^{2ip}} \varphi^p(\lambda^{i} x, 0) \\
&= \frac{\kappa_Y^p}{\lambda^{2p}} \cdot 0 \\
&= 0
\end{aligned}$$

for all $x \in X$. This implies that

$$|||H(x) - G(x)|||_Y^p = 0.$$

Then $G(x) = H(x)$ for all $x \in X$. This show that $G = H$. Thus, G is unique. ∎

Theorem 2.3. *Suppose that*

(1) *X is a real vector space and $(Y, \|\cdot\|_Y, k_Y)$ is a real quasi-Banach space.*
(2) *$\varphi : X \times X \to [0, \infty)$ is a function such that for all $x, y, \in X$,*

$$\lim_{n\to\infty} \lambda^n \varphi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0 \tag{2.30}$$

and

$$\sum_{i=1}^{\infty} \lambda^{ip} \varphi^p \left(\frac{x}{\lambda^i}, 0\right) < \infty \tag{2.31}$$

for all $x \in X$ with $p = \log_{2\kappa_Y} 2$.
(3) *$f : X \to Y$ is an odd function such that $f(0) = 0$ and satisfying*

$$\|D_\lambda f(x, y)\| \leq \varphi(x, y) \tag{2.32}$$

for all $x, y \in X$.

Then there exists a unique general mixed addtive and quadratic function $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\|_Y \leq \frac{\kappa_Y}{\lambda}(\tilde{\psi}(x))^{\frac{1}{p}} \tag{2.33}$$

where

$$\tilde{\psi}(x) = \sum_{i=1}^{\infty} \lambda^{ip} \varphi^p \left(\frac{x}{\lambda^i}, 0\right)$$

for all $x \in X$.

Proof. (The proof is similar to the proof of Theorem 2.1.)

Since f is odd function, we have $f(-x) = -f(x)$ for all $x \in X$. Letting $y = 0$ in (2.32), we get that

$$\|f(\lambda x) - \lambda f(x)\|_Y \leq \frac{1}{2}\varphi(x, 0) \tag{2.34}$$

for all $x \in X$. For any $n \in \mathbb{N}$, replacing x by $\frac{x}{\lambda^{n+1}}$ and multiplying λ^n both sides of (2.34), we obtain that

$$\left\|\lambda^n f\left(\frac{x}{\lambda^n}\right) - \lambda^{n+1} f\left(\frac{x}{\lambda^{n+1}}\right)\right\|_Y \leq \frac{\lambda^n}{2}\varphi\left(\frac{x}{\lambda^{n+1}}, 0\right) \tag{2.35}$$

for all $x \in X$ and all non-negative integer n. By the same argument of Theorem 2.1, for any $m, n \in \mathbb{N}$ with $m < n$, we have

$$\begin{aligned}
&\frac{1}{2}\left\|\lambda^n f\left(\frac{x}{\lambda^n}\right) - \lambda^m f\left(\frac{x}{\lambda^m}\right)\right\|_Y^p \\
\leq &\left|\left\|\lambda^n f\left(\frac{x}{\lambda^n}\right) - \lambda^m f\left(\frac{x}{\lambda^m}\right)\right\|\right|_Y^p \\
\leq &\left|\left\|\lambda^n f\left(\frac{x}{\lambda^n}\right) - \lambda^{n-1} f\left(\frac{x}{\lambda^{n-1}}\right)\right\|\right|_Y^p + \left|\left\|\lambda^{n-1} f\left(\frac{x}{\lambda^{n-1}}\right) - \lambda^{n-2} f\left(\frac{x}{\lambda^{n-2}}\right)\right\|\right|_Y^p + \\
&\cdots + \left|\left\|\lambda^{m+1} f\left(\frac{x}{\lambda^{m+1}}\right) - \lambda^m f\left(\frac{x}{\lambda^m}\right)\right\|\right|_Y^p \\
\leq &\left\|\lambda^n f\left(\frac{x}{\lambda^n}\right) - \lambda^{n-1} f\left(\frac{x}{\lambda^{n-1}}\right)\right\|_Y^p + \left\|\lambda^{n-1} f\left(\frac{x}{\lambda^{n-1}}\right) - \lambda^{n-2} f\left(\frac{x}{\lambda^{n-2}}\right)\right\|_Y^p + \\
&\cdots + \left\|\lambda^{m+1} f\left(\frac{x}{\lambda^{m+1}}\right) - \lambda^m f\left(\frac{x}{\lambda^m}\right)\right\|_Y^p \\
\leq &\frac{\lambda^{(n-1)p}}{2^p}\varphi^p\left(\frac{x}{\lambda^n}, 0\right) + \frac{\lambda^{(n-2)p}}{2^p}\varphi^p\left(\frac{x}{\lambda^{n-1}}, 0\right) + \ldots + \frac{\lambda^{mp}}{2^p}\varphi^p\left(\frac{x}{\lambda^{m+1}}, 0\right) \\
= &\frac{1}{2^p}\sum_{i=m}^{n-1} \lambda^{ip}\varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right) \\
\leq &\frac{1}{2^p}\sum_{i=m}^{\infty} \lambda^{ip}\varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right)
\end{aligned} \tag{2.36}$$

for all $x \in X$. Letting $n, m \to \infty$ in (2.36) and using (2.31), we get

$$\lim_{n\to\infty} \left\|\lambda^n f\left(\frac{x}{\lambda^n}\right) - \lambda^m f\left(\frac{x}{\lambda^m}\right)\right\|_Y = 0 \tag{2.37}$$

for all $x \in X$. This show that the sequence $\left\{\lambda^n f\left(\frac{x}{\lambda^n}\right)\right\}$ is Cauchy in $(Y, \|\cdot\|_Y, \kappa_Y)$ for all $x \in X$. Since $(Y, \|\cdot\|_Y, \kappa_Y)$ is a real quasi-Banach space, the sequence $\left\{\lambda^n f\left(\frac{x}{\lambda^n}\right)\right\}$ is convergent for all $x \in X$. So, we can define the function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right) \tag{2.38}$$

for all $x \in X$. Let $m = 0$ in (2.36), we get

$$\left|\left\|\lambda^n f\left(\frac{x}{\lambda^n}\right) - f(x)\right\|\right|_Y^p \leq \frac{1}{2^p}\sum_{i=0}^{n-1} \lambda^{ip}\varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right) \tag{2.39}$$

for all $x \in X$. By using the continuity $|\|\cdot\||_Y$, Theorem 1.3 and the inequality (2.39), we have

$$\begin{aligned}
|||Q(x)-f(x)|||_Y^p &= \left|\left|\left|\lim_{n\to\infty}\lambda^n f\left(\frac{x}{\lambda^n}\right)-f(x)\right|\right|\right|_Y^p\\
&= \left|\left|\left|\lim_{n\to\infty}\left(\lambda^n f\left(\frac{x}{\lambda^n}\right)-f(x)\right)\right|\right|\right|_Y^p\\
&= \lim_{n\to\infty}\left|\left|\left|\lambda^n f\left(\frac{x}{\lambda^n}\right)-f(x)\right|\right|\right|_Y^p\\
&\le \frac{1}{2^p}\sum_{i=0}^{\infty}\lambda^{ip}\varphi^p\left(\frac{x}{\lambda^{i+1}},0\right)\\
&= \frac{1}{2^p\lambda^p}\sum_{i=1}^{\infty}\lambda^{ip}\varphi^p\left(\frac{x}{\lambda^{i}},0\right)=\frac{1}{(2\lambda)^p}\tilde{\psi}(x)
\end{aligned} \tag{2.40}$$

for all $x \in X$. It follows Thereom 1.3 and (2.40) that

$$\|Q(x)-f(x)\|_Y \le 2\kappa_Y\,|||Q(x)-f(x)|||_Y \le \frac{2\kappa_y}{2\lambda}(\tilde{\psi}(x))^{\frac{1}{p}} = \frac{\kappa_Y}{\lambda}(\tilde{\psi}(x))^{\frac{1}{p}}$$

for all $x \in X$. Thus, (2.33) holds for all $x \in X$. Next, we will prove that Q is a general mixed additive and quadratic. By using the continuity of $|||\cdot|||$, we have

$$\begin{aligned}
&|||Q(\lambda x+y)+Q(\lambda x-y)-Q(x+y)-Q(x-y)-(\lambda-1)[(\lambda+2)Q(x)+\lambda Q(-x)]|||_Y\\
&= \left|\left|\left|\lim_{n\to\infty}\lambda^n f\left(\frac{\lambda x+y}{\lambda^n}\right)+\lim_{n\to\infty}\lambda^n f\left(\frac{\lambda x-y}{\lambda^n}\right)-\lim_{n\to\infty}\lambda^n f\left(\frac{x+y}{\lambda^n}\right)\right.\right.\right.\\
&\quad \left.\left.\left.-\lim_{n\to\infty}\lambda^n f\left(\frac{x-y}{\lambda^n}\right)-(\lambda-1)\left[(\lambda+2)\lim_{n\to\infty}\lambda^n f\left(\frac{x}{\lambda^n}\right)+\lambda\lim_{n\to\infty}\lambda^n f\left(-\frac{x}{\lambda^n}\right)\right]\right|\right|\right|_Y\\
&= \lim_{n\to\infty}\left|\left|\left|\lambda^n f\left(\frac{\lambda x+y}{\lambda^n}\right)+\lambda^n f\left(\frac{\lambda x-y}{\lambda^n}\right)-\lambda^n f\left(\frac{x+y}{\lambda^n}\right)\right.\right.\right.\\
&\quad \left.\left.\left.-\lambda^n f\left(\frac{x-y}{\lambda^n}\right)-(\lambda-1)\left[(\lambda+2)\lambda^n f\left(\frac{x}{\lambda^n}\right)+\lambda\cdot\lambda^n f\left(-\frac{x}{\lambda^n}\right)\right]\right|\right|\right|_Y\\
&\le \lim_{n\to\infty}\lambda^n\varphi\left(\frac{x}{\lambda^n},\frac{y}{\lambda^n}\right)\\
&= 0
\end{aligned} \tag{2.41}$$

for all $x \in X$. This implies that

$$Q(\lambda x+y)+Q(\lambda x-y)=Q(x+y)+Q(x-y)+(\lambda-1)[(\lambda+2)Q(x)+\lambda Q(-x)]$$

for all $x \in X$. Since f is an odd function and $f(0)=0$, from (2.22) we get that G is also odd and $G(0)=0$. Since G satisfy (2.26), G is quadratic.

Finally, we prove the uniqueness of G. Let $H : X \to Y$ be another general mixed additive and quadratic function satisfy

$$\|f(x)-H(x)\|_Y \le \frac{\kappa_Y}{\lambda}\left(\tilde{\psi}(x)\right)^{\frac{1}{p}}$$

for all $x \in X$. Since T is a general mixed additive and quadratic function, we can show that $T(x)=\lambda^n T\left(\frac{x}{\lambda^n}\right)$. The rest of the proof is similar to the proof of Theorem 2.1. ∎

Theorem 2.4. *Suppose that*

(1) *X is a real quasi-normed space and $(Y, \|\cdot\|_Y, k_Y)$ is a real quasi-Banach space.*
(2) *$\varphi : X \times X \to [0, \infty)$ is a function satisfying*

$$\lim_{n\to\infty} \frac{1}{\lambda^n} \varphi(\lambda^n x, \lambda^n y) = 0 \tag{2.42}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{\lambda^{ip}} \varphi^p\left(\lambda^i x, 0\right) < \infty \tag{2.43}$$

for all $x \in X$ with $p = \log_{2\kappa_Y} 2$.
(3) *$f : X \to Y$ is an odd function such that $f(0) = 0$ and satisfying*

$$\|D_\lambda f(x,y)\| \le \varphi(x,y) \tag{2.44}$$

for all $x, y \in X$.

Then there exists a unique general mixed additive and quadratic functional equation $G : X \to Y$ satisfying

$$\|f(x) - G(x)\|_Y \le \frac{\kappa_Y}{\lambda^2} \left(\tilde{\psi}(x)\right)^{\frac{1}{p}} \tag{2.45}$$

where

$$\tilde{\psi}(x) = \sum_{i=1}^{\infty} \frac{1}{\lambda^{2ip}} \varphi^p\left(\lambda^i x, 0\right)$$

for all $x \in X$.

Proof. By the same argument of Theorem 2.3, we have

$$\left\| \frac{1}{\lambda} f(\lambda x) - f(x) \right\|_Y \le \frac{1}{2\lambda} \varphi(x, 0) \tag{2.46}$$

for all $x \in X$. For any $n \in \mathbb{N}$, replacing x by $\lambda^n x$ and multiplying $\frac{x}{\lambda^n}$ both side of (2.46), we get that

$$\left\| \frac{1}{\lambda^{n+1}} f(\lambda^{n+1} x) - \frac{1}{\lambda^n} f(\lambda^n x) \right\|_Y \le \frac{1}{2\lambda^{n+1}} \varphi(\lambda^n, 0) \tag{2.47}$$

for all $x \in X$ and all non-negative integer n. By the same argument of Theorem 2.1, for any $m, n \in \mathbb{N}$ with $m < n$, we have

$$\begin{aligned} \frac{1}{2} \left\| \frac{1}{\lambda^n} f(\lambda^n x) - \frac{1}{\lambda^m} f(\lambda^m x) \right\|_Y^p &\le \left\|\left| \frac{1}{\lambda^n} f(\lambda^n x) - \frac{1}{\lambda^m} f(\lambda^m x) \right\|\right|_Y^p \\ &\le \frac{1}{2^p} \sum_{i=m}^{n-1} \frac{1}{\lambda^{(i+1)p}} \varphi^p\left(\lambda^i x, 0\right) \\ &\le \frac{1}{2^p} \sum_{i=m}^{\infty} \frac{1}{\lambda^{(i+1)p}} \varphi^p\left(\lambda^i x, 0\right) \end{aligned} \tag{2.48}$$

for all $x \in X$. Letting $n, m \to \infty$ in (2.48) and using (2.43), we get

$$\lim_{n\to\infty} \left\| \frac{1}{\lambda^n} f(\lambda^n x) - \frac{1}{\lambda^m} f(\lambda^m x) \right\|_Y = 0$$

for all $x \in X$. This show that the sequence $\left\{\frac{1}{\lambda^n} f(\lambda^n x)\right\}$ is Cauchy in $(Y, \|\cdot\|_Y, \kappa_Y)$ for all $x \in X$. Since $(Y, \|\cdot\|_Y, \kappa_Y)$ is a real quasi-Banach space, the sequence $\left\{\frac{1}{\lambda^n} f(\lambda^n x)\right\}$ is convergent for all $x \in X$. So, we can define the function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x) \tag{2.49}$$

for all $x \in X$. Let $m = 0$ in (2.48), we have

$$\left|\left|\left| \frac{1}{\lambda^n} f(\lambda^n x) - f(x) \right|\right|\right|_Y^p \le \frac{1}{2^p} \sum_{i=0}^{n-1} \frac{1}{\lambda^{(i+1)p}} \varphi^p \left(\lambda^i x, 0\right) \tag{2.50}$$

for all $x \in X$. By using the continuity of $|||\cdot|||_Y$ and the inequality (2.23), we have

$$\begin{aligned}
|||Q(x) - f(x)|||_Y^p &= \left|\left|\left| \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x) - f(x) \right|\right|\right|_Y^p \\
&= \left|\left|\left| \lim_{n\to\infty} \left(\frac{1}{\lambda^n} f(\lambda^n x) - f(x) \right) \right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \left|\left|\left| \frac{1}{\lambda^n} f(\lambda^n x) - f(x) \right|\right|\right|_Y^p \\
&\le \frac{1}{2^p} \sum_{i=0}^{\infty} \frac{1}{\lambda^{(i+1)p}} \varphi^p(\lambda^i x, 0) \\
&= \frac{1}{2^p \lambda^p} \tilde{\psi}(x)
\end{aligned} \tag{2.51}$$

for all $x \in X$. It follows Thereom 1.3 and (2.51) that

$$\|Q(x) - f(x)\|_Y \le 2\kappa_Y \, |||Q(x) - f(x)|||_Y \le \frac{2\kappa_y}{2\lambda} (\tilde{\psi}(x))^{\frac{1}{p}} = \frac{\kappa_Y}{\lambda} (\tilde{\psi}(x))^{\frac{1}{p}}$$

for all $x \in X$. Thus, (2.45) holds for all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.2. ∎

3. Stability of the Functional Equation (1.4) in Quasi-Banach Spaces

Now, we investigate the stability of the functional equation (1.4) in (β, p)-Banach spaces by using contractive subbadditive and expansively superadditive.

We recall that a subadditive function is a function $\varphi : A \to B$, having a domain A and a codomain $(B, \le)$ that are both closed under addition, with the following property:

$$\phi(x + y) \le \phi(x) + \phi(y)$$

for all $x, y \in A$. Now we say that a function $\phi : A \to B$ is contractively subadditive if there exists a constant L with $0 < L < 1$ such that

$$\phi(x + y) \le L \left(\phi(x) + \phi(y)\right)$$

for all $x, y \in A$. Then ϕ satisfies the following properties $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2^n x) \leq (2L)^n \phi(x)$. It follows by the contractively subadditive condition of ϕ that $\phi(\lambda x) \leq \lambda L \phi(x)$ and so $\phi\left(\lambda^i x\right) \leq (\lambda L)^i \phi(x)$ for all $i \in \mathbb{N}$, all $x \in A$ and all positive integer $\lambda \geq 2$.

Similarly, we say that a function $\phi : A \to B$ is expansively superadditive if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \geq \frac{1}{L}\left(\phi(x) + \phi(y)\right)$$

for all $x, y \in A$. Then ϕ satisfies the following properties $\phi(x) \leq \frac{L}{2}\phi(2x)$ and $\phi\left(\frac{x}{2^n}\right) \leq \left(\frac{L}{2}\right)^n \phi(x)$. We observe that an expansively superadditive mapping ϕ satisfies the following properties $\phi(\lambda x) \geq \frac{\lambda}{L}\phi(x)$ and so $\phi\left(\frac{x}{\lambda^i}\right) \leq \left(\frac{L}{\lambda}\right)^i \phi(x)$, $i \in \mathbb{N}$ for all $x \in A$ and all positive integer $\lambda \geq 2$.

Theorem 3.1. *Suppose that*

(1) *X is a real quasi normed space and $(Y, \|\cdot\|, k_Y)$ is a real quasi-Banach spaces.*
(2) *$\varphi : X \times X \to [0, \infty)$ is expansively superadditive with constant L satisfying $\lambda L < 1$.*
(3) *$f : X \to Y$ is an even function such that $f(0) = 0$ and satisfying*

$$\|D\lambda f(x,y)\| \leq \varphi(x,y) \tag{3.1}$$

for all $x, y \in X$.

Then there exists a unique general mixed additive and quadratic functional equation $Q : X \to Y$ satisfying

$$\|Q(x) - f(x)\|_Y \leq \frac{k_y L \varphi(x,0)}{\lambda(1-(\lambda L)^p)^{\frac{1}{p}}} \tag{3.2}$$

for all $x \in X$ where $p = \log_{2k_Y} 2$.

Proof. It follows from Theorem 2.1 that for any $m, n \in \mathbb{N}$ with $n > m$ we obtain that

$$\begin{aligned}
\frac{1}{2}\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|^p &\leq \left\|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|\right\|_Y^p \\
&\leq \frac{1}{2^p}\sum_{i=m}^{n-1} \lambda^{2ip}\varphi^p\left(\frac{x}{\lambda^{i+1}}, 0\right) \\
&\leq \frac{1}{2^p}\sum_{i=m}^{n-1} \lambda^{2ip}\left(\frac{L}{\lambda}\right)^{(i+1)p} \varphi^p(x,0) \\
&\leq \varphi^p(x,0)\left(\frac{L}{2\lambda}\right)^p \sum_{i=m}^{n-1} (\lambda L)^{ip} \\
&\leq \varphi^p(x,0)\left(\frac{L}{2\lambda}\right)^p \sum_{i=m}^{\infty} (\lambda L)^{ip}
\end{aligned} \tag{3.3}$$

for all $x \in X$. Letting $m \to \infty$ in (3.3), we get

$$\lim_{m\to\infty}\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - \lambda^{2m} f\left(\frac{x}{\lambda^m}\right)\right\|^p = 0$$

for all $x \in X$. This implies that the sequence $\left\{\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)\right\}$ is Cauchy in $(Y, \|\cdot\|, k_Y)$ for all $x \in X$. Since $(Y, \|\cdot\|, k_Y)$ is a real quasi-Banach spaces. Then the sequence $\left\{\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)\right\}$ is convergent. So we can define the function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \lambda^{2n} f\left(\frac{x}{\lambda^n}\right)$$

for all $x \in X$. It follows from (3.3) that for $m = 0$, we get that

$$\left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right\|\right|^p \leq \varphi^p(x, 0)\left(\frac{L}{2\lambda}\right)^p \sum_{i=0}^{n-1} (\lambda L)^{ip} \tag{3.4}$$

for all $x \in X$. By using the continuity $|\|\cdot\||_Y$ and (3.4), we have

$$\begin{aligned}
|\|Q(x) - f(x)\||_Y^p &= \left|\left\|\lim_{n\to\infty} \lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right\|\right|_Y^p \\
&= \left|\left\|\lim_{n\to\infty} \left(\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right)\right\|\right|_Y^p \\
&= \lim_{n\to\infty} \left|\left\|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right) - f(x)\right\|\right|_Y^p \\
&\leq \lim_{n\to\infty} \left(\varphi^p(x, 0)\left(\frac{L}{2\lambda}\right)^p \sum_{i=0}^{n-1} (\lambda L)^{ip}\right) \\
&= \varphi^p(x, 0)\left(\frac{L}{2\lambda}\right)^p \lim_{n\to\infty} \sum_{i=0}^{n-1} (\lambda L)^{ip} \\
&= \varphi^p(x, 0)\left(\frac{L}{2\lambda}\right)^p \sum_{i=0}^{\infty} (\lambda L)^{ip} \\
&= \left(\frac{L}{2\lambda}\right)^p \frac{\varphi^p(x, 0)}{1 - (\lambda L)^p}
\end{aligned} \tag{3.5}$$

for all $x \in X$. This implies that

$$|\|Q(x) - f(x)\||_Y \leq \frac{L\varphi(x, 0)}{2\lambda(1 - (\lambda L)^p)^{\frac{1}{p}}} \tag{3.6}$$

It follows from Theorem 1.3 and (3.6) that

$$\frac{1}{2k_Y} \|Q(x) - f(x)\|_Y \leq |\|Q(x) - f(x)\||_Y \leq \frac{L\varphi(x, 0)}{2\lambda(1 - (\lambda L)^p)^{\frac{1}{p}}}$$

for all $x \in X$, that is,

$$\|Q(x) - f(x)\|_Y \leq \frac{k_y L\varphi(x, 0)}{\lambda(1 - (\lambda L)^p)^{\frac{1}{p}}}$$

Thus, (3.2) holds for all $x \in X$. Next, we will show that Q is a general mixed additive and quadratic. By continuity of $|||\cdot|||$, we have

$$
\begin{aligned}
&|||Q(\lambda x+y)+Q(\lambda x-y)-Q(x+y)-Q(x-y)-(\lambda-1)[(\lambda+2)Q(x)+\lambda Q(-x)]|||_Y\\
&=\Bigg|\Bigg|\Bigg| \lim_{n\to\infty}\lambda^{2n} f\left(\frac{\lambda x+y}{\lambda^n}\right)+\lim_{n\to\infty}\lambda^{2n} f\left(\frac{\lambda x-y}{\lambda^n}\right)-\lim_{n\to\infty}\lambda^{2n} f\left(\frac{x+y}{\lambda^n}\right)\\
&\quad -\lim_{n\to\infty}\lambda^{2n} f\left(\frac{x-y}{\lambda^n}\right)-(\lambda-1)\left[(\lambda+2)\lim_{n\to\infty}\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)+\lambda\lim_{n\to\infty}\lambda^{2n} f\left(-\frac{x}{\lambda^n}\right)\right]\Bigg|\Bigg|\Bigg|_Y\\
&=\lim_{n\to\infty}\Bigg|\Bigg|\Bigg|\lambda^{2n} f\left(\frac{\lambda x+y}{\lambda^n}\right)+\lambda^{2n} f\left(\frac{\lambda x-y}{\lambda^n}\right)-\lambda^{2n} f\left(\frac{x+y}{\lambda^n}\right)\\
&\quad -\lambda^{2n} f\left(\frac{x-y}{\lambda^n}\right)-(\lambda-1)\left[(\lambda+2)\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)+\lambda\cdot\lambda^{2n} f\left(-\frac{x}{\lambda^n}\right)\right]\Bigg|\Bigg|\Bigg|_Y\\
&\le \lim_{n\to\infty}\lambda^{2n}\varphi\left(\frac{x}{\lambda^n},\frac{y}{\lambda^n}\right)\\
&\le \lim_{n\to\infty}\lambda^{2n}\left(\frac{L}{\lambda}\right)^n\varphi(x,y)\\
&=\varphi(x,y)\lim_{n\to\infty}(\lambda L)^n\\
&=0
\end{aligned}
$$

for all $x, y \in X$. This implies that Q is a general mixed addtive and quadratic. Finally, we will show that Q is unique. Let T be another general mixed addtive and quadratic functional equation and satisfies (3.2), that is,

$$\|T(x)-f(x)\|_Y \le \frac{k_y L\varphi(x,0)}{\lambda(1-(\lambda L)^p)^{\frac{1}{p}}}$$

for all $x \in X$. By the same argument of Theorem 2.1, we get that

$$
\begin{aligned}
|||Q(x)-T(x)|||_Y^p &= \Bigg|\Bigg|\Bigg|\lim_{n\to\infty}\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)-\lambda^{2n}T\left(\frac{x}{\lambda^n}\right)\Bigg|\Bigg|\Bigg|_Y^p\\
&=\lim_{n\to\infty}\Bigg|\Bigg|\Bigg|\lambda^{2n} f\left(\frac{x}{\lambda^n}\right)-\lambda^{2n}T\left(\frac{x}{\lambda^n}\right)\Bigg|\Bigg|\Bigg|_Y^p\\
&=\lim_{n\to\infty}\lambda^{2n}\Bigg|\Bigg|\Bigg| f\left(\frac{x}{\lambda^n}\right)-T\left(\frac{x}{\lambda^n}\right)\Bigg|\Bigg|\Bigg|_Y^p\\
&\le \lim_{n\to\infty}\lambda^{2n}\frac{k_y L\varphi\left(\frac{x}{\lambda^n},0\right)}{\lambda(1-(\lambda L)^p)^{\frac{1}{p}}}\\
&=\frac{k_y L}{\lambda(1-(\lambda L)^p)^{\frac{1}{p}}}\lim_{n\to\infty}\lambda^{2n}\varphi\left(\frac{x}{\lambda^n},0\right)\\
&=\frac{k_y L\varphi(x,0)}{\lambda(1-(\lambda L)^p)^{\frac{1}{p}}}\lim_{n\to\infty}(\lambda L)^n\\
&=0
\end{aligned}
\tag{3.7}
$$

This implies that

$$|||Q(x)-T(x)|||_Y^p = 0. \tag{3.8}$$

Then $Q(x) = T(x)$ for all $x \in X$. So, Q is unique. This completes the proof. ■

Theorem 3.2. *Suppose that*

(1) X *is a real quasi-normed space and* $(Y, \|\cdot\|, k_Y)$ *is a real quasi-Banach spaces.*
(2) $\varphi : X \times X \to [0, \infty)$ *is contractive superadditive with constant* L *satisfying* $\frac{L}{\lambda} < 1$.
(3) $f : X \to Y$ *is an even function such that* $f(0) = 0$ *and satisfying*

$$\|D\lambda f(x,y)\| \leq \varphi(x,y) \tag{3.9}$$

for all $x, y \in X$.

Then there exists a unique general mixed additive and quadratic functional equation $Q : X \to Y$ *satisfying*

$$\|Q(x) - f(x)\|_Y \leq \frac{k_Y L\varphi(x,0)}{\lambda\left(1-(\lambda L)^p\right)^{\frac{1}{p}}} \tag{3.10}$$

for all $x \in X$ *where* $p = \log_{2k_Y} 2$.

Proof. It follows from 2.5 in Theorem 2.1 that

$$\left\|f(\lambda x) - \lambda^2 f(x)\right\|_Y \leq \frac{1}{2}\varphi(x,0) \tag{3.11}$$

for all $x \in X$. Replacing x in (3.11) by $\lambda^n x$ and dividing both side the above inequality by $\frac{1}{\lambda^{2(n+1)}}$, we obtain that

$$\left\|\frac{1}{\lambda^{2(n+1)}} f\left(\lambda^{n+1}x\right) - \frac{1}{\lambda^{2n}} f\left(\lambda^n x\right)\right\|_Y \leq \frac{1}{2\lambda^{2(n+1)}}\varphi\left(\lambda^n x, 0\right) \tag{3.12}$$

for all $x \in X$. It follows from (3.12) that for any $m, n \in \mathbb{N}$ with $m < n$, we have

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) - \frac{1}{\lambda^{2m}} f\left(\lambda^m x\right)\right\|_Y^p \\
&\leq \left\|\left\|\frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) - \frac{1}{\lambda^{2m}} f\left(\lambda^m x\right)\right\|\right\|_Y^p \\
&\leq \left\|\left\|\frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) - \frac{1}{\lambda^{2(n-1)}} f\left(\lambda^{n-1} x\right)\right\|\right\|_Y^p + \left\|\left\|\frac{1}{\lambda^{2(n-1)}} f\left(\lambda^{n-1} x\right) - \frac{1}{\lambda^{2(n-2)}} f\left(\lambda^{n-2} x\right)\right\|\right\|_Y^p \\
&\quad \cdots + \left\|\left\|\frac{1}{\lambda^{2(m+1)}} f\left(\lambda^{m+1} x\right) - \frac{1}{\lambda^{2m}} f\left(\lambda^m x\right)\right\|\right\|_Y^p \\
&\leq \left\|\frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) - \frac{1}{\lambda^{2(n-1)}} f\left(\lambda^{n-1} x\right)\right\|_Y^p + \left\|\frac{1}{\lambda^{2(n-1)}} f\left(\lambda^{n-1} x\right) - \frac{1}{\lambda^{2(n-2)}} f\left(\lambda^{n-2} x\right)\right\|_Y^p \\
&\quad \cdots + \left\|\frac{1}{\lambda^{2(m+1)}} f\left(\lambda^{m+1} x\right) - \frac{1}{\lambda^{2m}} f\left(\lambda^m x\right)\right\|_Y^p
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{(2\lambda^{2n})^p}\varphi^p\left(\lambda^{n-1}x,0\right)+\frac{1}{(2\lambda^{2(n-1)})^p}\varphi^p\left(\lambda^{n-2}x,0\right)+\ldots+\frac{1}{(2\lambda^{2(m+1)})^p}\varphi^p\left(\lambda^{m}x,0\right)\\
&\leq \frac{1}{(2\lambda^{2})^p}\left(\frac{1}{\lambda^{2(n-1)p}}\varphi^p\left(\lambda^{n-1}x,0\right)+\frac{1}{\lambda^{2(n-2)p}}\varphi^p\left(\lambda^{n-2}x,0\right)+\ldots+\frac{1}{\lambda^{2mp}}\varphi^p\left(\lambda^{m}x,0\right)\right)\\
&\leq \frac{1}{(2\lambda^{2})^p}\sum_{i=m}^{n-1}\frac{1}{\lambda^{2ip}}\varphi^p\left(\lambda^{i}x,0\right)\\
&\leq \frac{1}{(2\lambda^{2})^p}\sum_{i=m}^{n-1}\frac{1}{\lambda^{2ip}}(\lambda L)^{ip}\varphi^p(x,0)\\
&\leq \frac{\varphi^p(x,0)}{(2\lambda^{2})^p}\sum_{i=m}^{n-1}\left(\frac{L}{\lambda}\right)^{ip}\\
&\leq \frac{\varphi^p(x,0)}{(2\lambda^{2})^p}\sum_{i=m}^{\infty}\left(\frac{L}{\lambda}\right)^{ip}
\end{aligned}\tag{3.13}
$$

for all $x \in X$. Taking limit $m \to \infty$ in (3.13), we have

$$
\lim_{m\to\infty}\left\|\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)-\frac{1}{\lambda^{2m}}f\left(\lambda^m x\right)\right\|_Y^p=0
$$

for all $x \in X$. This implies that the sequence $\left\{\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)\right\}$ is Cauchy in $(Y, \|\cdot\|, k_Y)$ for all $x \in X$. Since $(Y, \|\cdot\|, k_Y)$ is a real quasi-Banach spaces. Then the sequence $\left\{\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)\right\}$ is convergent. So we can define the function $Q : X \to Y$ by

$$
Q(x)=\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)\tag{3.14}
$$

for all $x \in X$. From the inequality (3.13), we know that

$$
\left|\left|\left|\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)-\frac{1}{\lambda^{2m}}f\left(\lambda^m x\right)\right|\right|\right|_Y^p\leq\frac{\varphi^p(x,0)}{(2\lambda^{2})^p}\sum_{i=m}^{n-1}\left(\frac{L}{\lambda}\right)^{ip}\tag{3.15}
$$

for all $x \in X$. Let $m = 0$ in (3.15), we get that

$$
\left|\left|\left|\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)-f(x)\right|\right|\right|_Y^p\leq\frac{\varphi^p(x,0)}{(2\lambda^{2})^p}\sum_{i=0}^{n-1}\left(\frac{L}{\lambda}\right)^{ip}\tag{3.16}
$$

By using the continuity $|||\cdot|||_Y$ and (3.16), we have

$$
\begin{aligned}
|||Q(x)-f(x)|||_Y^p&=\left|\left|\left|\lim_{n\to\infty}\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)-f(x)\right|\right|\right|_Y^p\\
&=\left|\left|\left|\lim_{n\to\infty}\left(\frac{1}{\lambda^{2n}}f\left(\lambda^n x\right)-f(x)\right)\right|\right|\right|_Y^p
\end{aligned}
$$

$$= \lim_{n\to\infty} \left|\left|\left| \frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) - f(x) \right|\right|\right|_Y^p$$
$$\leq \lim_{n\to\infty} \frac{\varphi^p(x,0)}{(2\lambda^2)^p} \sum_{i=0}^{n-1} \left(\frac{L}{\lambda}\right)^{ip}$$
$$\leq \frac{\varphi^p(x,0)}{(2\lambda^2)^p} \sum_{i=0}^{\infty} \left(\frac{L}{\lambda}\right)^{ip}$$
$$\leq \frac{\varphi^p(x,0)}{(2\lambda^2)^p \left(1 - \left(\frac{L}{\lambda}\right)^p\right)}$$

for all $x \in X$. This implies that

$$|||Q(x) - f(x)|||_Y \leq \frac{\varphi(x,0)}{2\lambda^2 \left(1 - \left(\frac{L}{\lambda}\right)^p\right)^{\frac{1}{p}}} \tag{3.17}$$

for all $x \in X$. It follows from Theorem 1.3 and (3.17) that

$$\frac{1}{2k_Y} \|Q(x) - f(x)\|_Y \leq |||Q(x) - f(x)|||_Y \leq \frac{\varphi(x,0)}{2\lambda^2 \left(1 - \left(\frac{L}{\lambda}\right)^p\right)^{\frac{1}{p}}}$$

for all $x \in X$, that is,

$$\|Q(x) - f(x)\|_Y \leq \frac{k_Y \varphi(x,0)}{\lambda^2 \left(1 - \left(\frac{L}{\lambda}\right)^p\right)^{\frac{1}{p}}}$$

for all $x \in X$. Thus, (3.10) holds for all $x \in X$. Next, we will show that Q is general mixed additive and quadratic. By continuity of $|||\cdot|||$, we have

$$|||Q(\lambda x + y) + Q(\lambda x - y) - Q(x+y) - Q(x-y) - (\lambda - 1)[(\lambda+2)Q(x) + \lambda Q(-x)]|||_Y$$
$$= \left|\left|\left| \lim_{n\to\infty} \frac{1}{\lambda^{2n}} f\left(\lambda^n(\lambda x + y)\right) + \lim_{n\to\infty} \frac{1}{\lambda^{2n}} f\left(\lambda^n(\lambda x - y)\right) - \lim_{n\to\infty} \frac{1}{\lambda^{2n}} f\left(\lambda^n(x+y)\right) \right.\right.\right.$$
$$\left.\left.\left. - \lim_{n\to\infty} \frac{1}{\lambda^{2n}} f\left(\lambda^n(x-y)\right) - (\lambda-1)\left[(\lambda+2) \lim_{n\to\infty} \frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) + \lambda \lim_{n\to\infty} \frac{1}{\lambda^{2n}} f\left(\lambda^n(-x)\right)\right] \right|\right|\right|_Y$$
$$= \lim_{n\to\infty} \left|\left|\left| \frac{1}{\lambda^{2n}} f\left(\lambda^n(\lambda x + y)\right) + \frac{1}{\lambda^{2n}} f\left(\lambda^n(\lambda x - y)\right) - \frac{1}{\lambda^{2n}} f\left(\lambda^n(x+y)\right) \right.\right.\right.$$
$$\left.\left.\left. - \frac{1}{\lambda^{2n}} f\left(\lambda^n(x-y)\right) - (\lambda-1)\left[(\lambda+2) \frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) + \lambda \cdot \frac{1}{\lambda^{2n}} f\left(\lambda^n(-x)\right)\right] \right|\right|\right|_Y$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \left|\left|\left| f(\lambda(\lambda^n x) + \lambda^n y) + f\left(\lambda(\lambda^n x) - \lambda^n y\right) - f\left(\lambda^n x + \lambda^n y\right)\right)\right.\right. \\
&\quad \left.\left. - f\left(\lambda^n x - \lambda^n y\right)\right) - (\lambda-1)\left[(\lambda+2) f\left(\lambda^n x\right) + \lambda f\left(\lambda^n(-x)\right)\right]\right|\right|\right|_Y \\
&\le \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \varphi\left(\lambda^n x, \lambda^n y\right) \\
&\le \lim_{n\to\infty} \left(\frac{L}{\lambda}\right)^n \varphi(x,y) \\
&= \varphi(x,y) \lim_{n\to\infty} \left(\frac{L}{\lambda}\right)^n \\
&= 0
\end{aligned}
$$

for all $x, y \in X$. This implies that Q is a general mixed addtive and quadratic functional equation. Finally, we will show that Q is unique. Let T be another general mixed addtive and quadratic functional equation and satisfies (3.10), that is,

$$
\|T(x) - f(x)\|_Y \le \frac{k_Y \varphi(x,0)}{\lambda^2 \left(1 - \left(\frac{L}{\lambda}\right)^p\right)^{\frac{1}{p}}}
$$

for all $x \in X$. By the same argument of Theorem 2.1, we get that

$$
\begin{aligned}
|||Q(x) - T(x)|||_Y^p &= \left|\left|\left| \lim_{n\to\infty} \frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) - \frac{1}{\lambda^{2n}} T\left(\lambda^n x\right)\right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \left|\left|\left| \frac{1}{\lambda^{2n}} f\left(\lambda^n x\right) - \frac{1}{\lambda^{2n}} T\left(\lambda^n x\right)\right|\right|\right|_Y^p \\
&= \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \left|\left|\left| f\left(\lambda^n x\right) - T\left(\lambda^n x\right)\right|\right|\right|_Y^p \\
&\le \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \frac{k_Y \varphi(\lambda^n x, 0)}{\lambda^2 \left(1 - \left(\frac{L}{\lambda}\right)^p\right)^{\frac{1}{p}}} \\
&= \frac{k_Y}{\lambda^2 \left(1 - \left(\frac{L}{\lambda}\right)^p\right)^{\frac{1}{p}}} \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \varphi\left(\lambda^n x, 0\right) \\
&= \frac{k_y L \varphi\left(x, 0\right)}{\lambda(1 - (\lambda L)^p)^{\frac{1}{p}}} \lim_{n\to\infty} \left(\frac{L}{\lambda}\right)^n \\
&= 0
\end{aligned}
\tag{3.18}
$$

for all $x \in X$. This implies that

$$
|||Q(x) - T(x)|||_Y^p = 0 \tag{3.19}
$$

for all $x \in X$. Then $Q(x) = T(x)$ for all $x \in X$. So, Q is unique. This completes the proof. ∎

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