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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

The Modified SP and Noor Iterations with Shrinking Projection Methods for Three G-Nonexpansive Mappings in Hilbert Spaces with Graphs

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Abstract In this work, we propose two new iterative schemes by modifying the shrinking projection method with Noor and SP iterations. The strong convergence theorems are given for obtaining a common fixed point of three G-nonexpansive mappings in a Hilbert space with a directed graph under some suitable conditions. Finally, we give some numerical examples for supporting our main theorems and compare the rate of convergence of some examples under the same conditions.

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1. Introduction

Let C be a nonempty subset of a real Hilbert space H. Let $\triangle$ denote the diagonal of the cartesian product $C \times C$, i.e., $\triangle = \{(x, x) : x \in \triangle\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \triangle$. We assume G has no parallel edge. So we can identify the graph G with the pair $(V(G), E(G))$.

A mapping $S : C \to C$ is said to be

- G-contraction if S satisfies the conditions:

(G1) S is edge-preserving, i.e.,

$$(x, y) \in E(G) \Rightarrow (Sx, Sy) \in E(G),$$

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(G2) S decreases weights of edges of G, i.e., there exists $\delta \in (0,1)$ such that

$$(x,y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \delta \|x - y\|.$$

- G-nonexpansive if S satisfies the condition (G1) and

(G3) S non-increases weights of edges of G, i.e.,

$$(x,y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \|x - y\|.$$

The fixed point set of S is denoted by $F(S)$ that is $F(S) = \{x \in C : x = Sx\}$. It is well known that a G-nonexpansive is nonexpansive ($S : C \to C$ is nonexpansive if and only if $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$). In 2008, Jachymski [1] introduced the notion of G-contraction and studied combining two concepts of fixed point theory and graph theory in a metric space with a directed graph. Moreover, he obtained existence theorem under some conditions. By using Jachymski idea, Aleomraninejad et al. [2] introduced the concept of G-nonexpansive mappings in Banach spaces with directed graphs and presented some iterative scheme results for G-contractive and G-nonexpansive mappings. Later on, a class of G-nonexpansive mappings in both Hilbert spaces and Banach spaces is more general than that of G-contractions. Several authors have investigated fixed point theorems for nonexpansive mappings and the structure of their fixed point sets on both Hilbert spaces and Banach spaces, see [3–18]. In 2016, Tripak [19] proved the weak and strong convergence of a sequence $\{x_n\}$ generated by the Ishikawa iteration to some common fixed points of two G-nonexpansive mappings defined on a Banach space endowed with a directed graph. Common fixed points of some nonlinear mappings have been studied by many authors.
In 2018, Suparatulatorn et al. [20] used the concept of the work of [19, 24], modified the following iteration scheme:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n S_1 x_n, \\ x_{n+1} = (1-\beta_n)S_1 x_n + (1-\beta_n)S_2 y_n, \ n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $S_1, S_2 : C \to C$ are G-nonexpansive mappings. Also, they proved weak and strong convergence for approximating common fixed points of two G-nonexpansive mappings in a uniformly convex Banach space X endowed with a graph under this iteration.

Recently, Sridarat et al. [25] proved weak and strong convergence theorems of SP iteration[26] for common fixed point of three G-nonexpansive mappings in uniformly convex Banach spaces endowed with a directed graph under some suitable control conditions. Moreover, they gave some numerical examples for confirming the main theorem and compared convergence rate between SP iteration and Noor iteration [27]. The following iterative process is known as SP iteration:

$$\begin{cases} x_1 \in H \text{ be an arbitrarily}, \\ x_{n+1} = \alpha_n z_n + (1-\alpha_n)S_1 z_n, \\ z_n = \beta_n y_n + (1-\beta_n)S_2 y_n, \\ y_n = \gamma_n x_n + (1-\gamma_n)S_3 x_n, \end{cases}$$

and the Noor iteration is defined inductively by

$$\begin{cases} x_1 \in H \text{ be an arbitrarily,} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_1 z_n, \\ z_n = \beta_n x_n + (1-\beta_n) S_2 y_n, \\ y_n = \gamma_n x_n + (1-\gamma_n) S_3 x_{n,}\ n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1] and C is a convex subset of a normed space X and $S_1, S_2, S_3 : C \to C$ are three G-nonexpansive mappings. They compared the convergence speed of Noor, and SP iteration, and obtained the SP iteration converges faster than the Noor iteration.

In 2008, Takahashi et al. [28] introduced the following modification of Mann's iteration method [29] which is called shrinking projection method for finding a common fixed point of a countable family of nonexpansive mappings $\{S_n\}$.

$$\begin{cases} u_0 \in H \text{ be an arbitrarily,} \\ C_1 = C, u_1 = P_{C_1} x_0, \\ y_n = \alpha_n u_n + (1-\alpha_n) S_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0. \end{cases}$$

They proved that if $\alpha_n \leq a$ for all $n \geq 1$ and for some $0 < a < 1$, then the sequence $\{u_n\}$ converges strongly to $P_{F(S)} x_0$.

Inspired by Sridarat et al. [25] and Takahashi et al. [28], we modify the shrinking projection method combining SP and Noor iterations. We present two difference convergence theorems in Hilbert spaces with a directed graph. Numerical examples are given to show its efficiency in Euclidian spaces $\mathbb{R}^3$. Some comparison to various methods are also provided in this paper.

2. Preliminaries and Lemmas

This section contains some definitions and lemmas that play an essential role in our analysis. The strong (weak) convergence of a sequence $\{x^k\}_{k \in \mathbb{N}}$ to x is denoted by $x^k \to x$ ($x^k \rightharpoonup x$), respectively.

Definition 2.1. The symbol G^{-1} is called the conversion of a graph G and it is a graph obtained from G by reversing the direction of edges as:

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Definition 2.2. The sequence $\{x_j\}_{j=0}^{N}$ of $N+1$ vertices is called a path in G from x to y of length $N (N \in \mathbb{N} \cup 0)$, where $x_0 = x, x_N = y$ and $(x_j, x_{j+1}) \in E(G)$ for $j = 0, 1, ..., N-1$.

Definition 2.3. If there is a path between any two vertices of the graph G, then a graph G is said to be connected.

Definition 2.4. If (x, y) and $(y, z) \in E(G)$, then $(x, z) \in E(G)$, This property is called the transitivity of a directed graph $G = (V(G), E(G))$.

Definition 2.5. Let $G = (V(G), E(G))$ be a directed graph. The set of edges $E(G)$ is said to be convex if for any $(x, y), (z, w) \in E(G)$ and for each $t \in (0, 1)$, then $(tx + (1-t)z, ty + (1-t)w) \in E(G)$.

Definition 2.6. Let $x_0 \in V(G)$ and A subset of $V(G)$. We say that
(i) A is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in A$;
(ii) A dominates x_0 if for each $x \in A, (x, x_0) \in E(G)$.

Lemma 2.7. If the sequence $\{x_n\}$ in a Banach space X converges weakly to $x \in X$, such that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X, y \neq x$. Then X is said to satisfy the Opial's condition.

Lemma 2.8. [21] *Let C be a nonempty, closed and convex subset of a Hilbert space H and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. Let $T : C \to C$ be a G-nonexpansive mapping and $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ for some $x \in C$. If, there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in N$ and $\{x_n - T_{x_n}\} \to y$ for some $y \in H$. Then $(I - T)x = y$.*

Lemma 2.9. *Let H be a real Hilbert space. Then*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,$$

for all $t \in [0, 1]$ and $x, y \in H$.

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C y\rangle \leq 0$ holds for all $x \in H$ and $y \in C$.

Lemma 2.10. [22] *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

Lemma 2.11. [23] *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection from H onto C. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \forall y \in C.$$

3. Main Results

In this section, by using the shrinking projection method, we obtain two different strong convergence theorems for finding the same common fixed point of three G-nonexpasive mappings in real Hilbert spaces with graphs under some suitable conditions.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $S_1, S_2, S_3 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \cap$*

$F(S_3) \neq \emptyset$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1, 2, 3.$. Let $\{x_n\}$ be sequence generated by $x_1 \in C$, $C_1 = C$

$$\begin{cases} y_n = (1-\mu_n)x_n + \mu_n S_1 x_n, \\ z_n = (1-\beta_n)y_n + \beta_n S_2 y_n, \\ w_n = (1-\alpha_n)z_n + \alpha_n S_3 z_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1; \forall n \geq 1. \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\} \subset (0,1)$. Assume that the following conditions hold :
(i) *$\{x_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$, then $(x_{n_k}, w) \in E(G)$;*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$.
Then the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. We split the proof into five steps.

Step 1. Show that $P_{C_{n+1}} x_1$ well-defined for each $x_1 \in C$. As shown in Theorem 3.2 of Tiammee et al. [5], $F(S_i)$ is convex for all $i = 1, 2, 3$. It follows from the assumption that F is closed and convex. Hence, $P_F x_1$ is well-defined. We see that $C_1 = C$ is closed and convex. Assume that C_n is closed and convex. From the definition of C_{n+1} and Lemma 2.10, we get C_{n+1} is closed and convex. Let $p \in F$. Since $\{x_n\}$ dominates p and S_1 is edge-preserving, we have $(S_1 x_n, p) \in E(G)$. This implies that $(y_n, p) = ((1-\mu_n)x_n + \mu_n S_1 x_n, p) \in E(G)$ and S_2 is edge-preserving $(S_2 y_n, p) \in E(G)$. This implies that $(z_n, p) = ((1-\beta_n)y_n + \beta_n S_2 y_n, p) \in E(G)$ by $E(G)$ is convex. Since S_1, S_2, S_3 is edge-preserving,we have

$$\begin{aligned} \|w_n - p\| &\leq (1-\alpha)\|z_n - p\| + \alpha_n \|S_3 z_n - p\| \\ &\leq (1-\alpha_n\|z_n - p\| + \alpha_n \|z_n - p\| \\ &= \|z_n - p\| \\ &\leq (1-\beta_n)\|y_n - p\| + \beta_n \|S_2 y_n - p\| \\ &\leq (1-\beta_n)\|y_n - p\| + \beta_n \|y_n - p\| \\ &= \|y_n - p\| \\ &\leq (1-\mu_n)\|x_n - p\| + \mu_n \|S_1 x_n - p\| \\ &\leq (1-\mu_n)\|x_n - p\| + \mu_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

We can conclude that $p \in C_{n+1}$, so $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} x_1$ is well-defined.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Since F is a nonempty, closed and convex subset of H, there exists a unique $v \in F$ such that $v = P_F x_1$. From $x_n = P_{C_n} x_1$ and $x_{n+1} \in C_n, \forall n \in \mathbb{N}$, we get

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \forall n \in \mathbb{N}. \tag{3.1}$$

On the other hand, as $F \subset C_n$, we obtain

$$\|x_n - x_1\| \leq \|v - x_1\|, \forall n \in \mathbb{N}. \tag{3.2}$$

It follows from (3.1) and (3.2) that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Step 3. Show that $x_n \to w \in C$ as $n \to \infty$. For $m > n$, by the definition of C_n, we see that $x_m = P_{C_m} x_1 \in C_m \subset C_n$. From Lemma 2.11, we have

$$\|x_n - x_m\|^2 \leq \|x_n - x_1\|^2 - \|x_1 - x_m\|^2.$$

We obtain that $\{x_n\}$ is a Cauchy sequence. Hence, there exists $w \in C$ such that $x_n \to w$ as $n \to \infty$. In particular, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

Step 4. Show that $w \in F$. Since $x_{n+1} \in C_n$, it follows from (3.3) that

$$\|w_n - x_n\| \leq \|w_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \to 0,$$

as $n \to \infty$. From $\{x_n\}$ dominates p for all $p \in F$ and Lemma 2.9, we get

$$\begin{aligned}
\|w_n - P\|^2 &\leq \|(1-\alpha_n)(z_n - p) + \alpha_n(S_3 z_n - p)\|^2 \\
&= \alpha_n\|S_3 z_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2 \\
&\leq \|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2 \\
&= \|(1-\beta_n)(y_n - p) + \beta_n(S_2 y_n - p)\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2 \\
&= \beta_n\|S_2 y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|S_2 y_n - y_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2 \\
&\leq \|y_n - p\|^2 - \beta_n(1-\beta_n)\|S_2 y_n - y_n\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2 \\
&= \|(1-\mu_n)(x_n - p) + \mu_n(S_1 x_n - p)\|^2 - \beta_n(1-\beta_n)\|S_2 y_n - y_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2 \\
&= \mu_n\|S_1 x_n - p\|^2 + (1-\mu_n)\|x_n - p\|^2 - \mu_n(1-\mu_n)\|S_1 x_n - x_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|S_2 y_n - y_n\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2 \\
&\leq \|x_n - p\|^2 - \mu_n(1-\mu_n)\|S_1 x_n - x_n\|^2 - \beta_n(1-\beta_n)\|S_2 y_n - y_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_n - p\|^2 - \|w_n - p\|^2 &\geq \mu_n(1-\mu_n)\|S_1 x_n - x_n\|^2 + \beta_n(1-\beta_n)\|S_2 y_n - y_n\|^2 \\
&\quad +\alpha_n(1-\alpha_n)\|S_3 z_n - z_n\|^2.
\end{aligned} \tag{3.4}$$

From our assumptions (i)-(iii) and (3.4) , we have

$$\lim_{n\to\infty} \|S_1 x_n - x_n\| = \lim_{n\to\infty} \|S_2 y_n - y_n\| = \lim_{n\to\infty} \|S_3 z_n - z_n\| = 0 \tag{3.5}$$

From (3.5), we have

$$\begin{aligned}
\|y_n - x_n\| &= \|(1-\mu_n)(x_n - x_n) + \mu_n(S_1 x_n - x_n)\| \\
&= \mu_n\|S_1 x_n - x_n\| \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{3.6}$$

It follows from (3.5) that

$$\begin{aligned} \|w_n - z_n\| &= \|(1-\alpha_n)(z_n - z_n) + \alpha_n(S_3 z_n - z_n)\| \\ &= \alpha_n\|S_3 z_n - z_n\| \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$\|z_n - x_n\| = \|z_n - w_n\| + \|w_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{3.8}$$

By Lemma 2.8, it follows from our assumption (i), and (3.5)-(3.8) that $w \in F$.

Step 5. Show that $w = P_F x_1$. From $x_n = P_{C_n} x_1$, we have

$$\langle x_1 - x_n, x_n - p\rangle \geq 0, \forall p \in C_n.$$

By taking the limit $x_n \to w$, we have

$$\langle x_1 - w, w - p\rangle \geq 0, \forall p \in C_n.$$

Since $F \subset C_n$, so $w = P_F x_1$. ■

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $S_1, S_2, S_3 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \cap F(S_3) \neq \emptyset$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be sequence generated by $x_1 \in C$, $C_1 = C$*

$$\begin{cases} y_n = (1-\mu_n)x_n + \mu_n S_1 x_n, \\ z_n = (1-\beta_n)x_n + \beta_n S_2 y_n, \\ w_n = (1-\alpha_n)x_n + \alpha_n S_3 z_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1; \forall n \geq 1. \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\} \subset (0,1)$. Assume that the following conditions hold :
(i) $\{x_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$, then $(x_{n_k}, w) \in E(G)$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$.
Then the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. We split the proof into four steps.

Step 1. By the same proof in Step 1 of Theorem 3.1, we get $P_F x_1$ is well-define and C_{n+1} is closed and convex. Let $p \in F$. Since $\{x_n\}$ dominates p and S_1 is edge-preserving, we have $(S_1 x_n, p) \in E(G)$. This implies that $(y_n, p) = ((1-\mu_n)x_n + \mu_n S_1 x_n, p) \in E(G)$ and S_2 is edge-preserving $(S_2 y_n, p) \in E(G)$. This implies that $(z_n, p) = ((1-\beta_n)x_n +$

$\beta_n S_2 y_n, p) \in E(G)$ by $E(G)$ is convex. Since S_3 is edge-preserving,we have

$$\begin{aligned}
\|w_n - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|S_3 z_n - p\| \\
&\leq (1-\alpha)\|x_n - p\| + \alpha_n\|z_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n((1-\beta_n)\|x_n - p\| + \beta_n\|S_2 y_n - p\|) \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n((1-\beta_n)\|x_n - p\| + \beta_n\|y_n - p\|) \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n((1-\beta_n)\|x_n - p\| + \beta_n((1-\mu_n)\|x_n - p\| \\
&\quad +\mu_n\|S_1 x_n - p\|) \\
&\leq \|x_n - p\|.
\end{aligned}$$

We can conclude $p \in C_{n+1}$ and $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} x_1$ is well-defined.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Since F is a nonempty, closed and convex subset of H, there exists a unique $v \in F$ such that $v = P_F x_1$. From $x_n = P_{C_n} x_1$ and $x_{n+1} \in C_n, \forall n \in \mathbb{N}$, we get

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \forall n \in \mathbb{N}. \tag{3.9}$$

On the other hand, as $F \subset C_n$, we obtain

$$\|x_n - x_1\| \leq \|v - x_1\|, \forall n \in \mathbb{N}. \tag{3.10}$$

It follows from (3.9) and (3.10) that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Step 3. Show that $x_n \to w \in C$ as $n \to \infty$. For $m > n$ by the definition of C_n, we see that $x_m = P_{C_m} x_1 \in C_m \subset C_n$. From Lemma 2.11, we have

$$\|x_n - x_m\|^2 \leq \|x_n - x_1\|^2 - \|x_1 - x_m\|^2.$$

From Step 3. we obtain that $\{x_n\}$ is a Cauchy sequence. Hence, there exists $w \in C$ such that $x_n \to w$ as $n \to \infty$ particular, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}$$

Step 4. Show that $w \in F$. Since $x_{n+1} \in C_n$, it follows from (3.11) that

$$\|w_n - x_n\| \leq \|w_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \to 0, \tag{3.12}$$

as $n \to \infty$. From $\{x_n\}$ dominates p for all $p \in F$ and Lemma 2.9, we get

$$\begin{aligned}
\|w_n - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|S_3 z_n - z_n\|^2 \\
&= \alpha_n\|S_3 z_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - x_n\|^2 \\
&\leq \alpha_n\|z_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - x_n\|^2 \\
&= \alpha_n\|(1-\beta_n)(x_n - p) + \beta_n(S_2 y_n - p)\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|S_3 z_n - x_n\|^2 \\
&= \alpha_n(\beta_n\|S_2 y_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|S_2 y_n - x_n\|^2) \\
&\quad +(1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_2 z_3 - x_n\|^2 \\
&\leq \alpha_n(\beta_n\|y_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|S_2 y_n - x_n\|^2) \\
&\quad +(1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_2 z_3 - x_n\|^2 \\
&\leq \alpha_n\beta_n\|y_n - p\|^2 + \alpha_n(1-\beta_n)\|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\|S_2 y_n - x_n\|^2 \\
&\quad +(1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_2 z_3 - x_n\|^2 \\
&= \alpha_n\beta_n(\mu_n\|S_1 x_n - p\|^2 + (1-\mu_n)\|x_n - p\|^2 - \mu_n(1-\mu_n)\|S_1 x_n - x_n\|^2) \\
&\quad +\alpha_n(1-\beta_n)\|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\|S_2 y_n - x_n\|^2 \\
&\quad +(1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - x_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n\beta_n\mu_n(1-\mu_n)\|S_1 x_n - x_n\|^2 \\
&\quad -\alpha_n\beta_n(1-\beta_n)\|S_2 y_n - x_n\|^2 - \alpha_n(1-\alpha_n)\|S_3 z_n - x_n\|^2.
\end{aligned}$$

This impiles that

$$\begin{aligned}
\|x_n - p\|^2 - \|w_n - p\|^2 &\geq \alpha_n\beta_n\mu_n(1-\mu_n)\|S_1 x_n - x_n\|^2 - \alpha_n\beta_n(1-\beta_n)\|S_2 y_n - x_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|S_3 z_n - x_n\|^2. \qquad (3.13)
\end{aligned}$$

From our assumptions (i)-(iii) and (3.13) , we have

$$\lim_{n\to\infty}\|S_1 x_n - x_n\| = \lim_{n\to\infty}\|S_1 y_n - x_n\| = \lim_{n\to\infty}\|S_1 z_n - x_n\| = 0. \qquad (3.14)$$

This impiles that

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|S_1 x_n - x_n\| = 0. \qquad (3.15)$$

It follow from (3.14) and (3.15) that

$$\|S_2 y_n - y_n\| \leq \|S_2 y_n - x_n\| + \|x_n - y_n\| \to 0, \text{ as } n \to \infty. \qquad (3.16)$$

It follows from (3.14) that

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|S_2 y_n - x_n\| = 0. \qquad (3.17)$$

From (3.14) and (3.17), we have

$$\|S_3 z_n - z_n\| \leq \|S_3 z_n - x_n\| + \|x_n - z_n\| \to 0 \text{ as } n \to \infty. \qquad (3.18)$$

By Lemma 2.8 and assumption (i), we obtain $w \in F$ from (3.14), (3.16) and (3.18).

By the same proof in Step 5 of Theorem 3.1, we get $w = P_F x_1$. ■

4. Numerical Experiments

We give example and numerical results for supporting our theorem. Moreover, we compere convergence rate of all iterations in Theorem 3.1 and Theorem 3.2.

Example 4.1. Let $H = \mathbb{R}^3$ and $C = [0,\infty)\times[0,\infty)\times[0,\infty)$. Assume that $(x,y) \in E(G)$ if and only if $x_1, y_1 \leq 0.3$, $0.3 \leq x_2, y_2 \leq 1.7$ and $0.5 \leq x_3, y_3$ or $x = y$ for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in C$. Define mappings $S_1, S_2, S_3 : C \to C$ by

$$\begin{aligned} S_1x &= (\frac{\sin^2 x}{7}, 1, 1); \\ S_2x &= (0, \log(\frac{x}{1.54}) + 1, 1); \\ S_3x &= (0, 1, \tan\frac{(x-1)}{\sqrt{7.45}} + 1) \end{aligned}$$

for all $x = (x_1, x_2, x_3) \in C$. It's easy to check that S_1, S_2 and S_3 are G-nonexpensive and $F(S_1) \cap F(S_2) \cap F(S_3) = \{(0, 1, 1)\}$. On the other hand, S_1 is not nonexpansive since for $x = (0.31, 1, 2)$ and $y = (0.22, 1, 2)$, this implies that $\|S_1x - S_1y\| > 0.40 > \|x - y\|$. S_2 is not nonexpansive since for $x = (5, -0.5, 2.11)$ and $y = (5, -0.5, 2.28)$, we have $\|S_2x - S_2y\| > 0.08 > \|x - y\|$. Moreover, S_3 is not nonexpansive since for $x = (1, 1.19, 0.2)$ and $y = (1, 1.02, 0.2)$, we have $\|S_3x - S_3y\| > 0.30 > \|x - y\|$.

We provide a numerical example of Theorem 3.1 and Theorem 3.2, and choose $\alpha_n = \beta_n = \mu_n$. The stopping criterion is defined by $\|x_{n+1} - x_n\| < 10^{-9}$. The results of the proposed algorithm are shown in Table 1 and Figures 1–2.

Table 1. Comparison of the methods in Theorem 3.1 and Theorem 3.2 of Example 4.1 by Choice 1 and Choice 2.

		$x_1 = (0.16, 1, 1.48)$		$x_1 = (0.16, 1, 1.48)$	
$\{\alpha_n\}$		SP-S	Noor-S	SP-S	Noor-S
$\frac{4n^2+12}{20n^2+10}$	No. of Iter	77	248	71	219
	Cpu (Time)	0.0464	0.0191	0.0400	0.0201
$\frac{8n^2+14}{20n^2+10}$	No. of Iter	43	119	40	99
	Cpu (Time)	0.0361	0.0161	0.0349	0.0174
$\frac{12n^2+16}{20n^2+10}$	No. of Iter	35	65	32	57
	Cpu (Time)	0.0354	0.0156	0.0375	0.0158
$\frac{16n^2+18}{20n^2+10}$	No. of Iter	32	42	28	36
	Cpu (Time)	0.0352	0.0149	0.0353	0.0257

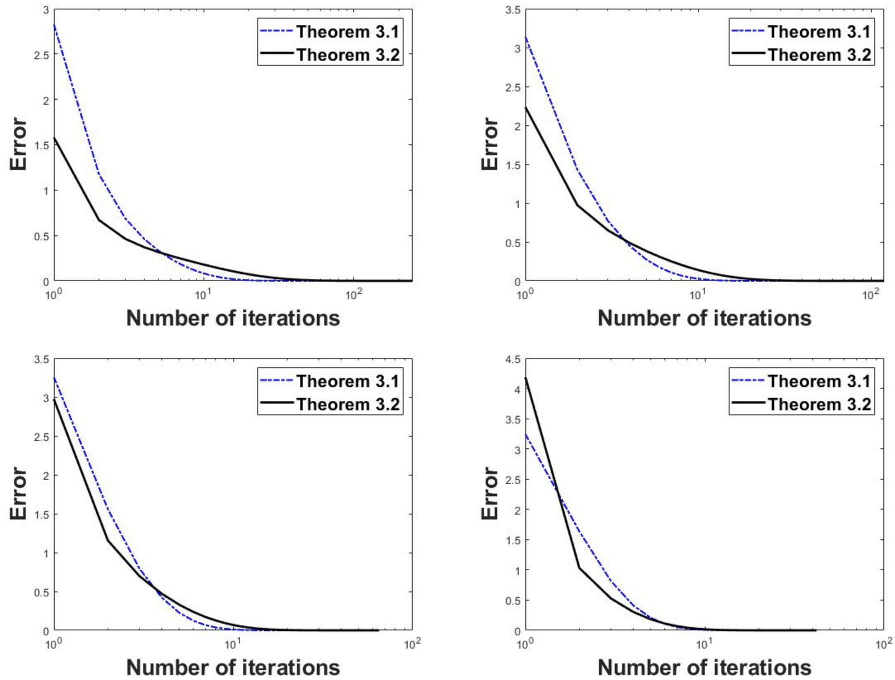

FIGURE 1. Error plotting for an initial point $(0.49, 1.61, 7.48)$ in Table 1.

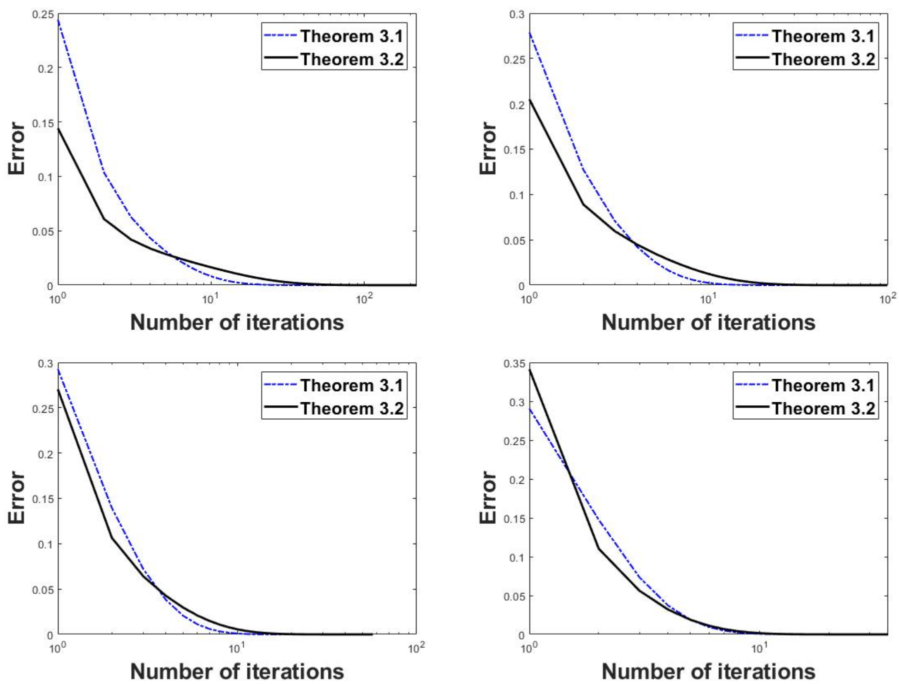

FIGURE 2. Error plotting for an initial point $(0.36, 0.75, 1.39)$ in Table 1.

Remark 4.2. From Figure 1 and Figure 2, it is shown that the modified SP iteration has requires a small number of iteration than the modified Noor iteration, while the modified Noor iteration getting CPU time smaller than modified SP iteration.

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References

[1] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (2008) 1359–1373.

[2] S.M.A. Aleomraninejad, S. Rezapour, N. Shahzad, Some fixed point result on a metric space with a graph, Topol. Appl. 159 (3) (2012) 659–663.

[3] M.R. Alfuraidan, Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph, Fixed Point Theory Appl. (2015) https://doi.org/10.1186/s13663-015-0294-5.

[4] M.R. Alfuraidan, M.A. Khamsi, Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph, Fixed Point Theory Appl. (2015) https://doi.org/10.1186/s13663-015-0294-5.

[5] J. Tiammee, A. Kaewkhao, S. Suantai, On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, Fixed Point Theory Appl. (2015) https://doi.org/10.1186/s13663-015-0436-9.

[6] P. Cholamjiak, S. Suantai, Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions, J. Glob. Optim. 54 (2012) 185–197.

[7] S. Suantai, W. Cholamjiak, P. Cholamjiak, An implicit Iteration process for a finite family of multi-valued mappings in Banach spaces, Appl. Math. Lett. 25 (2012) 1656–1660.

[8] Y. Shehu, P. Cholamjiak, Another look at the split common fixed point problem for demicontractive operators, RACSAM. 110 (2016) 201–218.

[9] R. Suparatulatorn, P. Cholamjiak, S. Suantai, On solving the minimization problem and the fixed point problem for nonexpansive mappings in CAT(0) spaces, Optim. Meth. Softw. 32 (2017) 182–192.

[10] P. Cholamjiak, W. Cholamjiak, Fixed point theorems for hybrid multivalued mappings in Hilbert spaces, J. Fixed Point Theory Appl. 18 (2016) 673–688.

[11] P. Cholamjiak, S. Suantai, Weak Convergence Theorems for a Countable Family of Strict Pseudocontractions in Banach Spaces, Fixed Point Theory Appl. (2010) https://doi.org/10.1155/2010/632137.

[12] P. Cholamjiak, S. Suantai, Strong convergence for a countable family of strict pseudocontractions in q-uniformly smooth Banach spaces, Comput. Math. Appl. 62 (2011) 787–796.

[13] R. Suparatulatorn, P. Cholamjiak, The modified S-iteration process for nonexpansive mappings in CAT(κ) spaces, Fixed Point Theory Appl. (2016) Article no. 25.

[14] P. Sunthrayuth, P. Cholamjiak, Iterative methods for solving quasi-variational inclusion and fixed point problem in q-uniformly smooth Banach spaces, Numer. Algor. 78 (2018) 1019–1044.
[15] P. Senakka, P. Cholamjiak, Approximation method for solving fixed point problem of Bregman strongly nonexpansive mappings in reflexive Banach spaces, Ricerche Mat. 65 (2016) 209–220.
[16] S. Khatoon, I. Uddin, N. Pakkaranang, N. Wairojjana, Common fixed points of modified Picard-S iteration process involving two G-nonexpansive mapping in CAT(0) space with directed graph, Thai J. Math. 18 (1) 1–13.
[17] N. Pakkaranang, P. Kumam and Y. J. Cho, 2018, Proximal point algorithms for solving convex minimization problem and common fixed points problem of asymptotically quasi-nonexpansive mappings in CAT(0) spaces with convergence analysis, Numer. Algorithms, 78 (3), 827-845.
[18] P. Thounthong, N. Pakkaranang, Y.J. Cho, W. Kumam, P. Kumam, 2020, The numerical reckoning of modified proximal point methods for minimization problems in non-positive curvature metric spaces, Int. J. Comput. Math. 97 (1-2) 245–262.
[19] O. Tripak, Common fixed points of G-nonexpansive mappings on Banach spaces with a graph, Fixed Point Theory Appl. (2016) Article no. 87.
[20] R. Suparatulatorn, W. Cholamjiak, S. Suantai, A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs, Numer Algor. 77 (2018) 479–490.
[21] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal.Appl. 67 (1979) 274–276.
[22] T.H. Kim, H.K. Xu, Strongly convergence of modified Mann iterations for with asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 64 (2006) 1140–1152.
[23] K. Nakajo, W. Takahashi, Strongly convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372–379.
[24] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex. Anal. 8 (2007) 61–79.
[25] P. Sridarat, R. Suparaturatorn, S. Suantai, Y.J. Cho, Convergence analysis of SP-iteration for G-nonexpansive mappings with directed graphs, Bull. Malays. Math. Sci. Soc. 42 (5), 2361–2380.
[26] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235 (2011) 3006–3014.
[27] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217–229.
[28] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (1) (2008) 276–286.
[29] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.