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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Regularity of the Semigroups of Transformations with a Fixed Point Set

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Abstract For a nonempty set X, let $T(X)$ and $P(X)$ denote respectively the full transformation semigroup on X and the partial transformation semigroup on X. For a nonempty subset S of X, let

$$T_{\mathcal{F}(S)}(X) = \{\alpha \in T(X) \mid x\alpha = x \text{ for all } x \in S\},$$
$$P_{\mathcal{F}(S)}(X) = \{\alpha \in P(X) \mid x\alpha = x \text{ for all } x \in \operatorname{dom}\alpha \cap S\}.$$

Then $T_{\mathcal{F}(S)}(X)$ is a regular subsemigroup of $T(X)$ and $P_{\mathcal{F}(S)}(X)$ is a subsemigroup of $P(X)$ which need not be regular. In this paper, a necessary and sufficient condition for an element of $P_{\mathcal{F}(S)}(X)$ to be regular is given. Furthermore, we characterize the left regular and right regular elements of the semigroups $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$ and made use of these results to deduce the left regularity and right regularity of them.

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1. Introduction and Preliminaries

An element x of a semigroup S is said to be *regular* if $x = xyx$ for some $y \in S$, *left regular* if $x = yx^2$ for some $y \in S$ and *right regular* if $x = x^2y$ for some $y \in S$. In fact, if an element x of S is both left and right regular, then x is regular. An element x of a semigroup S is called an *idempotent* of S if $x^2 = x$. Then an idempotent of S is regular, left regular and right regular. We call S a *regular semigroup* if every element of S is regular. *Left [Right] regular semigroups* are defined similarly. For regularity, left regularity and right regularity of semigroups, one does not imply the others. However, if a semigroup S is both left and right regular, then S is regular. As we know, regularity is an important notion and it is every extensively studied in semigroup theory.

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For a nonempty set X, let $T(X)$ and $P(X)$ denote the set of all mappings from X into itself and the set of all mappings from a subset of X into X, respectively. Then, under the composition of mappings, $P(X)$ is a semigroup having $T(X)$ as its subsemigroup. The semigroups $T(X)$ and $P(X)$ are called the *full transformation semigroup* on X and the *partial transformation semigroup* on X, respectively. The domain and the range (image) of $\alpha \in P(X)$ are denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. For an element x in the domain of $\alpha \in P(X)$, the image of x under α shall be written as $x\alpha$. Notice that $\operatorname{dom}\alpha = \bigcup_{x\in\operatorname{ran}\alpha} x\alpha^{-1}$. For $A \subseteq \operatorname{dom}\alpha$, denote by $\alpha_{|_A}$ the restriction of α to A. The identity mapping on X is denoted by 1_X. Recall that for $\alpha, \beta \in P(X)$,

$$\operatorname{dom}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha,$$
$$\operatorname{ran}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{ran}\beta \text{ and}$$
$$\text{for } x \in X,\ x \in \operatorname{dom}(\alpha\beta) \quad \Leftrightarrow \quad x \in \operatorname{dom}\alpha \text{ and } x\alpha \in \operatorname{dom}\beta.$$

It is well-known that $T(X)$ and $P(X)$ are regular semigroups. A characterization of regularity, left regularity and right regularity on subsemigroups of $T(X)$ have been widely studied. See [1–8], for example.

For a nonempty subset S of X, let

$$T_S(X) = \{\alpha \in T(X) \mid S\alpha \subseteq S\},$$
$$P_S(X) = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap S)\alpha \subseteq S\}.$$

Then $T_S(X)$ and $P_S(X)$ are clearly subsemigroups of $T(X)$ and $P(X)$, respectively. The semigroup $T_S(X)$ was introduced and studied by Magill [9] in 1966. In [4], Nenthein et al. provided some characterizations of regular elements in $T_S(X)$. The characterizations of left regular and right regular elements on $T_S(X)$ are given in [10].

For a nonempty subset S of X, let

$$T_{\mathcal{F}(S)}(X) = \{\alpha \in T(X) \mid x\alpha = x \text{ for all } x \in S\},$$
$$P_{\mathcal{F}(S)}(X) = \{\alpha \in P(X) \mid x\alpha = x \text{ for all } x \in \operatorname{dom}\alpha \cap S\}.$$

Then $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$ are subsemigroups of $T_S(X)$ and $P_S(X)$, respectively. Also, $T_{\mathcal{F}(S)}(X)$ is a subsemigroup of $P_{\mathcal{F}(S)}(X)$. We call $T_{\mathcal{F}(S)}(X)$ the *full transformation semigroup with a fixed point set* S and $P_{\mathcal{F}(S)}(X)$ the *partial transformation semigroup with a fixed point set* S. In 2013, Honyam and Sanwong [11] showed that $T_{\mathcal{F}(S)}(X)$ is a regular semigroup.

In this paper, we give an example to show that the semigroup $P_{\mathcal{F}(S)}(X)$ need not be regular and provide a necessary and sufficient condition for an element of $P_{\mathcal{F}(S)}(X)$ to be regular. We also investigate a condition for which of the semigroup $P_{\mathcal{F}(S)}(X)$ to be regular. In addition, we give a necessary and sufficient conditions for the elements of the semigroups $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$ to be left regular and right regular. These conditions are then applied to determine the left regularity and right regularity of $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$.

2. Regularity of $P_{\mathcal{F}(S)}(X)$

It is obvious that if $|X| = 1$, then $P_{\mathcal{F}(S)}(X)$ is a regular semigroup. The following example show that the semigroup $P_{\mathcal{F}(S)}(X)$ need not be regular when $|X| > 1$.

Example 2.1. Let $X = \mathbb{N}$ and $S = 2\mathbb{N}$. Define $\alpha \in P_{\mathcal{F}(S)}(X)$ by

$$x\alpha = \begin{cases} x & \text{if } x \in \{2,4,6\}, \\ x+1 & \text{if } x \in 2\mathbb{N}+1 \text{ and } x \geq 7. \end{cases}$$

Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in P_{\mathcal{F}(S)}(X)$. Then $7\alpha = 7\alpha\beta\alpha = 8\beta\alpha$. Since α is injective, we get $8\beta = 7$ which contradicts to the fact that $\beta \in P_{\mathcal{F}(S)}(X)$. Hence α is not a regular element of $P_{\mathcal{F}(S)}(X)$.

The following theorem gives a necessary and sufficient condition for an element of $P_{\mathcal{F}(S)}(X)$ to be regular.

Theorem 2.2. *For $\alpha \in P_{\mathcal{F}(S)}(X)$, α is regular if and only if* $\operatorname{dom}\alpha \cap S = \operatorname{ran}\alpha \cap S$.

Proof. To prove the forward implication, we assume that α is a regular element of $P_{\mathcal{F}(S)}(X)$. Then there exists $\beta \in P_{\mathcal{F}(S)}(X)$ such that $\alpha = \alpha\beta\alpha$. It is clear that $\operatorname{dom}\alpha \cap S \subseteq \operatorname{ran}\alpha \cap S$. Let $y \in \operatorname{ran}\alpha \cap S$. Then $y = x\alpha$ for some $x \in \operatorname{dom}\alpha$. Since $x \in \operatorname{dom}\alpha = \operatorname{dom}(\alpha\beta\alpha)$, it follows that $x\alpha \in \operatorname{dom}\beta$ and $x\alpha\beta \in \operatorname{dom}\alpha$. Then $y = x\alpha \in \operatorname{dom}\beta \cap S$, so $y\beta = y$ which implies that $y = y\beta = x\alpha\beta \in \operatorname{dom}\alpha$. This shows that $\operatorname{ran}\alpha \cap S \subseteq \operatorname{dom}\alpha \cap S$. Hence $\operatorname{dom}\alpha \cap S = \operatorname{ran}\alpha \cap S$.

For the other implication, suppose that $\operatorname{dom}\alpha \cap S = \operatorname{ran}\alpha \cap S$. For each $x \in \operatorname{ran}\alpha \setminus S$, choose $y_x \in \operatorname{dom}\alpha$ such that $y_x\alpha = x$. Define $\beta : \operatorname{ran}\alpha \to X$ by

$$x\beta = \begin{cases} x & \text{if } x \in \operatorname{ran}\alpha \cap S = \operatorname{dom}\alpha \cap S, \\ y_x & \text{if } x \in \operatorname{ran}\alpha \setminus S. \end{cases}$$

Clearly, $\beta \in P_{\mathcal{F}(S)}(X)$ and $\operatorname{ran}\beta \subseteq \operatorname{dom}\alpha$. Since $(\operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha = \operatorname{dom}\beta$ and $\operatorname{ran}\beta \subseteq \operatorname{dom}\alpha$, it follows that $\operatorname{dom}\alpha = \operatorname{dom}(\alpha\beta\alpha)$. Let $x \in \operatorname{dom}\alpha$.

Case 1: $x\alpha \in S$. Then $x\alpha \in \operatorname{ran}\alpha \cap S = \operatorname{dom}\alpha \cap S$, so $(x\alpha)\beta = x\alpha = (x\alpha)\alpha$. Hence $x\alpha\beta\alpha = (x\alpha)\alpha = x\alpha$.
Case 2: $x\alpha \notin S$. Then $x\alpha \in \operatorname{ran}\alpha \setminus S$, so $(x\alpha)\beta = y_{x\alpha}$. Hence $x\alpha\beta\alpha = y_{x\alpha}\alpha = x\alpha$.

This shows that $\alpha = \alpha\beta\alpha$. Consequently, α is regular in $P_{\mathcal{F}(S)}(X)$. ∎

Next, we characterize when $P_{\mathcal{F}(S)}(X)$ is a regular semigroup where $|X| > 1$.

Theorem 2.3. *Let $|X| > 1$. Then $P_{\mathcal{F}(S)}(X)$ is a regular semigroup if and only if $S = X$.*

Proof. Suppose that $S \neq X$. Let $x \in X \setminus S$ and let $s \in S$. Define $\alpha : \{x\} \to X$ by $x\alpha = s$. Then $\alpha \in P_{\mathcal{F}(S)}(X)$, $\operatorname{dom}\alpha \cap S = \emptyset$ and $\operatorname{ran}\alpha \cap S = \{s\}$. It follows from Theorem 2.2 that α is not a regular element of $P_{\mathcal{F}(S)}(X)$. Hence $P_{\mathcal{F}(S)}(X)$ is not a regular semigroup.

Conversely, if $S = X$, then every element of $P_{\mathcal{F}(S)}(X)$ is idempotent and hence $P_{\mathcal{F}(S)}(X)$ is a regular semigroup. ∎

3. Left Regularity of $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$

Before we determine the left regular elements of $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$, it is convenient to have the following lemma.

Lemma 3.1. *If α is a left regular element of $P(X)$, then* $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$.

Proof. Assume that α is left regular in $P(X)$. Then $\alpha = \beta\alpha^2$ for some $\beta \in P(X)$. Clearly, $\operatorname{ran}(\alpha^2) \subseteq \operatorname{ran}\alpha$. Let $y \in \operatorname{ran}\alpha$. Then $y = x\alpha$ for some $x \in \operatorname{dom}\alpha$. Since $x \in \operatorname{dom}\alpha = \operatorname{dom}(\beta\alpha^2)$, we obtain $x\beta \in \operatorname{dom}(\alpha^2)$. Hence $y = x\alpha = x\beta\alpha^2 \in \operatorname{ran}(\alpha^2)$. ∎

Now, we investigate the condition under which an element of $T_{\mathcal{F}(S)}(X)$ is left regular.

Theorem 3.2. *For $\alpha \in T_{\mathcal{F}(S)}(X)$, α is left regular if and only if* $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$.

Proof. Assume that α is a left regular element of $T_{\mathcal{F}(S)}(X)$. Since $T_{\mathcal{F}(S)}(X) \subseteq T(X) \subseteq P(X)$, we have α is left regular in $P(X)$. By Lemma 3.1, $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$.

Conversely, suppose that $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$. For each $x \in X = \bigcup_{y \in \operatorname{ran}\alpha} y\alpha^{-1}$, choose $y_x \in \operatorname{ran}\alpha$ such that $x \in y_x\alpha^{-1}$, that is, $x\alpha = y_x$. Since $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$, we can choose $d_{y_x} \in X$ such that $d_{y_x}\alpha^2 = y_x$. Then we have $d_{y_x}\alpha^2 = x\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x & \text{if } x \in S, \\ d_{y_x} & \text{if } x \in X \setminus S. \end{cases}$$

Then $\beta \in T_{\mathcal{F}(S)}(X)$. Let $x \in X$. If $x \in S$, then $x\alpha = x = x\beta$, so $x\beta\alpha^2 = x\alpha^2 = (x\alpha)\alpha = x\alpha$. If $x \notin S$, then $x\beta = d_{y_x}$ which implies that $x\beta\alpha^2 = d_{y_x}\alpha^2 = x\alpha$. This shows that $\alpha = \beta\alpha^2$. Hence α is a left regular element of $T_{\mathcal{F}(S)}(X)$, completing the proof. ∎

Theorem 3.3. *If $|X| \leq 2$, then $T_{\mathcal{F}(S)}(X)$ is a left regular semigroup.*

Proof. Assume that $|X| \leq 2$. If $|X| = 1$, then $T_{\mathcal{F}(S)}(X)$ contains exactly one element. It is clear that $T_{\mathcal{F}(S)}(X)$ is a left regular semigroup. Suppose that $|X| = 2$. Let $X = \{a, b\}$. If $S = X$, then $T_{\mathcal{F}(S)}(X) = \{1_X\}$ and hence $T_{\mathcal{F}(S)}(X)$ is left regular. Assume that $S \neq X$.

Case 1: $S = \{a\}$. Then we have

$$T_{\mathcal{F}(S)}(X) = \left\{ \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right\}.$$

We see that $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$ for all $\alpha \in T_{\mathcal{F}(S)}(X)$. By Theorem 3.2, $T_{\mathcal{F}(S)}(X)$ is a left regular semigroup.

Case 2: $S = \{b\}$. Then

$$T_{\mathcal{F}(S)}(X) = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix} \right\}.$$

It is easy to see that $T_{\mathcal{F}(S)}(X)$ is a left regular semigroup by Theorem 3.2. ∎

Next, we use Theorem 3.2 to investigate the condition under which the semigroup $T_{\mathcal{F}(S)}(X)$ is left regular where $|X| > 2$.

Theorem 3.4. *Let $|X| > 2$. Then $T_{\mathcal{F}(S)}(X)$ is a left regular semigroup if and only if $S = X$ or $|X \setminus S| = 1$.*

Proof. Suppose that $S \neq X$ and $|X \setminus S| > 1$. Let $a, b \in X \setminus S$ be such that $a \neq b$ and let $c \in S$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} x & \text{if } x \in S, \\ b & \text{if } x = a, \\ c & \text{otherwise.} \end{cases}$$

Then $\alpha \in T_{\mathcal{F}(S)}(X), \operatorname{ran}\alpha = S \cup \{b\}$ and $\operatorname{ran}(\alpha^2) = (S \cup \{b\})\alpha = S$. Since $b \notin S$, $\operatorname{ran}\alpha \neq \operatorname{ran}(\alpha^2)$. Then α is not a left regular element of $T_{\mathcal{F}(S)}(X)$ by Theorem 3.2.

For the converse, if $S = X$, then $T_{\mathcal{F}(S)}(X) = \{1_X\}$ and hence $T_{\mathcal{F}(S)}(X)$ is a left regular semigroup. Assume that $|X \setminus S| = 1$. Let $\alpha \in T_{\mathcal{F}(S)}(X)$. Then $\operatorname{ran}\alpha = S$ or $\operatorname{ran}\alpha = X$. If $\operatorname{ran}\alpha = S$, then $\operatorname{ran}(\alpha^2) = (\operatorname{ran}\alpha)\alpha = S\alpha = S = \operatorname{ran}\alpha$, so α is left regular in $T_{\mathcal{F}(S)}(X)$ by Theorem 3.2. If $\operatorname{ran}\alpha = X$, then $\operatorname{ran}(\alpha^2) = (\operatorname{ran}\alpha)\alpha = X\alpha = \operatorname{ran}\alpha$, so α is left regular in $T_{\mathcal{F}(S)}(X)$ by Theorem 3.2. Hence $T_{\mathcal{F}(S)}(X)$ is a left regular semigroup. ■

Next, we give a characterization of the left regular elements in the semigroup $P_{\mathcal{F}(S)}(X)$.

Theorem 3.5. *For $\alpha \in P_{\mathcal{F}(S)}(X)$, α is left regular if and only if $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$.*

Proof. Assume that α is a left regular element of $P_{\mathcal{F}(S)}(X)$. Since $P_{\mathcal{F}(S)}(X) \subseteq P(X)$, we obtain α is left regular in $P(X)$. By Lemma 3.1, $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$.

Conversely, suppose that $\operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$. For each $x \in \operatorname{dom}\alpha = \bigcup_{y\in\operatorname{ran}\alpha} y\alpha^{-1}$, choose $y_x \in \operatorname{ran}\alpha$ such that $x \in y_x\alpha^{-1}$. Since $y_x \in \operatorname{ran}\alpha = \operatorname{ran}(\alpha^2)$, we can choose $d_{y_x} \in \operatorname{dom}(\alpha^2)$ such that $d_{y_x}\alpha^2 = y_x$. We then have $d_{y_x}\alpha^2 = x\alpha$. Define $\beta : \operatorname{dom}\alpha \to X$ by

$$x\beta = \begin{cases} x & \text{if } x \in \operatorname{dom}\alpha \cap S, \\ d_{y_x} & \text{if } x \in \operatorname{dom}\alpha \setminus S. \end{cases}$$

Then $\beta \in P_{\mathcal{F}(S)}(X)$. If $x \in \operatorname{dom}\alpha \cap S$, then $x\alpha = x \in \operatorname{dom}\alpha$ which implies that $x \in \operatorname{dom}(\alpha^2)$. Hence $\operatorname{ran}\beta \subseteq \operatorname{dom}(\alpha^2)$. We then have $\operatorname{dom}(\beta\alpha^2) = \big(\operatorname{ran}\beta \cap \operatorname{dom}(\alpha^2)\big)\beta^{-1} = (\operatorname{ran}\beta)\beta^{-1} = \operatorname{dom}\beta = \operatorname{dom}\alpha$. Let $x \in \operatorname{dom}\alpha$. If $x \in S$, then $x\alpha = x = x\beta$, so $x\beta\alpha^2 = x\alpha^2 = (x\alpha)\alpha = x\alpha$. If $x \notin S$, then $x\beta = d_{y_x}$ which implies that $x\beta\alpha^2 = d_{y_x}\alpha^2 = x\alpha$. This shows that $\alpha = \beta\alpha^2$. Hence α is a left regular element of $P_{\mathcal{F}(S)}(X)$. ■

It is easy to see that if $|X| = 1$, then $P_{\mathcal{F}(S)}(X)$ is a left regular semigroup. We end this section by characterizing when $P_{\mathcal{F}(S)}(X)$ is a left regular semigroup where $|X| > 1$.

Theorem 3.6. *Let $|X| > 1$. Then $P_{\mathcal{F}(S)}(X)$ is a left regular semigroup if and only if $S = X$.*

Proof. Assume that $S \neq X$. Let $a \in S$ and $b \in X \setminus S$. Define $\alpha : \{b\} \to X$ by $b\alpha = a$. Then $\alpha \in P_{\mathcal{F}(S)}(X)$ and $\operatorname{ran}\alpha \neq \operatorname{ran}(\alpha^2)$, so α is not left regular in $P_{\mathcal{F}(S)}(X)$ by Theorem 3.5.

Conversely, if $S = X$, then every element of $P_{\mathcal{F}(S)}(X)$ is idempotent and hence $P_{\mathcal{F}(S)}(X)$ is a left regular semigroup. ■

4. Right Regularity of $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$

In this section, we give characterizations of the right regular elements of the semigroups $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$. In addition, the right regularity of $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$ is determined by making use of our characterizations.

The following lemma is needed to determine the right regular elements of $T_{\mathcal{F}(S)}(X)$ and $P_{\mathcal{F}(S)}(X)$.

Lemma 4.1. *If α is a right regular element of $P(X)$, then $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha_{|\operatorname{ran}\alpha}$ is injective.*

Proof. Assume that α is a right regular element of $P(X)$. Then $\alpha = \alpha^2\beta$ for some $\beta \in P(X)$. Let $y \in \operatorname{ran}\alpha$. Then $y = x\alpha$ for some $x \in \operatorname{dom}\alpha$. Since $x \in \operatorname{dom}\alpha = \operatorname{dom}(\alpha^2\beta)$, we get $y = x\alpha \in \operatorname{dom}\alpha$. Hence $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$. Let $x, y \in \operatorname{ran}\alpha$ be such that $x\alpha = y\alpha$. Then there are $a, b \in \operatorname{dom}\alpha$ such that $x = a\alpha$ and $y = b\alpha$. Thus $x = a\alpha = a\alpha^2\beta = x\alpha\beta = y\alpha\beta = b\alpha^2\beta = b\alpha = y$. This shows that $\alpha_{|\operatorname{ran}\alpha}$ is injective. ∎

Now, we give a characterization of the right regular elements in $T_{\mathcal{F}(S)}(X)$.

Theorem 4.2. *For $\alpha \in T_{\mathcal{F}(S)}(X)$, α is right regular if and only if $\alpha_{|\operatorname{ran}\alpha}$ is injective.*

Proof. Assume that α is a right regular element of $T_{\mathcal{F}(S)}(X)$. Since $T_{\mathcal{F}(S)}(X) \subseteq P(X)$, we have α is right regular in $P(X)$. By Lemma 4.1, $\alpha_{|\operatorname{ran}\alpha}$ is injective.

Conversely, suppose that $\alpha_{|\operatorname{ran}\alpha}$ is injective. For each $x \in \operatorname{ran}(\alpha^2) = (\operatorname{ran}\alpha)\alpha$, there is a unique $y_x \in \operatorname{ran}\alpha$ such that $y_x\alpha = x$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} y_x & \text{if } x \in \operatorname{ran}(\alpha^2), \\ x & \text{if } x \in X \setminus \operatorname{ran}(\alpha^2). \end{cases}$$

Let $x \in S \subseteq \operatorname{ran}(\alpha^2)$. Then $x = x\alpha \in \operatorname{ran}\alpha$, so $y_x = x$ by the uniqueness of y_x. Then $x\beta = y_x = x$. Hence $\beta \in T_{\mathcal{F}(S)}(X)$. Let $x \in X$. Since $x\alpha^2 \in \operatorname{ran}(\alpha^2)$ and $x\alpha \in \operatorname{ran}\alpha$ such that $(x\alpha)\alpha = x\alpha^2$, we get $y_{x\alpha^2} = x\alpha$. Then $x\alpha^2\beta = y_{x\alpha^2} = x\alpha$. We have that $\alpha = \alpha^2\beta$. This proves that α is a right regular element of $T_{\mathcal{F}(S)}(X)$. ∎

Theorem 4.3. *If $|X| \leq 2$, then $T_{\mathcal{F}(S)}(X)$ is a right regular semigroup.*

Proof. Assume that $|X| \leq 2$. If $|X| = 1$, then $T_{\mathcal{F}(S)}(X)$ contains only one element and hence it is a right regular semigroup. Suppose that $|X| = 2$. If $S = X$, then $T_{\mathcal{F}(S)}(X) = \{1_X\}$, so it is right regular. If $S \neq X$, then we have

$$T_{\mathcal{F}(S)}(X) = \left\{ \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right\} \text{ or } T_{\mathcal{F}(S)}(X) = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix} \right\}$$

where $X = \{a, b\}$. We see that $\alpha_{|\operatorname{ran}\alpha}$ is injective for all $\alpha \in T_{\mathcal{F}(S)}(X)$. By Theorem 4.2. $T_{\mathcal{F}(S)}(X)$ is a right regular semigroup. ∎

Next, we characterize when $T_{\mathcal{F}(S)}(X)$ is a right regular semigroup where $|X| > 2$.

Theorem 4.4. *Let $|X| > 2$. Then $T_{\mathcal{F}(S)}(X)$ is a right regular semigroup if and only if $S = X$ or $|X \setminus S| = 1$.*

Proof. Assume that $S \neq X$ and $|X \setminus S| > 1$. Let $a \in S$ and let $b, c \in X \setminus S$ such that $b \neq c$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} x & \text{if } x \in S, \\ a & \text{if } x = b, \\ b & \text{otherwise.} \end{cases}$$

Then $\alpha \in T_{\mathcal{F}(S)}(X)$ and $\operatorname{ran}\alpha = S \cup \{b\}$. Since $a, b \in \operatorname{ran}\alpha$ and $a\alpha = a = b\alpha$ but $a \neq b$, it follows that $\alpha_{|\operatorname{ran}\alpha}$ is not injective. Then α is not right regular in $T_{\mathcal{F}(S)}(X)$ by Theorem 4.2. Hence $T_{\mathcal{F}(S)}(X)$ is not a right regular semigroup.

Conversely, if $S = X$, then $T_{\mathcal{F}(S)}(X) = \{1_X\}$ and hence $T_{\mathcal{F}(S)}(X)$ is a right regular semigroup. Assume that $|X \setminus S| = 1$. Let $\alpha \in T_{\mathcal{F}(S)}(X)$. Then $\alpha = 1_X$ or $\operatorname{ran}\alpha = S$. It is easy to see that $\alpha_{|\operatorname{ran}\alpha}$ is injective. So α is right regular in $T_{\mathcal{F}(S)}(X)$ by Theorem 4.2. Hence $T_{\mathcal{F}(S)}(X)$ is a right regular semigroup. ■

Theorem 4.5. *For $\alpha \in P_{\mathcal{F}(S)}(X)$, α is right regular if and only if $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha_{|\operatorname{ran}\alpha}$ is injective.*

Proof. Suppose that α is right regular in $P_{\mathcal{F}(S)}(X)$. Since $P_{\mathcal{F}(S)}(X) \subseteq P(X)$, we get α is right regular in $P(X)$. By Lemma 4.1, $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha_{|\operatorname{ran}\alpha}$ is injective.

For the converse, assume that $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha_{|\operatorname{ran}\alpha}$ is injective. Since $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$, we get $\operatorname{ran}(\alpha^2) = (\operatorname{ran}\alpha \cap \operatorname{dom}\alpha)\alpha = (\operatorname{ran}\alpha)\alpha$. For each $x \in \operatorname{ran}(\alpha^2)$, there is a unique $y_x \in \operatorname{ran}\alpha$ such that $y_x\alpha = x$. Define $\beta : \operatorname{ran}(\alpha^2) \to X$ by

$$x\beta = y_x \text{ for all } x \in \operatorname{ran}(\alpha^2).$$

Let $x \in \operatorname{dom}\beta \cap S$. Then $x \in \operatorname{ran}(\alpha^2) \cap S$. Since $\operatorname{ran}(\alpha^2) \subseteq \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$, we have $x \in \operatorname{ran}\alpha$ and $x\alpha = x$. By the uniqueness of y_x, we obtain $y_x = x$. Then $x\beta = y_x = x$. Hence $\beta \in P_{\mathcal{F}(S)}(X)$. Clearly, $\operatorname{dom}(\alpha^2\beta) \subseteq \operatorname{dom}\alpha$. If $x \in \operatorname{dom}\alpha$, then $x\alpha \in \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $x\alpha^2 \in \operatorname{ran}(\alpha^2) = \operatorname{dom}\beta$, so $x \in \operatorname{dom}(\alpha^2\beta)$. It follows that $\operatorname{dom}\alpha = \operatorname{dom}(\alpha^2\beta)$. Let $x \in \operatorname{dom}\alpha$. Since $x\alpha^2 \in \operatorname{ran}(\alpha^2)$ and $x\alpha \in \operatorname{ran}\alpha$ such that $(x\alpha)\alpha = x\alpha^2$, we get $y_{x\alpha^2} = x\alpha$. Then $x\alpha^2\beta = y_{x\alpha^2} = x\alpha$. This shows that $\alpha = \alpha^2\beta$. Hence α is a right regular element of $P_{\mathcal{F}(S)}(X)$. ■

Clearly, $P_{\mathcal{F}(S)}(X)$ is right regular when $|X| = 1$. As a consequence of Theorem 4.5, a necessary and sufficient condition for the semigroup $P_{\mathcal{F}(S)}(X)$ with $|X| > 1$ to be right regular semigroup can be given as follows :

Theorem 4.6. *Let $|X| > 1$. Then $P_{\mathcal{F}(S)}(X)$ is a right regular semigroup if and only if $S = X$.*

Proof. Assume that $S \neq X$. Let $a \in S$ and $b \in X \setminus S$. Define $\alpha : \{b\} \to X$ by $b\alpha = a$. Then $\alpha \in P_{\mathcal{F}(S)}(X)$ and $\operatorname{ran}\alpha \nsubseteq \operatorname{dom}\alpha$, so α is not right regular in $P_{\mathcal{F}(S)}(X)$ by Theorem 4.5.

Conversely, if $S = X$, then every element of $P_{\mathcal{F}(S)}(X)$ is idempotent and hence $P_{\mathcal{F}(S)}(X)$ is a left regular semigroup. ■

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