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Hyperidentities in $(xx)y \approx x(yx)$ Graph Algebras of Type $(2,0)$

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Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). A graph G satisfies an identity $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. G is called a $(xx)y \approx x(yx)$ graph if $\underline{A(G)}$ satisfies the equation $(xx)y \approx x(yx)$. An identity $s \approx t$ of terms s and t of any type τ is called a hyperidentity of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identities hold in $\underline{A}$.

In this paper we characterize $(xx)y \approx x(yx)$ graph algebras, identities and hyperidentities in $(xx)y \approx x(yx)$ graph algebras.

Keywords : identities, hyperidentities, term, normal form term, binary algebra, graph algebras, $(xx)y \approx x(yx)$ graph algebras.

1 Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely using the concept of a hypersubstitution.

We fix a type $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is $n_i - ary$. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping $\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$ which assigns to every $n_i - ary$ operation symbol f_i an $n_i - ary$ term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$. The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras have been invented in [9] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of arcs $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u,v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

Graph identities were characterized in [3] by using the rooted graph of a term t, where the vertices correspond to the variables occurring in t. Since on a graph algebra we have one nullary and one binary operation, $\sigma(f)$ in this case is a binary term in $W_\tau(X)$, i.e., a term built up from variables of a two-element alphabet and a binary operation symbol f corresponding to the binary operation of the graph algebra. Any term over the class of all graph algebras were shown in [7], can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term t. Graph hyperidentities in [1] and associative graph hyperidentities in [6] were charaterized by using normal form graph hypersubstitutions.

We say that a graph $G = (V, E)$ is called $(xx)y \approx x(yx)$ if the corresponding graph algebra $\underline{A(G)}$ satisfied the identity $(xx)y \approx x(yx)$. In this paper we characterize $(xx)y \approx x(yx)$ graph algebras, identities and hyperidentities in $(xx)y \approx x(yx)$ graph algebras.

2 $(xx)y \approx x(yx)$ graph algebras.

We begin with a more precise definition of terms of the type of graph algebras.

Definition 2.1. The set $W_\tau(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...,$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then $f(t_1, t_2)$ is a term; instead of $f(t_1, t_2)$ we will write $t_1 t_2$, for short;

(iii) $W_\tau(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_\tau(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs is called a *trivial term*.

Definition 2.2. To each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set $Var(t)$ of all variables occurring in t, and where the edge set $E(t)$ is defined inductively by

$E(t) = \phi$ if t is a variable and $E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$.

when $t = t_1t_2$ is a compound term and $L(t_1)$, $L(t_2)$ are the leftmost variables in t_1 and t_2, respectively.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, to every trivial term t we assign the empty graph ϕ.

Definition 2.3. We say that a graph $G = (V, E)$ *satisfies an identity* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e., we have $s = t$ for every assignment $\overline{V(s) \cup V(t) \to V \cup \{\infty\})}$, and in this case, we write $G \models s \approx t$.

Definition 2.4. Let $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V^{'}$ carrying edges to edges ,that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E^{'}$.

In [3] it was proved:

Proposition 2.5. *Let s and t be non-trivial terms from $W_\tau(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G iff it is a homomorphism from $G(t)$ into G.

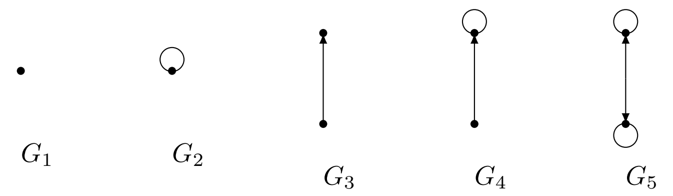
Proposition 2.5 gives a method to check whether a graph $G = (V, E)$ satisfies the equation $s \approx t$. Hence, we can check whether a graph $G = (V, E)$ has a $(xx)y \approx x(yx)$ graph algebra.

Proposition 2.6. *Let $G = (V, E)$ be a graph. Then G has $(xx)y \approx x(yx)$ graph algebra if and only if for any a,b $\in V$ if $(a, b) \in E$ then $(a, a) \in E$ if and only if $(b, a) \in E$.*

Proof. Suppose $G = (V, E)$ has a $(xx)y \approx x(yx)$ graph algebra. Let s and t be non trivial terms such that $s = (xx)y$, $t = x(yx)$. Let a,b $\in$ V such that $(a, a), (a, b) \in E$. Let $h : V(s) \to V$ be a function such that $h(x) = a$, $h(y) = b$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 2.5, we have that h is a homomorphism from $G(t)$ into G. Since $(y, x) \in E(t)$, so $(h(y), h(x)) = (b, a) \in E$. By the same way, if $(a, b), (b, a) \in E$, then we can prove that $(a, a) \in E$.

Conversely, assume $G = (V, E)$ is a graph such that for every $a, b \in V$ if $(a, b) \in E$,then $(a, a) \in E$ if and only if $(b, a) \in E$. Let s and t be non trivial terms such that $s = (xx)y$, $t = x(yx)$. Suppose $h : V(s) \to V$ is a homomorphism from $G(s)$ into G. Since $(x, x), (x, y) \in E(s)$, so $(h(x), h(x)), (h(x), h(y)) \in E$. By assumption, we get $(h(y), h(x)) \in E$. Therefore, h is a homomorphism from $G(t)$ into G. Suppose that h is a homomorphism from $G(t)$ into G. Since $(x, y), (y, x) \in E(t)$, so $(h(x), h(y)), (h(y), h(x)) \in E$. By assumption, we get $(h(x), h(x)) \in E$. Hence h is a homomorphism from $G(s)$ into G. By Proposition 2.5, we get that $\underline{A(G)}$ satisfies $s \approx t$. ■

By Proposition 2.6, the following graphs have $(xx)y \approx x(yx)$ graph algebras:



and all graphs such that each component of every subgraph induced by at most two vertices is one of these graphs.

3 Identities in $(xx)y \approx x(yx)$ graph algebras.

Graph identities were characterized in [3] by the following proposition:

Proposition 3.1. *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras iff either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Further it was proved.

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $h : X \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

In [6] the following lemma was proved:

Lemma 3.3. *Let $G = (V, E)$ be a graph, let t be a term and let*

$$h : X \longrightarrow V \cup \{\infty\}$$

be an evaluation of the variables. Then:

(i) If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is complete, then $h(t) = h(L(t))$.

(ii) If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is disconnected, then $h(t) = \infty$.

Now we apply our results to characterize all identities in the class of all $(xx)y \approx x(yx)$ graph algebras. Clearly, if s and t are trivial or $s = t = x, (x \in X)$ then $s \approx t$ is an identity in the class of all $(xx)y \approx x(yx)$ graph algebras. So we consider the case that s and t are non-trivial and different from variables. Before to do this let us introduce some notation about graph.

For any $t \in W_\tau(X)$, $x \in V(t)$ we defined $V_x(t) = \{y \in V(t) \mid$ there exists a dipath from y to x in $G(t)\}$ and $V_t = \{x \in V(t) |$ for each $z \in V_x(t)$ there is no a vertex which is both in-neighbor and out-neighbor in $G(t)\}$.

Theorem 3.4. *Let s and t be terms. Then $s \approx t$ is an identity in the class of all $(xx)y \approx x(yx)$ graph algebras if and only if the following conditions are satisfied:*
(i) $L(s) = L(t)$,
(ii) $V(s) = V(t)$,
(iii) $V_s = V_t$,
(iv) for any $x, y \in V(s), x \neq y$. Then $(x, y) \in E(s)$ or $(y, x) \in E(s), y \notin V_s$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t), y \notin V_t$.

Proof. Let s and t be non-trivial terms and suppose that $s \approx t$ is an identity in the class of all $(xx)y \approx x(yx)$ graph algebras. Since the complete graph is an $(xx)y \approx x(yx)$ graph it follows that $V(s) = V(t)$ and $L(s) = L(t)$.

Suppose $V_s \neq V_t$. Then we can suppose that there exist $x \in V_s$ but $x \notin V_t$. Consider the graph $G = (V, E)$ such that $V = V_x(s) \cup \{x, 0\}$ and $E = E_1 \cup E_2 \cup \{(0, 0)\}$ where $E_1 = \{(z, 0) \mid z \in V_x(s) \cup \{x\}\}$ and E_2 is the set of all edges in the subgraph of $G(s)$ which induced by $V_x(s) \cup \{x\}$. Then G has an $(xx)y \approx x(yx)$ graph algebras. Let $h : V(s) \to V$ be a function such that $h(z) = z, \forall z \in V_x(s) \cup \{x\}$ and $h(y) = 0$, for all other $y \in V(s)$. We see that $h(s) = h(L(s))$ and $h(t) = \infty$. Hence $\underline{A(G)}$ does not satisfy $s \approx t$.

Suppose that there exist $x, y \in V(s)$ such that $(x, y) \in E(s)$ or $(y, x) \in E(s)$, $y \notin V_s$ but $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$ or $y \in V_t$. *CaseI.* Suppose that $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$. Consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$, $E = \{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1), (0, 1), (1, 0)\}$. Then G has $(xx)y \approx x(yx)$ graph algebras. Let $h : V(s) \to V$ be a function such that $h(x) = 0$, $h(y) = 2$, $h(z) = 1$, for all other $z \in V(s)$. We see that $h(t) = h(L(t))$, $h(s) = \infty$. *CaseII.* Suppose that $(x, y) \notin E(t)$ and $(y, x) \in E(t)$ and $y \in V_t$. By (iii) and the assumption, we have $y \in V_s$ and $(x, y) \in E(s)$. Consider the graph $G = (V, E)$ such that $V = V_y(t) \cup \{y, 0\}$ and $E = E'_1 \cup E'_2 \cup \{(0, 0)\}$ where $E'_1 = \{(z, 0) \mid z \in V_y(t)\{y\}\}$ and E'_2 is the set of all edges in the subgraph of $G(s)$ which induced by $V_y(t)\{y\}$. Then G has an $(xx)y \approx x(yx)$ graph algebras. Let $h : V(s) \to V$ be a function such that $h(z) = z, \forall z \in V_x(s)\{y\}$ and $h(w) = 0$, for all other $w \in V(s)$. We see that $h(s) = h(L(s))$ and $h(t) = \infty$. Hence $\underline{A(G)}$ does not satisfy $s \approx t$.

Conversely, suppose that s and t are non-trivial terms satisfying $(i) - (iv)$. Let $G = (V, E)$ be an $(xx)y \approx x(yx)$ graph. Suppose that a mapping $h : V(s) \to V$ is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. *Case* $x = y$. Then $(x, x) \in E(t)$, so $x \notin V_t$. By (iii) we get $x \notin V_s$. Since G is an $(xx)y \approx x(yx)$ graph and h is homomorphism from $G(s)$ into G. We have $(h(x), h(x)) \in E$. *Case* $x \neq y$. If $(x, y) \in E(s)$ then we get $(h(x), h(y)) \in E$. If $(x, y) \notin E(s)$, then by (iv) we get $(y, x) \in E(s), y \notin V_s$, so there exists $z \in V(s)$ such that there is a dipath from z to y and z has a vertex which is both in-neighbor and out-neighbor. Since h is homomorphism from $G(s)$ into G and $G = (V, E)$ is an $(xx)y \approx x(yx)$ graph,we have $(h(x), h(y)) \in E$. Hence h is homomorphism from $G(t)$ into G. By the same way, if $h : V(s) \to V$ is a homomorphism from $G(t)$ into G, then we can prove that it is a homomorphism from $G(s)$ into G. By Proposition 2.5, we get that $A(G)$ satisfies $s \approx t$. ■

4 Hyperidentities in $(xx)y \approx x(yx)$ graph algebras

Let $\mathcal{ACG}$ be the classes of all $(xx)y \approx x(yx)$ graph algebras and let $Id(\mathcal{ACG})$ be the set of all identities satisfied in $\mathcal{ACG}$. Now we want to make precise the concept of a hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \to W_\tau(X_2)$, where f is the operation symbol corresponding to the binary operation of a graph algebra is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_\tau(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a $(xx)y \approx x(yx)$ *graph hyperidentity* iff for all graph hypersubstitutions σ, $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{ACG}$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $\mathcal{ACG}$ we can restrict ourselves to a (small) subset of $Hyp(\mathcal{G})$ - the set of all graph hypersubstitutions.

In [4] the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{ACG}$*-equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an identity in $\mathcal{ACG}$. In this case we write $\sigma_1 \sim_{\mathcal{ACG}} \sigma_2$.

In [2] (see also [4]) the following lemma was proved:

Lemma 4.4. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id(\mathcal{ACG})$ and $\sigma_1 \sim_{\mathcal{ACG}} \sigma_2$ then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id(\mathcal{ACG})$.*

Therefore, it is enough to consider the quotient set $Hyp(\mathcal{G})/\sim_{\mathcal{ACG}}$.

In [7] it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now, we want to describe how to construct the normal form term. Let t be a non-trivial term. The *normal form term* of t is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e. $(x, x_{i_j}) \in E(t), 1 \le j \le k(x)$) ordered by increasing indices $i_1 \le ... \le i_{k(x)}$ and let s_x be the term $(...((xx_{i_1})x_{i_2})...x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. While $Z \neq \phi$ continue this procedure. The resulting term is the normal form $NF(t)$.
The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

In [1] the following definition was given:

Definition 4.5. The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution*. Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id(\mathcal{ACG})$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_\tau(X_2)$, one obtains $\sigma_t \sim_{\mathcal{ACG}} \sigma_{NF(t)}$.

In [1] all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table.

normal form term	graph hypers.	normal form term	graph hypers.
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$x_2((x_1x_1)x_2)$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$(x_2(x_1x_2))x_2$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

By Theorem 3.4, we have the following relations:
(i) $\sigma_6 \sim_{\mathcal{ACG}} \sigma_{10} \sim_{\mathcal{ACG}} \sigma_{12} \sim_{\mathcal{ACG}} \sigma_{14} \sim_{\mathcal{ACG}} \sigma_{16} \sim_{\mathcal{ACG}} \sigma_{18}$,
(ii) $\sigma_7 \sim_{\mathcal{ACG}} \sigma_{11} \sim_{\mathcal{ACG}} \sigma_{13} \sim_{\mathcal{ACG}} \sigma_{15} \sim_{\mathcal{ACG}} \sigma_{17} \sim_{\mathcal{ACG}} \sigma_{19}$.
Let $M_{\mathcal{ACG}}$ be the set of all normal form graph hypersubstitutions in $\mathcal{ACG}$. Then

$$M_{\mathcal{ACG}} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9\},$$

and we defined the product of them as follows.

Definition 4.6. The *product* $\sigma_{1N} \circ_N \sigma_{2N}$ of two normal form graph hypersubstitutions is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.

The following table gives the multiplication of elements in $M_{\mathcal{ACG}}$.

$\circ_N$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
σ_1	σ_1	σ_1	σ_2	σ_1	σ_2	σ_2	σ_1	σ_2	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2	σ_2	σ_2	σ_1
σ_3	σ_3	σ_1	σ_2	σ_3	σ_4	σ_4	σ_3	σ_4	σ_3	σ_4
σ_4	σ_4	σ_1	σ_2	σ_3	σ_4	σ_3	σ_4	σ_4	σ_4	σ_3
σ_5	σ_5	σ_1	σ_2	σ_3	σ_4	σ_0	σ_9	σ_7	σ_7	σ_6
σ_6	σ_6	σ_1	σ_2	σ_3	σ_4	σ_7	σ_6	σ_7	σ_6	σ_7
σ_7	σ_7	σ_1	σ_2	σ_3	σ_4	σ_6	σ_7	σ_7	σ_7	σ_6
σ_8	σ_8	σ_1	σ_2	σ_3	σ_4	σ_9	σ_6	σ_7	σ_8	σ_9
σ_9	σ_9	σ_1	σ_2	σ_3	σ_4	σ_8	σ_9	σ_7	σ_7	σ_6

In [1] the concept of a leftmost normal form graph hypersubstitution was defined.

Definition 4.7. A graph hypersubstitution σ is called *$leftmost$* hypersubstitution if $L(\sigma(f)) = x_1$.

The set $M_{L(\mathcal{ACG})}$ of all leftmost normal form graph hypersubstitutions in $M_{\mathcal{ACG}}$ contains exactly the following elements;

$$M_{L(\mathcal{ACG})} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8\}.$$

In [5] the concept of a proper hypersubstitution of a class of algebras was introduced.

Definition 4.8. A hypersubstitution σ is called *proper with respect to a class $\mathcal{K}$ of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id(\mathcal{K})$ for all $s \approx t \in Id(\mathcal{K})$.

A graph hypersubstitution with the property that $\sigma(f)$ contains both variables x_1 and x_2 is called *regular*. It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid M_{reg}.

We want to prove that $\{\sigma_0, \sigma_6\, \sigma_8\}$ is the set of all proper normal form graph hypersubstitutions with respect to $\mathcal{ACG}$. In [1] the following lemma was proved.

Lemma 4.9. *For each non-trivial term $s, (s \neq x \in X_2)$ and for all $u, v \in X_2$, we have (i) $E(\hat{\sigma_6}[s]) = E(s) \cup \{(u,u) | (u,v) \in E(s)\}$,*
(ii) $E(\hat{\sigma_8}[s]) = E(s) \cup \{(v,v) | (u,v) \in E(s)\}$.

Then we obtain:

Theorem 4.10. $\{\sigma_0, \sigma_6, \sigma_8\}$ *is the set of all proper graph hypersubstitution with respect to the class $\mathcal{ACG}$ of $(xx)y \approx x(yx)$ graph algebras.*

Proof. If $s \approx t \in Id(\mathcal{ACG})$ and $s,\ t$ are trivial terms, then for every graph hypersubstitutions $\sigma \in \{\sigma_6, \sigma_8\}$ the term are $\hat{\sigma}[s]$ and $\hat{\sigma}[t]$ are also trivial terms and thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id(\mathcal{ACG})$. In the same manner, we see that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id(\mathcal{ACG})$ for every $\sigma \in \{\sigma_6, \sigma_8\}$, if $s = t = x$.

Now, assume that s and t are non-trivial terms, different from variables, and $s \approx t \in Id\mathcal{ACG}$. Then $(i) - (iv)$ of Theorem 3.4 hold.

For σ_6 and σ_8, we get $L(\hat{\sigma}_6[s]) = L(s) = L(t) = L(\hat{\sigma}_6[t])$ and $L(\hat{\sigma}_8[s]) = L(s) = L(t) = L(\hat{\sigma}_8[t])$. Since σ_6 and σ_8 are regular, we have $V(\hat{\sigma}_6[s]) = V(s) = V(t) = V(\hat{\sigma}_6[t])$ and $V(\hat{\sigma}_8[s]) = V(s) = V(t) = V(\hat{\sigma}_8[t])$. By Lemma 4.9, we get $E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u) \mid (u,v) \in E(s)\}$, $E(\hat{\sigma}_6[t]) = E(t) \cup \{(u,u) \mid (u,v) \in E(t)\}$, $E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v) \mid (u,v) \in E(s)\}$ and $E(\hat{\sigma}_8[t]) = E(t) \cup \{(v,v) \mid (u,v) \in E(t)\}$. Since $L(s) = L(t)$ and s, t are different from variables, we obtain $(L(s), y) \in E(s)$ and $(L(t), z) \in E(t)$ for some $y, z \in V(s)$. Hence $(L(s), L(s)) \in E(\hat{\sigma}_6[s])$ and $(L(t), L(t)) \in E(\hat{\sigma}_6[t])$. Thus $V_{\hat{\sigma}_6[s]} = V_{\hat{\sigma}_6[t]} = \emptyset$. For any $x, y \in V(s),\ x \neq y$. Suppose that $(x,y) \in E(\hat{\sigma}_6[s])$ or $(y,x) \in E(\hat{\sigma}_6[s]),\ y \notin V_{\hat{\sigma}_6[s]}$. If $(x,y) \in E(\hat{\sigma}_6[s])$, then by Lemma 4.9 we get $(x,y) \in E(s)$. By assumption and Theorem 3.4 we get $(x,y) \in E(t)$ or $(y,x) \in E(t),\ y \notin V_t$. Hence $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x) \in$

$E(\hat{\sigma}_6[t]),\ y \notin V_{\hat{\sigma}_6[t]}$. If $(y,x) \in E(\hat{\sigma}_6[s]),\ y \notin V_{\hat{\sigma}_6[s]}$, then $(y,x) \in E(s)$. By assumption and Theorem 3.4 we get $(y,x) \in E(t)$ or $(x,y) \in E(t),\ x \notin V_t$. Hence $(y,x) \in E(\hat{\sigma}_6[t])$ or $(x,y) \in E(\hat{\sigma}_6[t]),\ x \notin V_{\hat{\sigma}_6[t]}$. Since $V_{\hat{\sigma}_6[t]} = \emptyset$, we get $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x) \in E(\hat{\sigma}_6[t]),\ y \notin V_{\hat{\sigma}_6[t]}$. By the same way if $(x,y) \in E(\hat{\sigma}_6[t])$ or $(y,x) \in E(\hat{\sigma}_6[t]),\ y \notin V_{\hat{\sigma}_6[t]}$, then we can prove that $(x,y) \in E(\hat{\sigma}_6[s])$ or $(y,x) \in E(\hat{\sigma}_6[s]),\ y \notin V_{\hat{\sigma}_6[s]}$. By Theorem 3.4, we get that $\hat{\sigma_6}[s] \approx \hat{\sigma_6}[t] \in Id\mathcal{ACG}$.

For σ_8, since for any $x' \in V(s),\ x' \neq L(s)$ there exist $y', z' \in V(s)$ such that $(y', x') \in E(s)$ and $(z', x') \in E(t)$. By Lemma 4.9, we get $(x', x') \in E(\hat{\sigma}_8[s])$ and $(x', x') \in E(\hat{\sigma}_8[t])$. Hence $x' \notin V_{\hat{\sigma}_8[s]}$ and $x' \notin V_{\hat{\sigma}_8[t]}$. If $L(s) \notin V_{\hat{\sigma}_8[s]}$, then we get $(L(s), L(s)) \in E(s)$ or there exist $y \in V(s),\ y \neq L(s)$ such that $(y, L(s)) \in E(s)$. If $(L(s), L(s)) \in E(s)$, then $L(s) \notin V_s$. Since $V_s = V_t$, we get $L(s) \notin V_t$. Then $L(s) \notin V_{\hat{\sigma}_8[t]}$. If $(y, L(s)) \in E(s)$, then by (iv) we have $(y, L(s)) \in E(t)$ or $(L(s), y) \in E(t),\ L(s) \notin V_t$. Hence $(L(s), L(s)) \in E(\hat{\sigma}_8[t])$ or $L(s) \notin V_{\hat{\sigma}_8[t]}$. Thus $L(s) \notin V_{\hat{\sigma}_8[t]}$. By the same way, if $L(s) \in V_{\hat{\sigma}_8[t]}$, then we can prove that $L(s) \notin V_{\hat{\sigma}_8[s]}$. Thus $V_{\hat{\sigma}_8[s]} = V_{\hat{\sigma}_8[t]}$. For any $x, y \in V(s),\ x \neq y$. Suppose that $(x,y) \in E(\hat{\sigma}_8[s])$ or $(y,x) \in E(\hat{\sigma}_8[s]),\ y \notin V_{\hat{\sigma}_8[s]}$. If $(x,y) \in E(\hat{\sigma}_8[s])$, then $(x,y) \in E([s])$. By (iv) we get $(x,y) \in E(t)$ or $(y,x) \in E(t),\ y \notin V_t$. Hence $(x,y) \in E(\hat{\sigma}_8[t])$ or $(y,x) \in E(\hat{\sigma}_8[t]),\ y \notin V_{\hat{\sigma}_8[t]}$. If $(y,x) \in E(\hat{\sigma}_8[s]),\ y \notin V_{\hat{\sigma}_8[s]}$, then $(y,x) \in E(s),\ y \notin V_{\hat{\sigma}_8[s]} = V_{\hat{\sigma}_8[t]}$. By (iv) we get $(y,x) \in E(t)$ or $(x,y) \in E(t),\ x \notin V_t$. Hence $(x,y) \in E(\hat{\sigma}_8[t])$ or $(y,x) \in E(\hat{\sigma}_8[t]),\ y \notin V_{\hat{\sigma}_8[t]}$. By the same way if $(x,y) \in E(\hat{\sigma}_8[t])$ or $(y,x) \in E(\hat{\sigma}_8[t]),\ y \notin V_{\hat{\sigma}_8[t]}$, then we can prove that $(x,y) \in E(\hat{\sigma}_8[s])$ or $(y,x) \in E(\hat{\sigma}_8[s]),\ y \notin V_{\hat{\sigma}_8[s]}$. By Theorem 3.4, we get $\hat{\sigma_8}[s] \approx \hat{\sigma_6}[t] \in Id(\mathcal{ACG})$.

For any $\sigma \notin \{\sigma_0, \sigma_6, \sigma_8\}$, we give an identity $s \approx t$ in $\mathcal{ACG}$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id(\mathcal{ACG})$. Clearly, if s and t are terms with different leftmost and different rightmost, then $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t]$, $\hat{\sigma_2}[s] \approx \hat{\sigma_2}[t]$, $\hat{\sigma_3}[s] \approx \hat{\sigma_3}[t]$ and $\hat{\sigma_4}[s] \approx \hat{\sigma_4}[t] \notin Id(\mathcal{ACG})$. Now, let $s = x_1(x_2x_1),\ t = x_1(x_2(x_1x_2))$. By Theorem 3.4, we get $s \approx t \in Id(\mathcal{ACG})$. If $\sigma \in \{\sigma_5, \sigma_7, \sigma_9\}$, then $L(\sigma(f)) = x_2$. We see that $L(\hat{\sigma}[s]) = x_1$ and $L(\hat{\sigma}[t]) = x_2$ for $\sigma \in \{\sigma_5, \sigma_7, \sigma_9\}$. Thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id(\mathcal{ACG})$. ∎

Now, we apply our results to characterize all hyperidentities in the class of all $(xx)y \approx x(yx)$ graph algebras. Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in $\mathcal{ACG}$ if and only if they have the same leftmost and the same rightmost and $x \approx x,\ x \in X$ is a hyperidentity in $\mathcal{ACG}$, too. So we consider the case that s and t are non-trivial and different from variables.

In [1] the concept of a *dual term* s^d of the non-trivial term s was defined in the following way:

If $s = x \in X$, then $x^d = x$, if $s = t_1t_2$, then $s^d = t_2^d t_1^d$. The dual term s^d can be obtained by application of the graph hypersubstitution σ_5, namely, $\hat{\sigma}_5[s] = s^d$.

Theorem 4.11. *An identity $s \approx t$ in $\mathcal{ACG}$, where $s,\ t$ are non-trivial and $s \neq x$, $t \neq x$, is a hyperidentity in $\mathcal{ACG}$ if and only if the dual $s^d \approx t^d$ is also an identity in $\mathcal{ACG}$.*

Proof. If $s \approx t$ is a hyperidentity in $\mathcal{ACG}$, then $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{ACG}$, i.e., $s^d \approx t^d$ is an identity in $\mathcal{ACG}$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{ACG}$ and that $s^d \approx t^d$ is an identity in $\mathcal{ACG}$, too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from $M_{\mathcal{ACG}}$.

If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8\}$, then σ is a proper and we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{ACG}$. By assumption, $\hat{\sigma}_5[s] = s^d \approx t^d = \hat{\sigma}_5[t]$ is an identity in $\mathcal{ACG}$.

For $\sigma_1, \sigma_2, \sigma_3$ and σ_4, we have $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$, $\hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t]$, $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$ and $\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]$. Because of $\sigma_6 \circ_N \sigma_5 = \sigma_7$, $\hat{\sigma}_6[\hat{\sigma}_5[t']] = \hat{\sigma}_7[{t'}^d]$ and $\sigma_8 \circ_N \sigma_5 = \sigma_9$, $\hat{\sigma}_8[\hat{\sigma}_5[t']] = \hat{\sigma}_9[{t'}^d]$ for all $t' \in W_\tau(X)$. So, we have $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$ and $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ are identities in $\mathcal{ACG}$. ■

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