Fixed and Periodic Point Results for Generalized Geraghty Contractions in Partial Metric Spaces with Application

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Abstract In this paper, by using the concept of α-Garaghty contractions, we introduce a new concept of $(\mathcal{F}, h)$-type Garaghty contraction via α-admissible and μ-subadmissible mappings. We prove some fixed point and periodic point results for such mappings in partial metric spaces. We also give an example for supporting the result. Moreover, an existence of solutions for nonlinear ordinary differential equation of second order with boundary conditions is obtained.

1. INTRODUCTION

Fixed point theory is a subfield of nonlinear functional analysis. It has been used as a tool in research areas of nonlinear sciences and mathematics. In 1922, Banach [1] proved a theorem, which is well known as " Banach's fixed point theorem " to establish the existence of solutions for integral equation.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a contractive mapping, that is, there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

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for all $x, y \in X$. Then T has a unique fixed point $z \in X$.

In 1973, Geraghty [2] studied a generalization of Banach's contraction principle in the setting of complete metric spaces observing the following interesting results.

We denote by Ω the class of all functions $\beta : [0, \infty) \to [0, 1)$ satisfying $\beta(t_n) \to 1$, implies $t_n \to 0$ as $n \to \infty$.

Theorem 1.2. *Let (X, d) be a metric space. Let $S : X \to X$ be a self mapping. Suppose that there exists $\beta \in \Omega$ such that for all $x, y \in X$,*

$$d(Sx, Sy) \leq \beta(d(x, y))d(x, y).$$

Then S has a unique fixed point $p \in X$ and $\{S^n x\}$ converges to p for each $x \in X$.

In 1994, Matthews [3] introduced the concept of partial metric space and studied the Banach's contraction principle in partial metric space. After, many authors studied fixed point results in such space. [4–6].

In 2013, the concept of α-admissible mappings is generalized by Hussian et al. [7]. Later in the same year, the concept of a pair of α-admissible is introduced by Abdeljawad [8]. Thereafter, many papers on Geraghty's contraction in both complete metric spaces and partial metric spaces appeared in the literature. [9–11].

Recently, Ansari [12] (and Ansari et al. [13]) introduced the notion of a pair $(\mathcal{F}, h)$ as an upper class of type I and type II and obtained some fixed point theorems which generalized many existing results. In this paper, we introduce the notion of $(\alpha, \mu), (\mathcal{F}, h)$-type Geraghty contraction in the setting of partial metric space and prove some fixed point and periodic point theorems for such contractions. Then, we apply our results to establishing the existence of solution for nonlinear ordinary differential equation of second order satisfying some boundary conditions.

2. Preliminaries

In this section, we give some definitions, examples and fundamental results.

Definition 2.1. [3] Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ satisfy following properties:

  (PM1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$;
  (PM2) $p(x, x) \leq p(x, y)$;
  (PM3) $p(x, y) = p(y, x)$;
  (PM4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$,

for all $x, y, z \in X$. Then p is called a partial metric on X and the pair (X, p) is known as partial metric space.

In 1995, Matthews [3] proved that every partial metric p on X induces a metric $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$.

Notice that a metric on a set X is a partial metric d such that $d(x, x) = 0$ for all $x \in X$.

Definition 2.2. [3] Let (X, p) be a partial metric space.

  (i) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, p) is called a Cauchy sequence if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists and is finite.

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.

Definition 2.3. [14] Let $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that S is α-admissible if $x, y \in X$, such that

$$\alpha(x, y) \geq 1 \implies \alpha(Sx, Sy) \geq 1.$$

Example 2.4. [15] Consider $X = [0, \infty)$ and define $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = 2x$, for all $x, y \in X$ and

$$\alpha(x, y) = \begin{cases} e^{y/x}, & \text{if } x \geq y, x \neq 0 \\ 0, & \text{if } x < y. \end{cases}$$

Then S is α-admissible.

Definition 2.5. Let $T : X \to X$ be a map and $\mu : X \times X \to [0, \infty)$ be a function. We say that T is μ-subadmissible if $x, y \in X$, $\mu(x, y) \leq 1$ implies that $\mu(Tx, Ty) \leq 1$.

Lemma 2.6. [3]

(i) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete.

(ii) A sequence $\{x_n\}$ in X converges to a point $x \in X$, with respect to $\tau(d_p)$ if and only if $\lim_{n \to \infty} p(x, x_n) = p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.

(iii) If $\lim_{n \to \infty} x_n = v$ such that $p(v, v) = 0$ then $\lim_{n \to \infty} p(x_n, y) = p(v, y)$ for every $y \in X$.

Lemma 2.7. [11] Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

In 2014 A.H. Ansari [12] introduced the concept of a pair $(\mathcal{F}, h)$ is an upper class.

Definition 2.8. [12] We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if $x \geq 1 \implies h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.

Example 2.9. [12, 13] Define $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

- $h(x, y) = (y + l)^x, l > 1$;
- $h(x, y) = (x + l)^y, l > 1$;
- $h(x, y) = x^n y, n \in \mathbb{N}$;
- $h(x, y) = y$;
- $h(x, y) = \frac{1}{n+1} (\sum_{i=0}^{n} x^i) y, n \in \mathbb{N}$;
- $h(x, y) = \left[\frac{1}{n+1} (\sum_{i=0}^{n} x^i) + l \right]^y, l > 1, n \in \mathbb{N}$,

for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 2.10. [12, 13] Let $h, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type I, if h is a function of subclass of type I and satisfying the following conditions;

(i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s,t) \leq \mathcal{F}(1,t)$;
(ii) $h(1,y) \leq \mathcal{F}(1,t) \Longrightarrow y \leq t$,

for all $t, y \in \mathbb{R}^+$.

Example 2.11. [12, 13] Define $h, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

- $h(x,y) = (y+l)^x, l > 1$ and $\mathcal{F}(s,t) = st + l$;
- $h(x,y) = (x+l)^y, l > 1$ and $\mathcal{F}(s,t) = (1+l)^{st}$;
- $h(x,y) = x^m y, m \in \mathbb{N}$ and $\mathcal{F}(s,t) = st$;
- $h(x,y) = y$ and $\mathcal{F}(s,t) = t$;
- $h(x,y) = \frac{1}{n+1}(\sum_{i=0}^{n} x^i)y, n \in \mathbb{N}$ and $\mathcal{F}(s,t) = st$;
- $h(x,y) = \left[\frac{1}{n+1}(\sum_{i=0}^{n} x^i) + l\right]^y, l > 1, n \in \mathbb{N}$ and $\mathcal{F}(s,t) = (1+l)^{st}$,

for all $x, y, s, t \in \mathbb{R}^+$. Then h the pair $(\mathcal{F}, h)$ is an upper class of type I.

3. Main Results

In this section, we prove some fixed point theorems for $(\alpha, \mu), (\mathcal{F}, h)$-type Geraghty contraction in complete partial metric space.

3.1. Fixed Point Results for $(\mathcal{F}, h)$-Type Geraghty Contraction

We begin with the following definition.

Definition 3.1. Let (X, p) be a partial metric space and let C be a nonempty subset of X. A mapping $T : C \to C$ is called an $(\alpha, \mu), (\mathcal{F}, h)$-type Garaghty contraction if there exist $\beta \in \Omega$ and $\alpha, \mu : C \times C \to [0, \infty)$ such that, for all $x, y \in C$, satisfying the following inequality:

$$h(\alpha(x,y), p(Tx, Ty)) \leq \mathcal{F}(\mu(x,y), \beta(p(x,y))p(x,y)), \tag{3.1}$$

where the pair $(\mathcal{F}, h)$ is an upper class of type I.

Example 3.2. Let $X = \mathbb{R}$ and $C = [0, 1]$. Define $p : C \times C \to [0, \infty)$ by

$$p(x,y) = \max\{x, y\}$$

for all $x, y \in C$. It is easy to check that p is a partial metric space and define $\alpha, \mu : C \times C \to [0, \infty)$ by

$$\alpha(x,y) = 1$$

and

$$\mu(x,y) = 2$$

for all $x, y \in C$.

Define the mapping $T : C \to C$ by

$$Tx = x^2$$

and define $\beta : [0, \infty) \to [0, 1)$ by $\beta(t) = \frac{1}{t+1}$ and $h, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $h(x, y) = y$, $\mathcal{F}(s, t) = st$, respectively, for all $s, t, x, y \in C$.

Now, we will show that the Definition (3.1) is true. We consider

$$h(\alpha(x, y), p(T(x), T(y))) = h(1, p(x^2, y^2)). \tag{3.2}$$

If $x > y$, then from (3.2), we obtain

$$\begin{aligned}
h(\alpha(x, y), p(T(x), T(y))) &= h(1, p(x^2, y^2)) \\
&= h(1, \max\{x^2, y^2\}) \\
&= h(1, x^2) \\
&= x^2 \\
&\leq \frac{2x}{x+1} \\
&= \mathcal{F}(2, \frac{x}{x+1}) \\
&= \mathcal{F}(2, \left(\frac{1}{x+1}\right)(x)) \\
&= \mathcal{F}(2, \beta(x)(x)) \\
&= \mathcal{F}(2, \beta(\max\{x, y\})(\max\{x, y\})) \\
&= \mathcal{F}(\mu(x, y), \beta(p(x, y))p(x, y)).
\end{aligned}$$

Similarly, for cases $x < y$ and $x = y$.
So, Definition (3.1) holds.

Theorem 3.3. *Let (X, p) be a complete partial metric space and let C be a nonempty subset of X and $\alpha, \mu : X \times X \to [0, \infty)$ be a function. Suppose that $T : C \to C$ be an $(\alpha, \mu), (\mathcal{F}, h)$-type Geraghty contraction satisfying the following conditions:*

   (i) T is α-admissible and μ-subadmissible;
   (ii) there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$;
   (iii) T is continuous.
Then T has a fixed point $x^* \in C$.

Proof. Let $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. In particular, if $x_n = Tx_n$ for all $n \in \mathbb{N}$, then T has a fixed point in C. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From condition (ii) and (i), we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \implies \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$$

and

$$\mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1 \implies \mu(Tx_0, Tx_1) = \mu(x_1, x_2) \leq 1.$$

By induction, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{3.3}$$

and

$$\mu(x_n, x_{n+1}) \leq 1. \tag{3.4}$$

Since T is $(\alpha, \mu), (\mathcal{F}, h)$-type Garaghty contraction, then for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
h(1, p(x_n, x_{n+1})) &\leq h(\alpha(x_n, x_{n+1}, p(Tx_{n-1}, Tx_n))) \\
&\leq \mathcal{F}(\mu(x_n, x_{n+1}), \beta(p(x_{x-1}, x_n))p(x_{n-1}, x_n)) \\
&\leq \mathcal{F}(1, \beta(p(x_{x-1}, x_n))p(x_{n-1}, x_n))
\end{aligned}
$$

which implies that

$$
\begin{aligned}
p(x_n, x_{n+1}) &\leq \beta(p(x_{x-1}, x_n))p(x_{n-1}, x_n) \tag{3.5} \\
&< p(x_{n-1}, x_n).
\end{aligned}
$$

That is

$$
p(x_n, x_{n+1}) < p(x_{n-1}, x_n).
$$

So, the sequence $\{p(x_n, x_{n+1})\}$ is nonnegative and decreasing. Now, we will show that $\lim_{n \to \infty} p(x_n, x_{n+1}) = 0$. It is clear that $\{p(x_n, x_{n+1})\}$ is a decreasing sequence. Thus, there exists $r > 0$, such that

$$
\lim_{n \to \infty} p(x_n, x_{n+1}) = r.
$$

From condition (3.5), we have

$$
\frac{p(x_n, x_{n+1})}{p(x_{n-1}, x_n)} \leq \beta(p(x_{n-1}, x_n)) < 1.
$$

By, taking $n \to \infty$, we have

$$
\beta(p(x_{n-1}, x_n)) \to 1 \text{ as } n \to \infty.,
$$

that is

$$
\lim_{n \to \infty} \beta(p(x_{n-1}, x_n)) = 1.
$$

By the property of $\beta \in \Omega$, which implies that

$$
\lim_{n \to \infty} p(x_{n-1}, x_n) = 0. \tag{3.6}
$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. That is. there exists $\varepsilon > 0$, we can find subsequence $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k, we have $m_k > n_k > k$,

$$
p(x_{m_k}, x_{n_k}) \geq \varepsilon
$$

and

$$
p(x_{m_k}, x_{n_{k-1}}) < \varepsilon.
$$

So, we have

$$
\begin{aligned}
\varepsilon &\leq p(x_{m_k}, x_{n_k}) \\
&\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) - p(x_{n_{k-1}}, x_{n_{k-1}}) \\
&\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) \\
&< \varepsilon + p(x_{n_{k-1}}, x_{n_k}).
\end{aligned}
$$

That is

$$
\varepsilon < \varepsilon + p(x_{n_{k-1}}, x_{n_k}), \tag{3.7}
$$

from conditions (3.6) and (3.7), we have

$$
\lim_{k \to \infty} p(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.8}
$$

By triangle inequality, we have

$$
\begin{aligned}
p(x_{m_k}, x_{n_k}) &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) - p(x_{m_{k+1}}, x_{m_{k+1}}) \\
&\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) \\
&\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}) - p(x_{n_{k+1}}, x_{n_{k+1}}) \\
&\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}),
\end{aligned}
$$

and

$$
\begin{aligned}
p(x_{m_{k+1}}, x_{n_{k+1}}) &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) - p(x_{m_k}, x_{m_k}) \\
&\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) \\
&\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}) - p(x_{n_k}, x_{n_k}) \\
&\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}).
\end{aligned}
$$

Taking $k \to \infty$ and from condition (3.6) and (3.8), we have

$$
\lim_{k \to \infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \varepsilon.
$$

Now, we consider

$$
\begin{aligned}
h(1, p(x_{n_{k+1}}, x_{m_{k+2}})) &\leq h(\alpha(x_{n_k}, x_{m_{k+1}}), p(x_{n_{k+1}}, x_{m_{k+2}})) \\
&= h(\alpha(x_{n_k}, x_{m_{k+1}}), p(T x_{n_k}, T x_{m_{k+1}})) \\
&\leq \mathcal{F}(\alpha(x_{n_k}, x_{m_{k+1}}), \beta(p(x_{n_k}, x_{m_{k+1}})) p(x_{n_k}, x_{m_{k+1}})) \\
&\leq \mathcal{F}(1, \beta(p(x_{n_k}, x_{m_{k+1}})) p(x_{n_k}, x_{m_{k+1}}))
\end{aligned}
$$

which implies that

$$
p(x_{n_{k+1}}, x_{m_{k+2}}) \leq \beta(p(x_{n_k}, x_{m_{k+1}})) p(x_{n_k}, x_{m_{k+1}}).
$$

That is

$$
\frac{p(x_{n_{k+1}}, x_{m_{k+2}})}{p(x_{n_k}, x_{m_{k+1}})} \leq \beta(p(x_{n_k}, x_{m_{k+1}})) < 1,
$$

taking $n \to \infty$ in the above inequality, we have

$$
\beta(p(x_{n_k}, x_{m_{k+1}})) \to 1 \text{ as } n \to \infty.
$$

By using the property of $\beta \in \Omega$, which implies that

$$
\lim_{k \to \infty} p(x_{n_k}, x_{m_{k+1}}) = 0 < \varepsilon,
$$

which is contradiction. So,

$$
\lim_{n,m \to \infty} p(x_n, x_m) = 0.
$$

That is $\{x_n\}$ is a Cauchy sequence in (X, p). By Lemma 2.6 (iii) $\{x_n\}$ is also Cauchy sequence in (X, d_p). Since (X, p) is complete, (X, d_p) is also complete. Thus there exists $x^* \in C \subset X$ such that $\lim_{n \to \infty} p(x_n, x^*) = 0 = p(x^*, x^*) = \lim_{n,m \to \infty} p(x_n, x_m)$ and since T is continuous, we have

$$
x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} T x_n = T \lim_{n \to \infty} x_n = T x^*.
$$

Thus x^* is a fixed point of T.

■

Theorem 3.4. *Let (X, p) be a complete partial metric space and let C be a nonempty subset of X and $\alpha, \mu : X \times X \to [0, \infty)$ be a function. Suppose that $T : C \to C$ be an $(\alpha, \mu), (\mathcal{F}, h)$-type Geraghty contraction satisfying the following conditions:*

(i) T is α-admissible and μ-subadmissible;

(ii) there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1, \mu(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point $x^* \in C$.

Proof. Let $x_0 \in C$ be such that $\alpha(x_0, Tx_0) \geq 1$ and let $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Following the proof of Theorem 3.3, we obtain $\{x_n\}$ is a Cauchy sequence in complete partial metric space Then there exists $x^* \in C$ such that $x_n \to x^*$ as $n \to \infty$.

We suppose that condition (iii) holds. So, $\alpha(x^*, Tx^*) \geq 1, \mu(x^*, Tx^*) \leq 1$. By (3.1), we have

$$
\begin{aligned}
h(1, p(Tx^*, x_{n+1})) &\leq h(\alpha(x^*, Tx^*), p(Tx^*, x_{n+1}) \\
&= h(\alpha(x^*, Tx^*), p(Tx^*, Tx_n) \\
&\leq \mathcal{F}(\mu(x^*, Tx^*), \beta(p(x^*, x_n))p(x^*, x_n)) \\
&\leq \mathcal{F}(1, \beta(p(x^*, x_n))p(x^*, x_n))
\end{aligned}
$$

which implies that

$$
p(Tx^*, x_{n+1}) \leq \beta(p(x^*, x_n))p(x^*, x_n).
$$

So, we have

$$
\begin{aligned}
p(Tx^*, x^*) &\leq p(Tx^*, x_{n+1}) + p(x_{n+1}, x^*) - p(x_{n+1}, x_{n+1}) \\
&\leq p(Tx^*, x_{n+1}) + p(x_{n+1}, x^*) \\
&\leq \beta(p(x^*, x_n))p(x^*, x_n) + p(x_{n+1}, x^*).
\end{aligned}
$$

That is

$$
p(Tx^*, x^*) \leq \beta(p(x^*, x_n))p(x^*, x_n) + p(x_{n+1}, x^*).
$$

Taking $n \to \infty$ in the above inequality, we obtain

$$
p(Tx^*, x^*) = 0.
$$

So from the definition of d_p we get $x^* = Tx^*$. Therefore x^* is a fixed point of T. ∎

Indeed, the uniqueness of fixed point, we considering the following condition:

(C): for all $x, y \in Fix(T), \alpha(x, y) \geq 1, \mu(x, y) \leq 1$.

Theorem 3.5. *From Theorem 3.3, if we add condition (C), then the uniqueness of fixed point is obtained.*

Proof. Suppose that $y^* \in C$ is an another fixed point in T such that $x^* \neq y^*$ and $Ty^* = y^*$. Then, we have

$$
\begin{aligned}
h(1, p(x^*, y^*)) &\leq h(\alpha(x^*, y^*), p(Tx^*, Ty^*)) \\
&\leq \mathcal{F}(\mu(x^*, y^*), \beta(p(x^*, y^*))p(x^*, y^*)) \\
&\leq \mathcal{F}(1, \beta(p(x^*, y^*))p(x^*, y^*))
\end{aligned}
$$

which implies that

$$p(x^*, y^*) \leq \beta(p(x^*, y^*))p(x^*, y^*)$$
$$< p(x^*, y^*),$$

which is contradiction. So, $x^* = y^*$ ∎

The Example 3.2 satisfies all the hypothesis of Theorem 3.4, so, T has unique fixed point $x = 1$. On the other hand, $p(x, y) = 0$ if and only if $x = y$ for all $x, y \in C$ in Theorem 3.3 and 3.4. Then, we deduce the following corollary:

Corollary 3.6. *Let (X, p) be a complete metric space and let C be a nonempty subset of X and $\alpha, \mu : X \times X \to [0, \infty)$ be a function. Suppose that $T : C \to C$ be an $(\alpha, \mu), (\mathcal{F}, h)$-type Geraghty contraction satisfying the following conditions:*

(i) T is α-admissible and μ-subadmissible;
(ii) there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$;
(iii) T is continuous.

Then T has a fixed point $x^* \in C$.

Corollary 3.7. *Let (X, p) be a complete metric space and let C be a nonempty subset of X and $\alpha, \mu : X \times X \to [0, \infty)$ be a function. Suppose that $T : C \to C$ be an $(\alpha, \mu), (\mathcal{F}, h)$-type Geraghty contraction satisfying the following conditions:*

(i) T is α-admissible and μ-subadmissible;
(ii) there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1, \mu(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point $x^* \in C$.

3.2. Periodic Point Results

In this section, we prove some periodic point results in complete partial metric space. We need the following definition.

Definition 3.8. [16] *A mapping $T : C \to C$ is said to have the property (P) if $Fix(T^n) = Fix(T)$ foe every $n \in \mathbb{N}$, where $Fix(T) := \{x \in X : Tx = x\}$.*

Theorem 3.9. *Let (X, p) be a complete partial metric space and let C be a nonempty subset of X and $\alpha, \mu : X \times X \to [0, \infty)$ be a function. Suppose that $T : C \to C$ be a mapping satisfying the following conditions:*

(i) there exist $\beta \in \Omega$ and $\alpha, \mu : C \times C \to [0, \infty)$ such that, such that

$$h(\alpha(x, y), p(Tx, Ty) \leq \mathcal{F}(\mu(x, y), \beta(p(x, y))p(x, y)$$

holds, for all $x, y \in C$.
(ii) T is α-admissible and μ-subadmissible;
(iii) there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$;
(iv) if $\{x_n\}$ is a sequence in C such that $\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ and $p(x_n, x) \to 0$ as $n \to \infty$. Then $p(Tx_nTx) \to 0$ as $n \to \infty$;
(v) if $z \in Fix(T^n)$ and $z \notin Fix(T)$, then $\alpha(T^{n-1}z, T^n z) \geq 1, \mu(T^{n-1z, T^n z}) \leq 1$

Then T has the property (P).

Proof. Let $x_0 \in C$ be such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$ and define sequence $\{x_n\}$ by $x_n = T^n x_n = Tx_{n-1}$. By (ii), we have $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$ and by induction, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{3.9}$$

and

$$\mu(x_n, x_{n+1}) \leq 1 \tag{3.10}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T. So, we assume that $x_n \neq x_{n+1}$ or $p(Tx_{n-1}, T^2 x_{n-1}) > 0$ for all $n \in \mathbb{N}$. From (3.9), (3.10) and (i), we have

$$\begin{aligned}
h(1, p(x_n, x_{n+1})) &= h(1, p(Tx_{n-1}, T^2 x_{n-1})) \\
&\leq h(\alpha(x_{n-1}, Tx_{n-1}), p(Tx_{n-1}, T^2 x_{n-1})) \\
&\leq \mathcal{F}(\mu(x_{n-1}, Tx_{n-1}), \beta(p(x_{n-1}, Tx_{n-1}))p(x_{n-1}, Tx_{n-1})) \\
&\leq \mathcal{F}(1, \beta(p(x_{n-1}, Tx_{n-1}))p(x_{n-1}, Tx_{n-1})),
\end{aligned}$$

which implies that

$$\begin{aligned}
p(x_n, x_{n+1}) &\leq \beta(p(x_{n-1}, Tx_{n-1}))p(x_{n-1}, Tx_{n-1}) \\
&= \beta(p(x_{n-1}, x_n))p(x_{n-1}, x_n) \\
&< p(x_{n-1}, x_n).
\end{aligned}$$

That is

$$p(x_n, x_{n+1}) < p(x_{n-1}, x_n).$$

Following the proof of Theorem 3.3, we obtain $\{x_n\}$ is a Cauchy sequence in complete partial metric space Then there exists $x^* \in C$ such that $x_n \to x^*$ as $n \to \infty$.

By (iv), we have $p(x_{n+1}, Tx^*) = p(Tx_n, Tx^*) \to 0$ as $n \to \infty$, that is $x^* = Tx^*$. So, T has a fixed point and $Fix(T^n) = Fix(T)$ is true for $n = 1$. Let $n > 1$ and suppose on contrary, that is, there exists $z \in Fix(T^n)$ and $z \notin Fix(T)$, such that $p(z, Tz) > 0$.

Now, from (i) and (v), we have

$$\begin{aligned}
h(1, p(z, Tz)) &= h(1, p(T^n z, T^n(Tz))) \\
&= h(1, p(T(T^{n-1}z), T^2(T^{n-1}z))) \\
&\leq h(\alpha(T^{n-1}z, T^n z), p(T(T^{n-1}z), T^2(T^{n-1}z))) \\
&\leq \mathcal{F}(\mu(T^{n-1}z, T^n z), \beta(p(T^{n-1}z, T^n z))p(T^{n-1}z, T^n z)) \\
&\leq \mathcal{F}(1, \beta(p(T^{n-1}z, T^n z))p(T^{n-1}z, T^n z))
\end{aligned}$$

which implies that

$$\begin{aligned}
p(z, Tz) &\leq \beta(p(T^{n-1}z, T^n z))p(T^{n-1}z, T^n z) \\
&\leq \beta(p(T^{n-2}z, T^{n-1}z))p(T^{n-2}z, T^{n-1}z) \\
&\vdots \\
&\leq \beta(p(z, Tz))p(z, Tz) \\
&< p(z, Tz).
\end{aligned}$$

That is

$$p(z, Tz) < p(z, Tz),$$

which is a contradiction. That is, $Fix(T^n) = Fix(T)$ and so T has the property (P).   ∎

4. An Application to Second Order Differential Equations

Consider the boundary value problem for second order differentiable problem of the from

$$\begin{cases} x''(t) = -f(t, x(t)), t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $I = [0, 1]$, $f \in C(I \times \mathbb{R}, \mathbb{R})$.

In this section, we are going to apply Theorem 3.3 to study of existence and uniqueness of splution for a type of second order differential equations. Our approach is inspired by Section 5 of [17]. It is known and easy to check, that problem (4.1) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t, s) f(s, x(s)) ds, \tag{4.2}$$

for $t \in I$, where G is the green function defined by

$$G(t, s) = \begin{cases} (1-t)s & \text{if } 0 \leq s \leq t \leq 1, \\ (1-s)t & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

This is, if $x \in C^2(I, \mathbb{R})$, then x is a solution of problem (4.1) if and only if it is a solution of the integral equation (4.2) Let $X = C(I)$ be the space of all continuous functions defined on I and $\| u \|_\infty = \max_{t \in I} |u(t)|$ for each $u \in X$.

Consider the partial metric p on X given by

$$p(x, y) = \| x - y \|_\infty + \| x \|_\infty + \| y \|_\infty \quad \text{for all } x, y \in X$$

Note that p is also a partial metric on X and that

$$d_p(x, y) := 2p(x, y) - p(x, x) - p(y, y) = 2 \| x - y \|_\infty.$$

Hence, (X, p) is a complete as the metric space $(X, \| \cdot \|_\infty)$ is complete.

Theorem 4.1. *Assume the following conditions:*

  (i) there exists continuous functions $\alpha : I \to \mathbb{R}^+$ and $\beta : I \to \mathbb{R}^+$ such that

$$|f(s, a) - f(s, b)| \leq 8\alpha(s)|a - b|, \text{ for } s \in I \text{ and } a, b \in \mathbb{R},$$
$$|f(s, a)| \leq 8\beta(s)|a|, \text{ for } s \in I \text{ and } a \in \mathbb{R};$$

  (ii) $\max_{s \in I} \alpha(s) = \lambda_1 < \frac{1}{3}$ and $\max_{s \in I} \beta(s) = \lambda_2 < \frac{1}{3}$.

Then the problem (4.1) has unique solution $u \in X = C(I, \mathbb{R})$.

Proof. Define the self-map $T : X \to X$ by

$$Tx(t) = \int_0^1 G(t, s)f(s, x(s))ds,$$

for all $x \in X$ and $t \in I$. Then, the problem (4.1) is equivalent to finding a fixed point u of T in X.

Let $x, y \in X$, we have

$$
\begin{aligned}
|Tx(t) - Ty(t)| &= | \int_0^1 G(t, s)f(s, x(s))ds - \int_0^1 G(t, s)f(s, y(s))ds| \\
&\leq \int_0^1 G(t, s)|f(s, x(s)) - f(s, y(s))|ds \\
&\leq 8 \int_0^1 G(t, s)\alpha(s)|x(s) - y(s)|ds \\
&\leq 8\lambda_1 \parallel x - y \parallel_\infty \sup_{t \in I} \int_0^1 G(t, s)ds \\
&= \lambda_1 \parallel x - y \parallel_\infty .
\end{aligned}
$$

Next, we recall that for each $t \in I$ one has $\int_0^1 G(t, s)ds = \frac{t(1-t)}{2}$, and then

$$\sup_{t \in I} \int_0^1 G(t, s)ds = \frac{1}{8}.$$

Therefore,

$$\parallel Tx - Ty \parallel_\infty \leq \lambda_1 \parallel x - y \parallel_\infty . \tag{4.3}$$

Moreover, we have

$$
\begin{aligned}
|Tx(t)| &= | \int_0^1 G(t, s)f(s, x(s))ds| \\
&= \int_0^1 G(t, s)|f(s, x(s))|ds \\
&\leq 8 \int_0^1 G(t, s)\beta(s)|x(s)|ds \\
&\leq 8\lambda_2 \parallel x \parallel_\infty \sup_{t \in I} \int_0^1 G(t, s)ds \\
&\leq \lambda_2 \parallel x \parallel_\infty .
\end{aligned}
$$

Thus,

$$\parallel Tx \parallel_\infty \leq \lambda_2 \parallel x \parallel_\infty . \tag{4.4}$$

Similarly,

$$\parallel Ty \parallel_\infty \leq \lambda_2 \parallel y \parallel_\infty . \tag{4.5}$$

Assuming $\beta(p(x,y)) = (\lambda_1 + 2\lambda_2) < 1$. Using (4.3)-(4.5), we have

$$
\begin{aligned}
p(Tx, Ty) =& \parallel Tx - Ty \parallel_\infty + \parallel Tx \parallel_\infty + \parallel Ty \parallel_\infty \\
\leq& \lambda_1 \parallel x - y \parallel_\infty + \lambda_2 \parallel x \parallel_\infty + \lambda_2 \parallel y \parallel_\infty \\
\leq& (\lambda_1 + 2\lambda_2)(\parallel x - y \parallel_\infty + \parallel x \parallel_\infty + \parallel y \parallel_\infty) \\
=& \beta(p(x,y))p(x,y).
\end{aligned}
$$

Then,

$$
p(Tx, Ty) \leq \beta(p(x,y))p(x,y) \tag{4.6}
$$

Also, define $\alpha(x,y) = 1$ and $\mu(x,y) = 1$ for all $x, y \in X$. Taking the function $h, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $h(x,y) = y$ and $\mathcal{F}(s,t) = t$ for all $x, y, s, t \in \mathbb{R}^+$.

Since

$$
p(x,y) = h(1, p(x,y)) = h(\alpha(x,y), p(x,y)),
$$

and

$$
\beta(p(x,y))p(x,y) = \mathcal{F}(1, \beta(p(x,y))p(x,y)) = \mathcal{F}(\mu(x,y), \beta(p(x,y))p(x,y)).
$$

Then

$$
h(\alpha(x,y), p(x,y)) \leq \mathcal{F}(\mu(x,y), \beta(p(x,y))p(x,y)).
$$

Therefore all hypothesis of Theorem 3.3 are satisfied, and so T has a unique fixed point $u \in X$, that is, the problem (4.1) has a unique solution $u \in C^2(I)$. ∎

Acknowledgements

References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math. 3 (1922) 133–181.

[2] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc. 40 (1973) 604–608.

[3] S. Matthews, Partial metric topology, Proceedings of the 11th Summer Conference on General Topology and Applications 183–197.

[4] L. Ciric, B. Samet, H. Aydi, C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, Appl. Math. Comput. 218 (2011) 2398–2406.

[5] D. Llic, V. Pavlivic, V. Rakocevic, Some new extensions of Banach's contraction principle to partial metric space, Appl. Math. Lett. 24 (8) (2011) 1326–1330.

[6] W. Shatanaei, H. Nashine, A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space, J. Nonlinear Sci Appl. 5 (2012) 37–43.

[7] N. Hussian, E. Karapinar, P. Salimi, F. Akbar, α-admissible mappings and related fixed point theorems, J. Inequal. Appl. 114 (2013) 1–11.

[8] T. Abdeljawad, Meir-Keeler α-contractive fixed and common fixed point theorems, Fixed Point Theory Appl. (2013) https://doi.org/10.1186/1687-1812-2013-19.

[9] S. Chondok, Some fixed point theorems for α, β-admissible Geraghty type contractive mappings and related results, Math. Sci. 9 (2015) 127–135.

[10] V. Rosa, P. Vetro, Fixed point for Geraghty contractions in partial metric spaces, J. Nonlinear Sci. Appl. 7 (2014) 1–10.

[11] S. Cho, J. Bar, E. Karapinar, Fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. (2013) https://doi.org/10.1186/1687-1812-2013-329.

[12] A. Ansari, Note on α-admissible mappings and related fixed point theorems, The 2nd Regional Conference on Mathematics and Applications (2014) 373–376.

[13] A. Ansari, S. Chandok, Make unification on α-admissible mappings and related fixed point theorems, Ser A: Appl. Math. Inform. and Mech. 8 (2016) 65–71.

[14] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anals. 75 (2012) 2154–2165.

[15] M. Kutbi, M. Arshad, A. Hussain, On modified α-η-contractive mappings, Abstr. Appl. Anal. (2014) Article ID 657858.

[16] M. Abbas, B. Ali, S. Romaguera, Fixed and periodic points of generalized contractions in metric space, Fixed Point Theory Appl. (2013) https://doi.org/10.1186/1687-1812-2013-243.

[17] H. Aydi, E. Karapinar, Fixed point results for generalized α-ψ-contractions in metric-like spaces and applications, Electronic J. Diff. Equations. 133 (2015) 1–15.