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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Iterative Methods for Solving the Monotone Inclusion Problem and the Fixed Point Problem in Banach Spaces

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Abstract In this work, we propose two iterative algorithms for solving the monotone inclusion problem and the fixed point problem of a relatively nonexpansive mapping in the framework of Banach spaces. We prove the strong convergence theorems of the proposed algorithms under some suitable assumptions. Furthermore, some numerical experiments of proposed algorithms to compressed sensing in signal recovery are presented. Our results improve and generalize many recent and important results in the literature.

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1. Introduction

Let E be a real Banach space. Consider the following so-called *monotone inclusion problem*: find $x^* \in E$ such that

$$0 \in (A+B)x^*, \tag{1.1}$$

where $A : E \to E$ and $B : E \to 2^E$ are single and set-valued mappings, respectively and 0 is a zero vector in E. In particular case, when $A = 0$, then the problem (1.1) becomes the inclusion problem introduced by Rockafellar [1] and when $E = \mathbb{R}^n$, then the problem

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(1.1) becomes the generalized equation introduced by Robinson [2]. The set of solutions of the problem (1.1) is denoted by $(A+B)^{-1}0$. Many practical nonlinear problems arising in applied sciences such as in machine learning, image processing, statistical regression and linear inverse problem can be formulated as this problem (see [3–5]).

A well-known method for solving the problem (1.1) in Hilbert spaces H, is the *forward-backward algorithm* [6] which is defined by the following manner:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = J_\lambda^B(x_n - \lambda A x_n), \ \forall n \geq 1, \end{cases} \tag{1.2}$$

where $J_\lambda^B := (I + \lambda B)^{-1}$ is a resolvent of B for $\lambda > 0$. Here, I denotes the identity operator of H. It was proved that the sequence generated by (1.2) converge weakly to a point in $(A+B)^{-1}0$ under the assumption that A is α-cocoercivity, that is,

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in H$$

and λ is chosen in $(0, 2\alpha)$.

In order to get strong convergence, Takashashi et al. [7] introduced the following modified forward–backward algorithm in Hilbert spaces H:

$$\begin{cases} x_1, u \in H, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n)), \ \forall n \geq 1, \end{cases} \tag{1.3}$$

where A is an α-cocoercive mapping on H and $\{\lambda_n\} \subset (0, \infty)$. They also proved the strong convergence of the generated by (1.3) converges strongly to a point in $(A+B)^{-1}0$ under appropriate conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

López et al. [8] established a strong convergence theorem of the forward-backward algorithm (1.2) in a q-uniformly smooth and uniformly convex Banach spaces E. They introduced a modified forward-backward algorithm with errors a_n and b_n in the following way:

$$\begin{cases} x_1, u \in E, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(J_{\lambda_n}^B (x_n - \lambda_n (A x_n + a_n)) + b_n), \ \forall n \geq 1, \end{cases} \tag{1.4}$$

where $J_{\lambda_n}^B := (I + \lambda_n B)^{-1}$ is the resolvent of an m-accretive operator B, A is an α-cocoercive mapping, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$. They also proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a point in $(A+B)^{-1}0$.

In recent years, various modifications of forward-backward algorithm have been constructed and modified by many authors in several settings (see, *e.g.*, [9–16]). It can be seen that, the cocoercivity of A of most of methods is strong assumption. To avoid this strong assumption, Tseng [17] introduced the following algorithm in Hilbert spaces H, later it is known as *Tseng's splitting algorithm*:

$$\begin{cases} x_1 \in H, \\ y_n = J_{\lambda_n}^B (I - \lambda_n A) x_n, \\ x_{n+1} = y_n - \lambda_n (A y_n - A x_n), \ \forall n \geq 1, \end{cases} \tag{1.5}$$

where A is Lipschitz continuous with a constant $L > 0$. It was shown that the sequence $\{x_n\}$ generated by (1.5) converges weakly to a solution of (1.1) provided the step-size λ_n is chosen in $\left(0, \frac{1}{L}\right)$.

On the other hand, the fixed point problem is problem of finding a point $x^* \in E$ such that

$$x^* = Tx^*, \tag{1.6}$$

where $T : E \to E$ is a nonlinear mapping. The set of solutions of problem (1.6) is denoted by $F(T) = \{x \in E : x = Tx\}$. In real life, many mathematical models have been formulated as this problem.

In this paper, we study the following problem: find $x^* \in E$ such that

$$x^* \in F(T) \cap (A + B)^{-1}0. \tag{1.7}$$

Currently, there have been many authors who interested in finding a common solution of the fixed point problem (1.6) and the monotone inclusion problem (1.1) (see, *e.g.*, [16, 18–23]).

Motivated by the works in the literature, we introduce two Halpern-Tseng type for solving the monotone inclusion problem and the fixed point problem of a relatively nonexpansive mapping in the framework of Banach spaces. We prove the strong convergence results of the proposed methods under some appropriate conditions. Finally, we provide numerical experiments to compressed sensing in signal recovery. The results presented in this paper are improve and generalize many known results in this direction.

2. Preliminaries

Let E be a real Banach space with its dual space E^*. We denote $\langle x, f\rangle$ by the value of a functional f in E^* at x in E, that is, $\langle x, f\rangle = f(x)$. For a sequence $\{x_n\}$ in E, the strong convergence and the weak convergence of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The set of all real numbers is denoted by $\mathbb{R}$, while $\mathbb{N}$ stands for the set of nonnegative integers. Let S_E denote the unit sphere of E. The space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. The space E is said to be *uniformly smooth* if the limit (2.1) converges uniformly in $x, y \in S_E$. It is said to be *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. The space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where δ_E is the modulus of convexity of E defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in S_E, \|x - y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. Let $p \geq 2$. The space E is said to be *p-uniformly convex* if there is a $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Let $1 < q \leq 2$. The space E is said to be *q-uniformly smooth* if there exists a $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$, where ρ_E is the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\tfrac{\|x+ty\|+\|x-ty\|}{2} - 1 : x, y \in S_E\right\}$$

for all $t \geq 0$. Let $1 < q \leq 2 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. It is observe that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [24]). If E is uniformly convex then E is reflexive and strictly convex and if E is uniformly smooth then E is reflexive and smooth (see [25]). Moreover, we know that for every $p > 1$, L_p and ℓ_p spaces are $\min\{p, 2\}$-uniformly smooth and $\max\{p, 2\}$-uniformly convex, while Hilbert space is 2-uniformly smooth and 2-uniformly convex (see [26] for more details).

Definition 2.1. Let C be a nonempty subset of E. Recall that a mapping $A : C \to E^*$ is said to be:

(i) *cocoercive* if there exists a constant $\gamma > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \gamma\|Ax - Ay\|^2$ for all $x, y \in C$;
(ii) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$;
(iii) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$;
(iv) *hemicontinuous* if for each $x, y \in C$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Remark 2.2. It is easy to see that if A is cocoercive, then A is monotone and Lipschitz continuous but converse is not true in general.

Definition 2.3. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*.

If E is a Hilbert space, then $J = I$ is the identity mapping on E. It is known that E is smooth if and only if J is single-valued from E into E^* and if E is a reflexive, smooth and strictly convex, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E. Moreover, if E is uniformly smooth then J is norm-to-norm uniformly continuous on bounded subsets of E (see [25] for more details).

Lemma 2.4. [27, 28] *(i) Let E be a 2-uniformly smooth Banach space. Then there exists a constant $\kappa > 0$ such that*

$$\|x - y\|^2 \leq \|x\|^2 - 2\langle y, Jx\rangle + \kappa\|y\|^2,\ \forall x, y \in E.$$

(ii) Let E be a 2-uniformly convex Banach space. Then there exists a constant $c > 0$ such that

$$\|x - y\|^2 \geq \|x\|^2 - 2\langle y, Jx\rangle + c\|y\|^2,\ \forall x, y \in E.$$

Remark 2.5. It is well-known that $\kappa = c = 1$ whenever E is a Hilbert space. Moreover, we refer to [28] for the exact values of the constants κ and c.

Next, we recall the following Lyapunov function which introduced in [29]:

Definition 2.6. Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall x, y \in E.$$

In the particular case in which E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2,\ \forall x, y \in E$$

and

$$\phi(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz) \leq \alpha\phi(x, y) + (1 - \alpha)\phi(x, z),\ \forall x, y, z \in E,\ \alpha \in [0, 1]. \quad (2.2)$$

In addition, the function ϕ satisfies the following *three point identity*:

$$\phi(x,y) = \phi(x,z) - \phi(y,z) + 2\langle y-x, Jy - Jz\rangle, \ \forall x,y,z \in E.$$

Lemma 2.7. [30] *Let E be a 2-uniformly convex Banach space. Then there exists a constant $c > 0$ such that*

$$c\|x-y\|^2 \leq \phi(x,y), \ \forall x,y \in E,$$

where c is the constant in Lemma 2.4 (ii).

Lemma 2.8. [31] *Let E be a uniformly convex Banach space. Then there exists a continuous strictly increasing convex function $g : [0,2r) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz) \leq \alpha\phi(x,y) + (1-\alpha)\phi(x,z) - \alpha(1-\alpha)g(\|Jy - Jz\|)$$

for all $\alpha \in [0,1]$, $x \in E$ and $y, z \in B_r := \{\omega : \|\omega\| \leq r\}$ for some $r > 0$.

The following important fact can be found in [32]. For two sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E. Then

$$\|x_n - y_n\| \to 0 \ \Leftrightarrow \ \|Jx_n - Jy_n\| \to 0 \ \Leftrightarrow \ \phi(x_n, y_n) \to 0. \tag{2.3}$$

Let C be a nonempty subset of a smooth Banach space E. A point $p \in C$ is a fixed point of T if $p = Tp$ and we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *relatively nonexpansive* if it satisfies the following conditions:

(i) $F(T) \neq \emptyset$;
(ii) $\phi(p, Tx) \leq \phi(p,x)$ for all $p \in F(T)$ and $x \in C$;
(iii) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, it follows that $p \in F(T)$.

Remark 2.9. If T satisfies (i) and (ii), then T is called *relatively quasi-nonexpansive*. In a Hilbert space H, we know that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$. Hence, if $T : C \to C$ is relatively quasi-nonexpansive, then it is quasi-nonexpansive, that is, $\|Tx - p\| \leq \|x-p\|$ for all $p \in F(T)$ and $x \in C$.

Lemma 2.10. [33] *Let E be a strictly convex and smooth Banach space. Let C be a closed and convex subset of E. If $T : C \to C$ be a relatively nonexpansive mapping, then $F(T)$ is closed and convex.*

We make use of the following mapping $V : E \times E^* \to \mathbb{R}$ studied in [29]:

$$V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \ \forall x \in E, \ x^* \in E^*.$$

Obviously, $V(x,x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.11. [29] *Let E be a reflexive, strictly convex and smooth Banach space. Then*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*), \ \forall x \in E, \ x^*, y^* \in E^*.$$

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty, closed convex subset of E. Then we know that for any $x \in E$, there exists a unique point $z \in C$ such that

$$\phi(z,x) = \min_{y \in C} \phi(y,x).$$

Such a mapping $\Pi_C : E \to C$ defined by $z = \Pi_C(x)$ is called the *generalized projection*. If E is a Hilbert space, then Π_C is coincident with the metric projection denoted by P_C.

Lemma 2.12. [29] *Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty, closed, and convex subset of E. For each $x \in E$ and $z \in C$. Then the following statements hold:*

(i) $z = \Pi_C(x)$ *if and only if* $\langle y - z, Jx - Jz\rangle \leq 0$, $\forall y \in C$.
(ii) $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x)$, $\forall y \in C$.

Let $B : E \to 2^{E^*}$ be a multi-valued mapping. The effective domain of B is denoted by $D(B) = \{x \in E : Bx \neq \emptyset\}$ and the range of B is also denoted by $R(B) = \bigcup\{Bx : x \in D(B)\}$. The set of zeros of B is denoted by $B^{-1}0 = \{x \in D(B) : 0 \in Bx\}$. A multi-valued mapping B from E into E^* is said to be *monotone* if

$$\langle x - y, u - v\rangle \geq 0,\ \forall x, y \in D(B),\ u \in Bx \text{ and } v \in By.$$

A monotone operator B on E is said to be *maximal* if its graph $G(B) = \{(x, y) \in E \times E^* : x \in D(B), y \in Bx\}$ is not properly contained in the graph of any other monotone operator on E. In other words, the maximality of B is equivalent to $R(J + \lambda B) = E^*$ for $\lambda > 0$ (see [34, Theorem 1.2]). It is known that if B is maximal monotone, then $B^{-1}0$ is closed and convex (see [35]). For a maximal monotone operator B, we define the resolvent of B by $J_\lambda^B(x) = (J + \lambda B)^{-1}Jx$ for $x \in E$ and $\lambda > 0$. It is also known that $B^{-1}0 = F(J_\lambda^B)$.

Lemma 2.13. [34] *Let E be a reflexive Banach space. Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded mapping. Let $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A + B$ is a maximal monotone mapping.*

Lemma 2.14. *Let E be a reflexive, strictly convex and smooth Banach space. Let $A : E \to E^*$ be a mapping and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then the following statements hold:*

(i) *Define a mapping* $T_\lambda x := J_\lambda^B \circ J^{-1}(J - \lambda A)x$ *for* $x \in E$ *and* $\lambda > 0$, *then* $F(T_\lambda) = (A + B)^{-1}0$.
(ii) $(A + B)^{-1}0$ *is closed and convex.*

Proof. (i) Let $x \in E$ and $\lambda > 0$. We see that

$$\begin{aligned} x = T_\lambda x \quad &\Leftrightarrow \quad x = J_\lambda^B \circ J^{-1}(J - \lambda A)x \\ &\Leftrightarrow \quad x = (J + \lambda B)^{-1}J \circ J^{-1}(J - \lambda A)x \\ &\Leftrightarrow \quad Jx - \lambda Ax \in Jx + \lambda Bx \\ &\Leftrightarrow \quad 0 \in (A + B)x \\ &\Leftrightarrow \quad x \in (A + B)^{-1}0. \end{aligned}$$

Hence $F(T_\lambda) = (A + B)^{-1}0$.

(ii) By Lemma 2.13, we know that $A + B$ is maximal monotone, then we can show that the set $(A + B)^{-1}0 = \{x \in E : 0 \in (A + B)x\}$ is closed and convex. ∎

Lemma 2.15. [36] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence of real numbers such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.16. [37] *Let* $\{a_n\}$ *be sequences of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists an increasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

In fact, $m_k := \max\{j \le k : a_j \le a_{j+1}\}$.

3. Main Results

In this section, we introduce two Halpern-Tseng type for finding a common solution of the monotone inclusion problem and the fixed point problem in Banach spaces. From now on, let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone operator. Let $T : E \to E$ be a relatively nonexpansive mapping. Assume that $\Omega := F(T) \cap (A+B)^{-1}0 \neq \emptyset$. To prove the strong convergence results, we also need to assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, such that $\{\beta_n\} \subset [a,b] \subset (0,1)$ for some $a, b > 0$ and $\lim_{n\to\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 1 Halpern-Tseng type algorithm

Step 0. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J_{\lambda_n}^B J^{-1}(Jx_n - \lambda_n A x_n). \tag{3.1}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.2}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n J u + (1-\alpha_n)(\beta_n J z_n + (1-\beta_n) JTz_n)). \tag{3.3}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 3.1. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3. Then*

$$\phi(p, z_n) \le \phi(p, x_n) - \Big(1 - \frac{\kappa \lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n), \ \forall p \in (A+B)^{-1}0,$$

where c *and* κ *are the constants in Lemma 2.4.*

Proof. Let $p \in (A+B)^{-1}0$. By Lemma 2.4 (ii), we have

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n))) \\
&= V(p, Jy_n - \lambda_n(Ay_n - Ax_n)) \\
&= \|p\|^2 - 2\langle p, Jy_n - \lambda_n(Ay_n - Ax_n)\rangle + \|Jy_n - \lambda_n(Ay_n - Ax_n)\|^2 \\
&\leq \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda_n\langle p, Ay_n - Ax_n\rangle + \|Jy_n\|^2 - 2\lambda_n\langle y_n, Ay_n - Ax_n\rangle \\
&\quad +\kappa\|\lambda_n(Ay_n - Ax_n)\|^2 \\
&= \|p\|^2 - 2\langle p, Jy_n\rangle + \|y_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, y_n) - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) + \phi(x_n, y_n) + 2\langle x_n - p, Jy_n - Jx_n\rangle - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle \\
&\quad +\kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) + \phi(x_n, y_n) - 2\langle y_n - x_n, Jy_n - Jx_n\rangle + 2\langle y_n - p, Jy_n - Jx_n\rangle \\
&\quad -2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) - \phi(y_n, x_n) + 2\langle y_n - p, Jy_n - Jx_n\rangle - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle \\
&\quad +\kappa\lambda_n^2\|Ay_n - Ax_n\|^2 + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) - \phi(y_n, x_n) + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&\quad -2\langle y_n - p, Jx_n - Jy_n + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.4)
\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}
\phi(p, z_n) &\leq \phi(p, x_n) - \Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n) + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&\quad -2\langle y_n - p, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.5)
\end{aligned}$$

We now show that

$$\langle y_n - p, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0.$$

From the definition of $\{y_n\}$, we note that $Jx_n - \lambda_n Ax_n \in Jy_n + \lambda_n By_n$. Since B is maximal monotone, there exists $v_n \in By_n$ such that $Jx_n - \lambda_n Ax_n = Jy_n + \lambda_n v_n$, it follows that

$$v_n = \frac{1}{\lambda_n}\big(Jx_n - Jy_n - \lambda_n Ax_n\big). \qquad (3.6)$$

Since $0 \in (A+B)p$ and $Ay_n + v_n \in (A+B)y_n$, it follows from Lemma 2.13 that $A+B$ is maximal monotone. Hence

$$\langle y_n - p, Ay_n + v_n\rangle \geq 0. \qquad (3.7)$$

Substituting (3.6) into (3.7), we have

$$\frac{1}{\lambda_n}\langle y_n - p, Jx_n - Jy_n - \lambda_n Ax_n + \lambda_n Ay_n\rangle \geq 0,$$

which implies that

$$\langle y_n - p, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0. \qquad (3.8)$$

Combining (3.5) and (3.8), we have

$$\phi(p, z_n) \leq \phi(p, x_n) - \Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n). \qquad (3.9)$$

∎

Theorem 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Suppose that $\{\lambda_n\}$ be a sequence in $\big(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\big)$ such that $\{\lambda_n\} \subset [a', b'] \subset \big(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\big)$ for some $a', b' > 0$. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = \Pi_\Omega(u)$.*

Proof. We first show that $\{x_n\}$ is bounded. Let $z \in \Omega$. Since $\lambda_n \in \big(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\big)$, we have $1 - \frac{\kappa\lambda_n^2 L^2}{c} > 0$. This implies by Lemma 3.1 that

$$\phi(z, z_n) \leq \phi(z, x_n). \tag{3.10}$$

Put $w_n = J^{-1}(\beta_n J z_n + (1-\beta_n)JTz_n)$ for all $n \in \mathbb{N}$. Thus by (2.2) and (3.10), we have

$$\begin{aligned} \phi(z, w_n) &\leq \beta_n \phi(z, z_n) + (1-\beta_n)\phi(z, Tz_n) \\ &\leq \beta_n \phi(z, z_n) + (1-\beta_n)\phi(z, z_n) \\ &\leq \phi(z, x_n). \end{aligned} \tag{3.11}$$

Using (3.11), we obtain

$$\begin{aligned} \phi(z, x_{n+1}) &\leq \alpha_n \phi(z, u) + (1-\alpha_n)\phi(z, w_n) \\ &\leq \alpha_n \phi(z, u) + (1-\alpha_n)\phi(z, x_n) \\ &\leq \max\{\phi(z, u), \phi(z, x_n)\} \\ &\vdots \\ &\leq \max\{\phi(z, u), \phi(z, x_1)\}. \end{aligned}$$

This implies that $\{\phi(z, x_n)\}$ is bounded. Applying Lemma 2.7, we have $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Let $x^* = \Pi_\Omega(u)$. From Lemma 2.8 and (3.9), we have

$$\begin{aligned} \phi(x^*, w_n) &\leq \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, Tz_n) - \beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|) \\ &\leq \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, z_n) - \beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|) \\ &\leq \beta_n \phi(x^*, z_n) + (1-\beta_n)\Big\{\phi(x^*, x_n) - \Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n)\Big\} \\ &\quad -\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|) \\ &\leq \phi(x^*, x_n) - (1-\beta_n)\Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n) \\ &\quad -\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|). \end{aligned} \tag{3.12}$$

Then we have

$$\begin{aligned} &\phi(x^*, x_{n+1}) \\ \leq\ & \alpha_n \phi(x^*, u) + (1-\alpha_n)\Big\{\phi(x^*, x_n) - (1-\beta_n)\Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n) \\ & -\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|)\Big\} \\ =\ & \alpha_n \phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n) - (1-\alpha_n)(1-\beta_n)\Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n) \\ & -(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|). \end{aligned}$$

This implies that

$$\begin{aligned}&(1-\alpha_n)(1-\beta_n)\Big(1-\frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n,x_n)+(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|)\\ \le\;& \phi(x^*,x_n)-\phi(x^*,x_{n+1})+\alpha_nK, \qquad (3.13)\end{aligned}$$

where $K=\sup_{n\in\mathbb{N}}\{|\phi(x^*,u)-\phi(x^*,x_n)|\}$.

The rest of the proof will be divided into two cases:

Case 1. Suppose that there exists $N\in\mathbb{N}$ such that $\phi(x^*,x_{n+1})\le\phi(x^*,x_n)$ for all $n\ge N$. This implies that $\lim_{n\to\infty}\phi(x^*,x_n)$ exists. By our assumptions, we have from (3.13) that

$$\lim_{n\to\infty}\phi(y_n,x_n)=0 \text{ and } \lim_{n\to\infty}g(\|Jz_n-JTz_n\|)=0. \qquad (3.14)$$

Consequently,

$$\lim_{n\to\infty}\|x_n-y_n\|=0 \text{ and } \lim_{n\to\infty}\|Jz_n-JTz_n\|=0. \qquad (3.15)$$

Moreover, we also have

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0. \qquad (3.16)$$

Since A is Lipschitz continuous, we have

$$\lim_{n\to\infty}\|Ax_n-Ay_n\|=0 \qquad (3.17)$$

and hence

$$\begin{aligned}\|Jz_n-Jy_n\| &= \lambda_n\|Ax_n-Ay_n\|\\ &\to 0. \qquad (3.18)\end{aligned}$$

Combining (3.16) and (3.18), we obtain

$$\begin{aligned}\|Jx_n-Jz_n\| &\le \|Jx_n-Jy_n\|+\|Jy_n-Jz_n\|\\ &\to 0. \qquad (3.19)\end{aligned}$$

Moreover from (3.15) and (3.19), we obtain

$$\begin{aligned}\|Jx_{n+1}-Jx_n\| &\le \|Jx_{n+1}-Jw_n\|+\|Jw_n-Jz_n\|+\|Jz_n-Jx_n\|\\ &= \alpha_n\|Ju-Jw_n\|+(1-\beta_n)\|JTz_n-Jz_n\|+\|Jz_n-Jx_n\|\\ &\to 0. \qquad (3.20)\end{aligned}$$

Then we have from (3.19) and (3.20) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0 \qquad (3.21)$$

and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.22)$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\hat{x}\in E$ and

$$\limsup_{n\to\infty}\langle x_n-x^*,Ju-Jx^*\rangle=\lim_{k\to\infty}\langle x_{n_k}-x^*,Ju-Jx^*\rangle.$$

From (3.21), we also have $z_{n_k} \rightharpoonup \hat{x}$. Since $\|z_n - Tz_n\| \to 0$ and $I - T$ is demi-closed at zero, we have $\hat{x} \in F(T)$. We next show that $\hat{x} \in (A+B)^{-1}0$. Let $(v, w) \in G(A+B)$, we have $w - Av \in Bv$. Since

$$(J - \lambda_{n_k} A)x_{n_k} \in (J + \lambda_{n_k} B)y_{n_k}.$$

It follows that

$$\frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \in By_{n_k}.$$

Since B is maximal monotone, we have

$$\left\langle v - y_{n_k}, w - Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \right\rangle \geq 0$$

Using the monotonicity of A, we have

$$\begin{aligned} \langle v - y_{n_k}, w \rangle &\geq \left\langle v - y_{n_k}, Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \right\rangle \\ &= \langle v - y_{n_k}, Av - Ax_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle \\ &= \langle v - y_{n_k}, Av - Ay_{n_k} \rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle \\ &\quad + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle \\ &\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle. \end{aligned}$$

Since $y_{n_k} \rightharpoonup \hat{x}$, it follows from (3.16) and (3.17) that

$$\langle v - \hat{x}, w \rangle \geq 0.$$

By the monotonicity of $A + B$, we get $0 \in (A+B)\hat{x}$, that is, $\hat{x} \in (A+B)^{-1}0$. So $\hat{x} \in \Omega := F(T) \cap (A+B)^{-1}0$. Thus we have

$$\limsup_{n\to\infty} \langle x_n - x^*, Ju - Jx^* \rangle = \langle \hat{x} - x^*, Ju - Jx^* \rangle \leq 0.$$

From (3.22), we also have

$$\limsup_{n\to\infty} \langle x_{n+1} - x^*, Ju - Jx^* \rangle \leq 0. \tag{3.23}$$

Finally, we show that $x_n \to x^*$. By Lemma 2.11, we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n)) \\ &= V(x^*, \alpha_n Ju + (1-\alpha_n)Jw_n) \\ &\leq V(x^*, \alpha_n Ju + (1-\alpha_n)Jw_n - \alpha_n(Ju - Jx^*)) \\ &\quad + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\ &= V(x^*, \alpha_n Jx^* + (1-\alpha_n)Jw_n) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\ &= \phi(x^*, J^{-1}(\alpha_n Jx^* + (1-\alpha_n)Jw_n)) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\ &\leq \alpha_n \phi(x^*, x^*) + (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\ &\leq (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle. \end{aligned} \tag{3.24}$$

This together with (3.23) and (3.24), so we can conclude by Lemma 2.15 that $\phi(x^*, x_n) \to 0$. Therefore, $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{\phi(x^*, x_{n_i})\}$ of $\{\phi(x^*, x_n)\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. By Lemma 2.16, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1}) \tag{3.25}$$

and

$$\phi(x^*, x_k) \leq \phi(x^*, x_{m_k}). \tag{3.26}$$

As proved in the **Case 1**, we obtain

$$\begin{aligned} &(1-\alpha_{m_k})(1-\beta_{m_k})\Big(1-\frac{\kappa\lambda_{m_k}^2 L^2}{c}\Big)\phi(y_{m_k}, x_{m_k}) \\ &+(1-\alpha_{m_k})\beta_{m_k}(1-\beta_{m_k})g(\|Jz_{m_k} - JTz_{m_k}\|) \\ \leq\ & \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) + \alpha_{m_k}K \\ \leq\ & \alpha_{m_k}K, \end{aligned}$$

where $K = \sup_{k\in\mathbb{N}}\{|\phi(x^*, u) - \phi(x^*, x_{m_k})|\}$. By our assumptions, we have

$$\lim_{k\to\infty} \phi(y_{m_k}, x_{m_k}) = 0 \text{ and } \lim_{k\to\infty} g(\|Jz_{m_k} - JTz_{m_k}\|) = 0.$$

Consequently,

$$\lim_{k\to\infty} \|x_{m_k} - y_{m_k}\| = 0 \text{ and } \lim_{k\to\infty} \|Jz_{m_k} - JTz_{m_k}\| = 0.$$

Using the same arguments as in the proof of **Case 1**, we can show that

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty} \langle x_{m_k+1} - x^*, Ju - Jx^*\rangle \leq 0.$$

From (3.24) and (3.25), we have

$$\begin{aligned} \phi(x^*, x_{m_k+1}) &\leq (1-\alpha_{m_k})\phi(x^*, x_{m_k}) + \alpha_{m_k}\langle x_{m_k+1} - x^*, Ju - Jx^*\rangle \\ &\leq (1-\alpha_{m_k})\phi(x^*, x_{m_k+1}) + \alpha_{m_k}\langle x_{m_k+1} - x^*, Ju - Jx^*\rangle. \end{aligned}$$

This implies that

$$\phi(x^*, x_{m_k+1}) \leq \langle x_{m_k+1} - x^*, Ju - Jx^*\rangle.$$

Then we have

$$\limsup_{k\to\infty} \phi(x^*, x_{m_k+1}) \leq 0. \tag{3.27}$$

Combining (3.26) and (3.27) we obtain

$$\limsup_{k\to\infty} \phi(x^*, x_k) \leq 0.$$

Hence $\limsup_{k\to\infty} \phi(x^*, x_k) = 0$ and so $x_k \to x^*$. This completes the proof. ∎

If we take $T = I$ in Theorem 3.2, then we obtain the following result regarding the monotone quasi-inclusion problem (1.1).

Corollary 3.3. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Assume that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, u \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(Jy_n - \lambda_n(Ay_n - Ax_n))), \ \forall n \geq 1, \end{cases} \tag{3.28}$$

where $\{\lambda_n\}$ is a sequence in $\left(0, \frac{1}{L}\right)$ such that $\{\lambda_n\} \subset [a', b'] \subset \left(0, \frac{1}{L}\right)$ for some $a', b' > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.28) converges strongly to $x^* \in (A+B)^{-1}0$, where $x^* = \Pi_{(A+B)^{-1}0}(u)$.

We next propose a strong convergence theorem of another modification of Tseng's splitting algorithm with line search for solving the monotone inclusion problem and the fixed point problem in Banach spaces. It is noted that this proposed algorithm does not required to know the Lipschitz constant of the Lipschitz continuous mapping.

Algorithm 2 Halpern-Tseng type algorithm with Armijo-type line search

Step 0. Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in \left(0, \sqrt{\frac{c}{\kappa}}\right)$. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \tag{3.29}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{3.30}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.31}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(\beta_n Jz_n + (1-\beta_n)JTz_n)). \tag{3.32}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 3.4. *The Armijo line search rule defined by (3.30) is well defined and*

$$\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma.$$

Proof. Since A is L-Lipschitz continuous on E, we have

$$\|Ax_n - A(J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n))\| \leq L\|x_n - J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n)\|.$$

Using the fact that $L > 0$ and $\mu > 0$, we get

$$\frac{\mu}{L}\|Ax_n - A(J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n))\| \leq \mu\|x_n - J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n)\|.$$

This implies that (3.30) holds for all $\gamma l^{m_n} \leq \frac{\mu}{L}$ and so λ_n is well defined. Obviously, $\lambda_n \leq \gamma$. If $\lambda_n = \gamma$, then the lemma is proved. Otherwise, if $\lambda_n < \gamma$, then we have from

(3.30) that

$$\|Ax_n - A(J^B_{\frac{\lambda_n}{l}} J^{-1}(Jx_n - \frac{\lambda_n}{l}Ax_n))\| \quad > \quad \frac{\mu}{\frac{\lambda_n}{l}}\|x_n - J^B_{\frac{\lambda_n}{l}} J^{-1}(Jx_n - \frac{\lambda_n}{l}Ax_n)\|.$$

Again by the L-Lipschitz continuity of A, we obtain $\lambda_n > \frac{\mu l}{L}$. This completes the proof. ■

Lemma 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then*

$$\phi(p, z_n) \leq \phi(p, x_n) - \Big(1 - \frac{\kappa\mu^2}{c}\Big)\phi(y_n, x_n), \ \forall p \in (A+B)^{-1}0,$$

where c and κ are the constants in Lemma 2.4.

Proof. From (3.30), we see that $\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_n}\|x_n - y_n\|$. By using the same arguments as in the proof of Lemma 3.1, we can show that this lemma holds. ■

Theorem 3.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$.*

Proof. By using the same arguments as in the proof of Theorem 3.2, we immediately obtain the proof. ■

If we take $T = I$ in Theorem 3.6, then we obtain the following result regarding the monotone quasi-inclusion problem (1.1).

Corollary 3.7. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone operator. Assume that $(A+B)^{-1}0 \neq \emptyset$. Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in \big(0, \sqrt{\frac{c}{\kappa}}\big)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, u \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(Jy_n - \lambda_n(Ay_n - Ax_n))), \ \forall n \geq 1, \end{cases} \tag{3.33}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n\|Ax_n - Ay_n\| \leq \mu\|x_n - y_n\|.$$

Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.33) converges strongly to $x^* \in (A+B)^{-1}0$, where $x^* = \Pi_{(A+B)^{-1}0}(u)$.

4. Numerical Experiments

In this section, we provide numerical experiments to the signal recovery in compressed sensing by using our proposed algorithms. Moreover, we also compare the mentioned algorithms with Tseng's splitting algorithm (1.5). In signal recovery, compressed sensing can be modeled as the following under determinated linear equation system:

$$y = Cx + \varepsilon \tag{4.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $C : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear observation operator. It is known that to solve (4.1) can be seen as solving the LASSO problem [5]:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Cx-y\|_2^2+\lambda\|x\|_1, \tag{4.2}$$

where $\lambda > 0$. In this case, we set $A = \nabla f$ the gradient of f, where $f(x) = \frac{1}{2}\|Cx-y\|_2^2$ and $B = \partial g$ the subdifferential of g, where $g(x) = \lambda\|x\|_1$. Then the LASSO problem (4.2) can be considered as the monotone quasi-inclusion problem (1.1). It is known that $\nabla f(x) = C^t(Cx - y)$ and it is $\|C\|^2$-Lipschitz continuous and monotone (see [3]). Moreover, ∂g is maximal monotone (see [1]).

In this experiment, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $C \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal-to-noise ratio (SNR)=40. The restoration accuracy is measured by the mean squared error (MSE) as follows:

$$E_n = \frac{1}{N}\|x_n - x\|_2^2 < 10^{-5}, \tag{4.3}$$

where x_n is an estimated signal of x. In our numerical test, we compare our Algorithm 3 and Algorithm 3 $(T = I)$ with Tseng's splitting algorithm (1.5).

We take $\alpha_n = \frac{1}{15(n+5)}$ and $\lambda_n = \frac{0.3}{\|C\|^2}$ in Algorithm 3 and take $\lambda_n = \frac{0.3}{\|C\|^2}$ in Tseng's splitting algorithm (1.5). For Alogorithm 3, we take $\alpha_n = \frac{1}{15(n+5)}$, $\gamma = 5$, $\mu = 0.5$, $l = 0.3$. The point u is chosen to be $(1, 1, 1, \dots, 1) \in \mathbb{R}^N$ and the starting point x_1 is randomly generated in $\mathbb{R}^N$. We perform the numerical test with the following four cases:

Case 1: $N = 512$, $M = 256$ and $m = 10$;
Case 2: $N = 1024$, $M = 512$ and $m = 30$;
Case 3: $N = 2048$, $M = 1024$ and $m = 60$;
Case 4: $N = 4096$, $M = 2048$ and $m = 100$.

The numerical results are reported as follows:

Table 1. The comparison of the proposed algorithms with Tseng's splitting algorithm

		Algorithm 3	Algorithm 3	Tseng's splitting algorithm
Case 1	No. of Iter.	1,850	4,864	5,689
Case 2	No. of Iter.	3,320	10,186	12,753
Case 3	No. of Iter.	7,126	19,076	24,666
Case 4	No. of Iter.	14,889	40,743	48,652

We next demonstrate the graphs of original signal and recovered signal by Algorithm 3, Algorithm 3 and Tseng's splitting algorithm. The number of iterations are reported in the Figures 1-8, respectively.

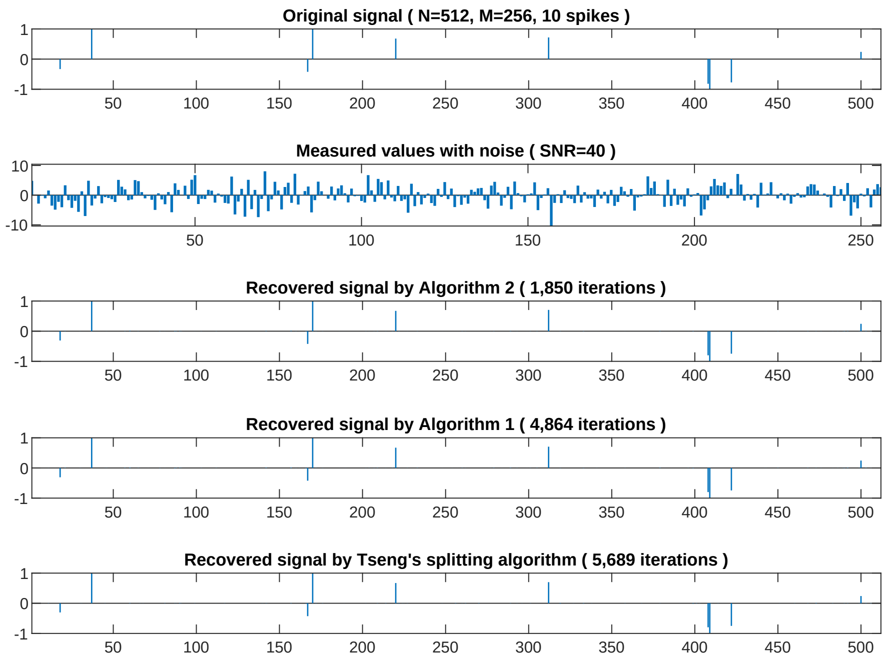

Figure 1: The comparison of recovered signal by using different algorithms in **Case 1**.

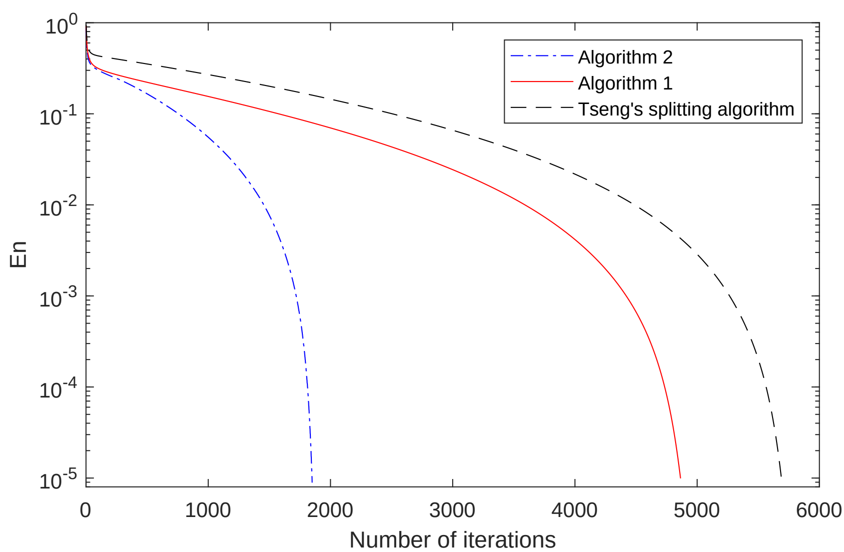

Figure 2: The plotting of MSE versus number of iterations in **Case 1**.

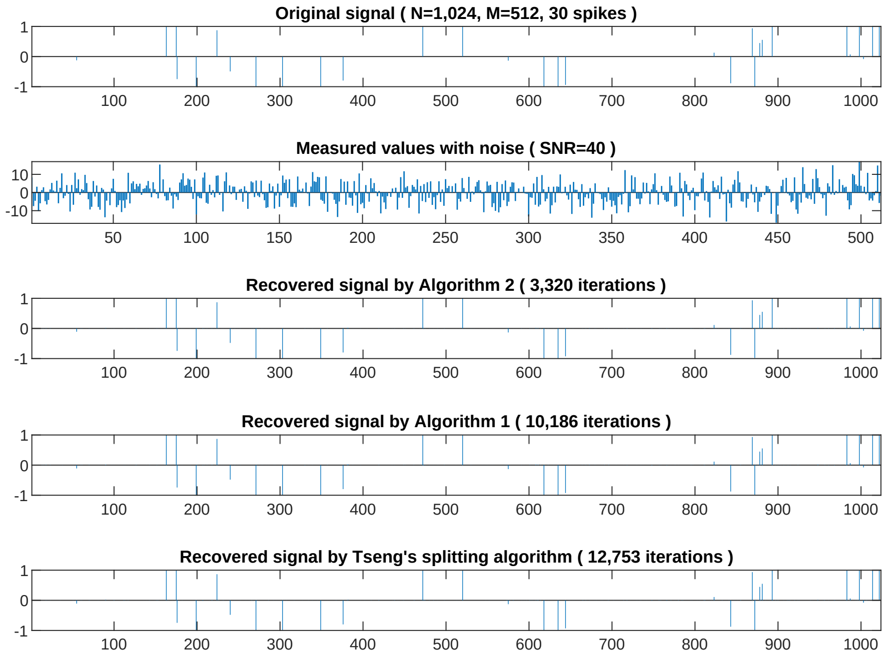

Figure 3: The comparison of recovered signal by using different algorithms in **Case 2**.

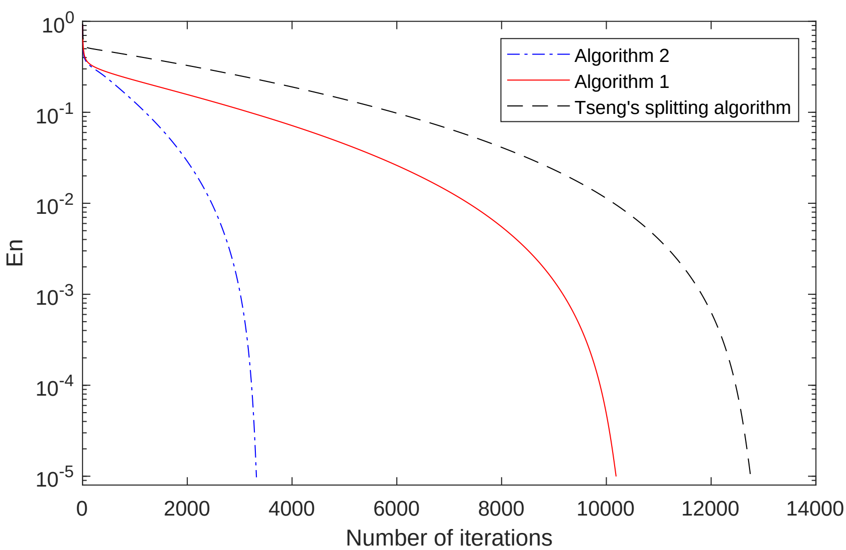

Figure 4: The plotting of MSE versus number of iterations in **Case 2**.

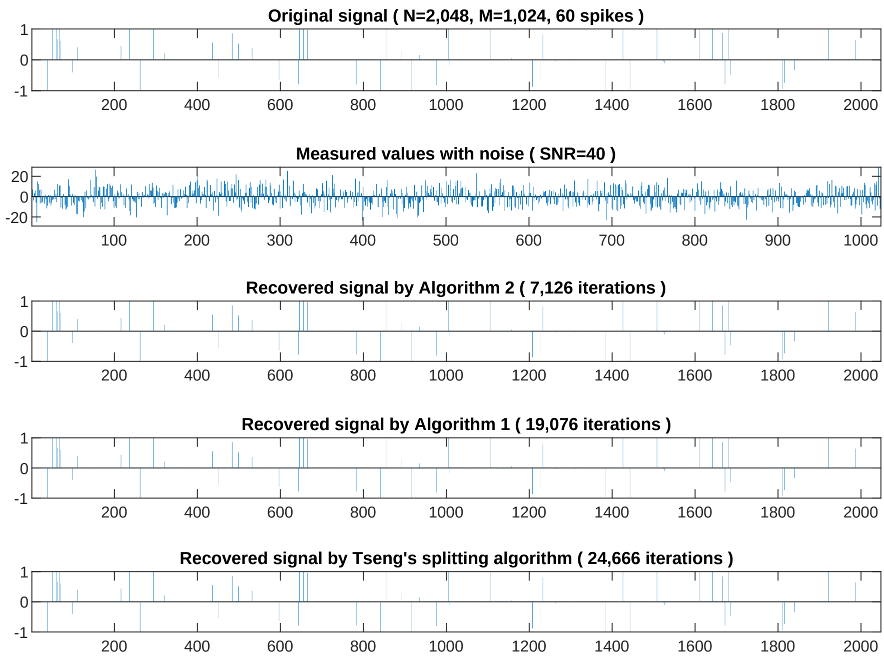

Figure 5: The comparison of recovered signal by using different algorithms in **Case 3**.

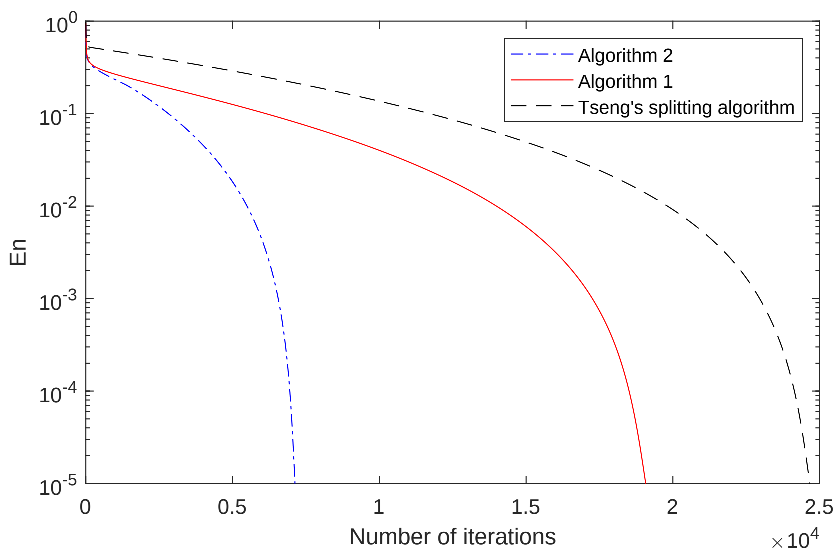

Figure 6: The plotting of MSE versus number of iterations in **Case 3**.

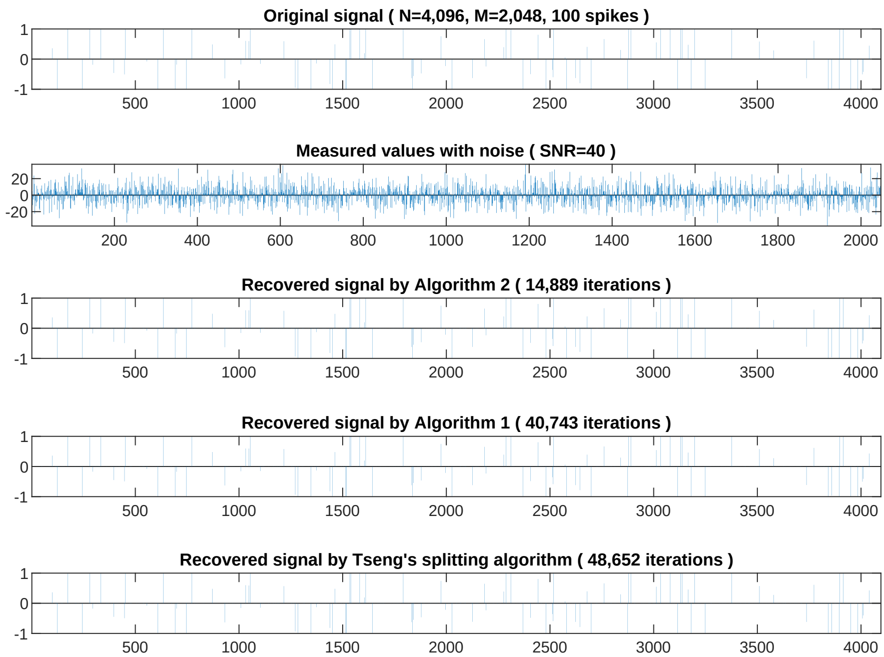

Figure7: The comparison of recovered signal by using different algorithms in **Case 4**.

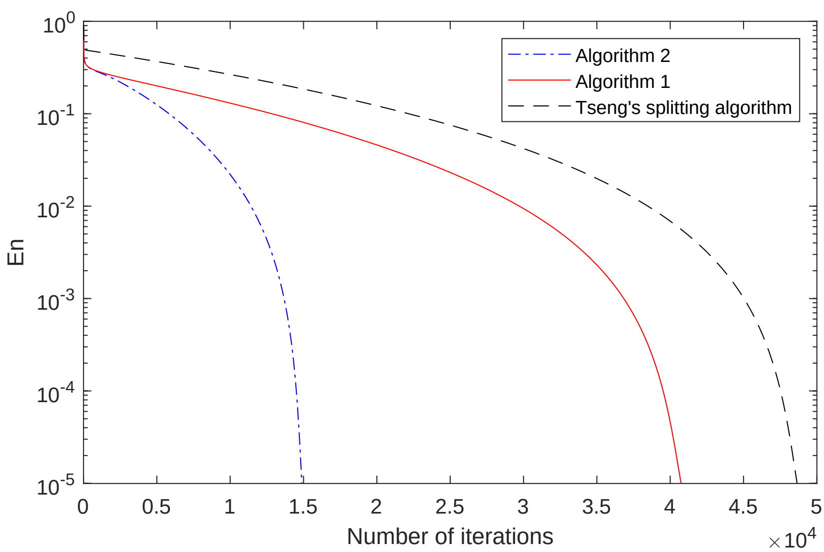

Figure 8: The plotting of MSE versus number of iterations in **Case 4**.

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