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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Adjacent-Vertex-Distinguishing-Total Choice Numbers

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Abstract Let ϕ be a proper total coloring of a graph G. Let $C(v) = \{\phi(v)\} \cup \{\phi(uv)|uv \in E(G)\}$ denote the set of colors assigned to a vertex v and those edges incident to v. If we have $C(u) \neq C(v)$ whenever $uv \in E(G)$, then ϕ is called an adjacent-vertex-distinguishing-total coloring or avd-total coloring. Let ${\chi''}_a(G)$ be the smallest integer k for which G has an avd-total coloring with k colors. In 2008, Wang and Wang [W. Wang, Y. Wang, Adjacent vertex distinguishing total colorings of outerplanar graphs, J. Comb. Optim. 19 (2010) 123–133] obtained many results about $\chi''_a(G)$ depending on the value of the maximum average degree.

A k-assignment L of G is a list assignment L with $|L(v)| = k$ for each vertex v and $|L(e)| = k$ for each edge e. A total-L-coloring is a proper total coloring ϕ of G such that $\phi(v) \in L(v)$ whenever $v \in V(G)$ and $\phi(e) \in L(e)$ whenever $e \in E(G)$. If G has a total-L-coloring such that $C(u) \neq C(v)$ for all $uv \in E(G)$, then G has an avd-total-L-coloring. Let $Ch''_a(G)$ be the smallest integer k such that G has an avd-total-L-coloring for every k-assignment L. In this paper, we strengthen results of Wang and Wang by giving analogous results for $Ch''_a(G)$.

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1. Introduction

In this paper only the simple, finite, and undirected graphs are examined. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. A *proper total coloring* ϕ is a mapping from $V(G) \cup E(G)$ to a set of colors such that any two adjacent vertices, any two adjacent edges, and any vertex and its incident edge receive different colors. Let $C(v) = \{\phi(v)\} \cup \{\phi(uv)|uv \in E(G)\}$ denote the set of colors assigned to a vertex v and those edges incident to v. A proper total coloring ϕ of G is an *adjacent-vertex-distinguishing-total coloring* (*avd-total coloring*), if $C(u) \neq C(v)$ whenever $uv \in E(G)$.

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The smallest integer k such that G has an avd-total coloring with k colors is called the *adjacent-vertex-distinguishing-total chromatic number*, denoted by $\chi''_a(G)$.

This coloring is related to a vertex-distinguishing proper edge coloring (a proper edge coloring with $C(u) \neq C(v)$ for each pair of distinct vertices u and v) which was discussed by Balister et al. [1], Bazgan [2], and Burris and Schelp [3]. In 2002, Zhang et al. [4] studied an adjacent vertex distinguishing proper edge coloring (a proper edge coloring with $C(u) \neq C(v)$ for each pair of adjacent vertices u and v). In 2005, avd-total coloring of graphs was introduced by Zhang et al. [5]. They obtained $\chi''_a(G)$ for graphs in many basic families such as paths, cycles, trees, wheels, stars, fans, complete graphs, and complete bipartite graphs. Additionally, they posed the following conjecture.

Conjecture 1.1. [5] If G is a graph with order at least two, then $\chi''_a(G) \leq \Delta(G) + 3$.

Subsequently, Wang [6] and Chen [7] independently verified the conjecture for the case $\Delta(G) = 3$. In 2009, Hulgan [8] presented a more concise proof for this result. Moreover, he also provided short proofs for the exact value of $\chi''_a(G)$ of complete graphs and cycles. In 2010, Wang and Wang [9] studied outerplanar graphs with $\Delta(G) \geq 3$ and proved that $\Delta(G) + 1 \leq \chi''_a(G) \leq \Delta(G) + 2$, whereas $\chi''_a(G) = \Delta(G) + 2$ if and only if G has two adjacent vertices of maximum degree. In 2014, Wang and Huang [10] extended the results to planar graphs. In 2015, Luiz et al. [11] verified the conjecture for complete equipartite graphs. Coker and Johannson [12] used a probabilistic approach to show that $\chi''_a(G) \leq \Delta(G) + c$ for some constant $c > 0$. Pedrotti and De Mello [13] confirmed the conjecture for indifference graphs. Chen et al. [14] obtained $\chi''_a(G)$ of mono-cycle graphs and square of cycles. Huang et al. [15] showed that $\chi''_a(G) \leq 2\Delta(G)$ for any graph with $\Delta(G) \geq 3$. In 2014, Papaioannou and Raftopoulou [16] constructed an algorithm that gives an avd-total coloring with seven colors to any 4-regular graph.

The length of a shortest cycle in G is called *girth* of a graph G, denoted by $g(G)$. The *maximum average degree* of G is defined by

$$\operatorname{mad}(G) = \max_{H \subseteq G} \left\{ \frac{2|E(H)|}{|V(H)|} \right\}.$$

The following lemma can be derived easily from the definition of maximum average degree.

Lemma 1.2. *If H is a subgraph of G, then* $\operatorname{mad}(H) \leq \operatorname{mad}(G)$.

The following fact is well-known.

Proposition 1.3. *If G is a planar graph, then* $\operatorname{mad}(G) < 2g(G)/(g(G) - 2)$.

In 2008, Wang and Wang [17] obtained following results about $\chi''_a(G)$ for graphs with smaller maximum average degree.

Theorem 1.4. [17] *Let G be a graph.*
(1) If $\operatorname{mad}(G) < 3$ *and* $\Delta(G) \geq 5$, *then* $\Delta(G) + 1 \leq \chi''_a(G) \leq \Delta(G) + 2$; *and* $\chi''_a(G) = \Delta(G) + 2$ *if and only if G has two vertices of maximum degree which are adjacent.*
(2) If $\operatorname{mad}(G) < 3$ *and* $\Delta(G) = 4$, *then* $\chi''_a(G) \leq 6$.
(3) If $\operatorname{mad}(G) < \frac{8}{3}$ *and* $\Delta(G) = 3$, *then* $\chi''_a(G) \leq 5$.

Applying proposition 1.3 to Theorem 1.4 yields the following corollary.

Corollary 1.5. [17]

Let G be a planar graph.

(1) If $g(G) \geq 6$ and $\Delta(G) \geq 5$, then $\Delta(G)+1 \leq \chi''_a(G) \leq \Delta(G)+2$; and $\chi''_a(G) = \Delta(G)+2$ if and only if G has two adjacent vertices of maximum degree.

(2) If $g(G) \geq 6$ and $\Delta(G) = 4$, then $\chi''_a(G) \leq 6$.

(3) If $g(G) \geq 8$ and $\Delta(G) = 3$, then $\chi''_a(G) \leq 5$.

The concept of list coloring was introduced independently by Vizing [18] and by Erdős, Rubin, and Taylor [19]. Each vertex (or edge) is assumed to have a list of legal colors that can be used where the lists may be different. Thereafter, many colorings are studied in the list analogous as a natural extension. In this paper, a k-assignment L of G is a list assignment L with $|L(v)| = k$ for each vertex v and $|L(e)| = k$ for each edge e. A *total-L-coloring* is a proper total coloring ϕ of G such that $\phi(v) \in L(v)$ whenever $v \in V(G)$ and $\phi(e) \in L(e)$ whenever $e \in E(G)$. We call that G has an *avd-total-L-coloring* if G has a total-L-coloring such that $C(u) \neq C(v)$ for all $uv \in E(G)$. The smallest integer k such that G has an avd-total-L-coloring for every k-assignment L, denoted by $Ch''_a(G)$, is called the *avd-total choice number*. For $H \subseteq G$, we let L_H denote a list L restricted to a subgraph H of G. In this paper, we strengthen Theorem 1.4 and thus Corollary 1.5 by giving analogous results for $Ch''_a(G)$. Naturally, some additional results are required.

2. Main Results

2.1. Graphs with Maximum Average Degree Less Than 3

Theorem 2.1. *If G is a graph with $\text{mad}(G) < 3$ and $K(G) = \max\{\Delta(G)+2, 6\}$, then $Ch''_a(G) \leq K(G)$.*

Proof. Assume that G is a minimal counterexample. Let $|L(v)| \geq K(G)$ for each vertex v and $|L(e)| \geq K(G)$ for each edge e in G. By minimality and Lemma 1.2, any proper subgraph G' of G has $Ch''_a(G') \leq K(G') \leq K(G)$. Thus there is an avd-total-$L_{G'}$-coloring ϕ of G'. The structure of G is analyzed in the claims below. After that we obtain a contradiction by using the discharging method.

Claim 1. There is no vertex of degree at most 3 is adjacent to a leaf.

Proof. Suppose to the contrary that G contains a vertex v with $d_G(v) \leq 3$ adjacent to a leaf. Without loss of generality, we may assume that $d_G(v) = 3$ and v_1, v_2, v_3 are neighbors of v where v_1 is a leaf. Let $G' = G - v_1$. Suppose that $\phi(v) = 1, \phi(vv_2) = 2, \phi(vv_3) = 3$. Since $|L(vv_1)| \geq 6$, we have $|L(vv_1) \setminus \{1,2,3\}| \geq 3$. Thus we can choose $\phi(vv_1) = a \in L(vv_1) \setminus \{1,2,3\}$ to obtain $C(v) = \{1,2,3,a\}$ such that $C(v_2) \neq C(v) \neq C(v_3)$. Finally, we can color v_1 from $L(v_1) \setminus \{1,a\}$. ■

Claim 2. There does not exist a path $x_1x_2x_3\ldots x_n$ or a cycle $x_1x_2x_3\ldots x_n$ where $x_1 = x_n$ with $d_G(x_1), d_G(x_n) \geq 3$ and $d_G(x_i) = 2$ for all $i = 2,3,\ldots,n-1$, where $n \geq 4$.

Proof. Suppose to the contrary that G contains such a path or a cycle. Let $G' = G - x_2x_3$.

If $n = 4$, we recolor x_2 with a color $a \in L(x_2) \setminus \{\phi(x_1), \phi(x_3), \phi(x_1x_2), \phi(x_3x_4)\}$, and color x_2x_3 with a color in $L(x_2x_3) \setminus \{a, \phi(x_3), \phi(x_1x_2), \phi(x_3x_4)\}$. Since $a \in C(x_2)$ but $a \notin C(x_3)$, we get $C(x_2) \neq C(x_3)$.

If $n \geq 5$, we recolor x_3x_4 with a color $a \in L(x_3x_4) \setminus \{\phi(x_2), \phi(x_4), \phi(x_5), \phi(x_4x_5)\}$, recolor x_3 with a color $b \in L(x_3) \setminus \{a, \phi(x_2), \phi(x_4), \phi(x_4x_5)\}$, and color x_2x_3 with a color in $L(x_2x_3) \setminus \{a, b, \phi(x_2), \phi(x_1x_2)\}$.

Since $\phi(x_2) \notin \{a, b, \phi(x_2x_3)\} = C(x_3)$ but $\phi(x_2) \in C(x_2)$, we get $C(x_2) \neq C(x_3)$. Since $a \in C(x_4) \setminus C(x_5)$ and $b \in C(x_3) \setminus C(x_4)$, we get $C(x_4) \neq C(x_5)$ and $C(x_3) \neq C(x_4)$. ■

Claim 3. There does not exist a k-vertex v, $k \geq 4$, with neighbors $v_1, v_2, v_3, \ldots, v_k$ such that $d_G(v_1) = 1, d_G(v_i) \leq 2$ for $2 \leq i \leq k-2$.

Proof. Suppose to the contrary that G contains such a vertex v. For $2 \leq i \leq k-2$, if v_i is a 2-vertex, we let $u_i \neq v$ be the second neighbor of v_i. Note that u_i has degree at least 3 by Claim 2 if it exists. Let $G' = G - v_1$. Without loss of generality, we may assume that $\phi(v) = 1, \phi(vv_i) = i$ for $i = 2, 3, \ldots, k$. Let $a, b \in L(vv_1) \setminus \{1, 2, 3, \ldots, k\}$. If $C(v_{k-1}) \neq \{1, 2, 3, \ldots, k, a\} \neq C(v_k)$, we color vv_1 with a. Thus $C(v) = \{1, 2, 3, \ldots, k, a\}$. If $C(v_{k-1}) \neq \{1, 2, 3, \ldots, k, b\} \neq C(v_k)$, we color vv_1 with b. Consequently, $C(v) = \{1, 2, 3, \ldots, k, b\}$. Hence $C(v) \neq C(v_i)$ for $i = k-1, k$. Assume that $C(v_{k-1}) = \{1, 2, 3, \ldots, k, a\}$ and $C(v_k) = \{1, 2, 3, \ldots, k, b\}$.

Case 1. $d(v_2) = 1$.

We recolor vv_2 with $s \in L(vv_2) \setminus \{1, 2, \ldots, k\}$ and choose $\phi(vv_1) \in L(vv_1) \setminus \{1, 2, \ldots, k, s\}$. Thus $2 \in C(v_{k-1}) \setminus C(v)$ and $2 \in C(v_k) \setminus C(v)$. Hence $C(v) \neq C(v_i)$ for $i = k-1, k$.

Finally, we color v_1 with a color in $L(v_1) \setminus \{\phi(v), \phi(vv_1)\}$ and we recolor v_2 with a color in $L(v_2) \setminus \{\phi(v), \phi(vv_2)\}$.

Case 2. $d(v_2) = 2$.

The proof is similar to Case 1 except we recolor vv_2 with $s \in L(vv_2) \setminus \{1, 2, ..., k, \phi(v_2u_2)\}$ and recolor v_2 with a color in $L(v_2) \setminus \{\phi(v), \phi(vv_2), \phi(u_2), \phi(v_2u_2)\}$. ■

Claim 4. A 2-vertex v is not adjacent to a 3-vertex u.

Proof. Suppose to the contrary that G contains a 2-vertex v adjacent to a 3-vertex u and another vertex w. Let $u_1, u_2 \neq v$ be the other neighbors of u. By Claims 1 and 2, $d_G(w) \geq 3$. Let $G' = G - uv$. Without loss of generality, we may assume that $\phi(u) = 1, \phi(uu_1) = 2, \phi(uu_2) = 3$. Since $|L(uv)| \geq 6$, we can choose $\phi(uv) = a \in L(uv) \setminus \{1, 2, 3\}$ such that $C(u_1) \neq \{1, 2, 3, a\} \neq C(u_2)$. Hence $C(u) \neq C(u_i)$ for $i = 1, 2$. Finally, we recolor v with a color in $L(v) \setminus \{1, a, \phi(w), \phi(vw)\}$. ■

Claim 5. A 4-vertex v is not adjacent to three 2-vertices.

Proof. Suppose to the contrary that G contains a 4-vertex v with neighbors v_1, v_2, v_3, v_4 such that $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$. Let $u_1 \neq v$ be the second neighbor of v_1. By Claims 1 and 2, we have $d_G(u_1) \geq 3$. Let $G' = G - vv_1$. Without loss of generality, we may assume that $\phi(v) = 1, \phi(vv_i) = i$ for $i = 2, 3, 4$. Since $|L(vv_1) \setminus \{1, 2, 3, 4\}| \geq 2$, we can choose $\phi(vv_1) = a \in L(vv_1) \setminus \{1, 2, 3, 4\}$ such that $C(v_4) \neq \{1, 2, 3, a\} = C(v)$. Finally, we recolor v_1 with a color in $L(v_1) \setminus \{1, a, \phi(u_1), \phi(v_1u_1)\}$. Since $d_G(u_1) \geq 3$ by Claim 2, we have $C(v_1) \neq C(u_1)$. ■

Claim 6. A 5-vertex v is not adjacent to five 2-vertices.

Proof. Suppose to the contrary that G contains a 5-vertex v adjacent to five 2-vertices v_1, v_2, v_3, v_4, v_5. For $1 \leq i \leq 5$, let $u_i \neq v$ be the second neighbor of v_i. We note that $d_G(u_i) \geq 3$ by Claims 1 and 2. Let $G' = G - vv_1$. First, we uncolor v, v_1, v_2, v_3, v_4, and v_5. Since $L(vv_1) \geq 6$, we can color vv_1 with a color in $L(vv_1) \setminus \{\phi(vv_2), \phi(vv_3), \phi(vv_4), \phi(vv_5), \phi(v_1u_1)\}$. Next, we color v with a color in $L(v) \setminus \{\phi(vv_1), \phi(vv_2), \phi(vv_3), \phi(vv_4), \phi(vv_5)\}$. Finally, we color v_i with a color in $L(v_i) \setminus \{\phi(v), \phi(u_i), \phi(vv_i), \phi(vu_i)\}$ for $1 \leq i \leq 5$.

Observe that $C(v) \neq C(v_i) \neq C(u_i)$ because $d_G(v) \neq d_G(v_i) \neq d_G(u_i)$. Therefore, we have an avd-total-L_G-coloring ϕ of G. ■

Let H be a graph obtained by removing all leaves of G. The properties of the graph H are collected in the following Claim 7:

Claim 7.
(1) Each vertex in H has degree at least 2.
(2) If $v \in V(G)$ with $2 \leq d_G(v) \leq 3$, then $d_H(v) = d_G(v)$.
(3) If $v \in V(H)$ with $d_H(v) = 2$, then $d_G(v) = 2$.
(4) If $v \in V(G)$ with $d_G(v) \geq 4$, then $d_H(v) \geq 3$.
(5) H cannot contain a 2-vertex adjacent to a 2-vertex or a 3-vertex.

Proof. (1) Suppose to the contrary that a vertex v in H has $d_H(v) \leq 1$. One can easily see that H does not contain a vertex v with $d_G(v) \leq 1$.
If $d_G(v) = 2$ or 3, then the vertex v cannot be adjacent to a leaf in G by Claim 1.
If $d_G(v) \geq 4$, then the vertex v is adjacent to at least $d_G(v) - 1$ leaves in G. This contradicts Claim 3. Hence each vertex v in H has $d_H(v) \geq 2$.

(2) Since a vertex v with $d_G(v) = 2$ or 3 is not adjacent to a leaf in G by Claim 1, we do not remove its neighbor in G. Hence $d_H(v) = d_G(v)$.

(3) Suppose to the contrary that $d_G(v) \geq 3$. If $d_G(v) = 3$, then the vertex v is adjacent to a leaf in G. This contradicts Claim 1.
If $d_G(v) \geq 4$, then the vertex v is adjacent to at least $d_G(v) - 2$ leaves in G. This contradicts Claim 3. Hence $d_G(v) = 2$.

(4) Assume that $v \in V(G)$ with $d_G(v) \geq 4$. Then we have that the vertex v is adjacent to at most $d_G(v) - 3$ leaves in G by Claim 3. Thus $d_H(v) \geq 3$.

(5) Suppose to the contrary that a 2-vertex u in H is adjacent to a 2-vertex v. By Claim 7(3), we have $d_G(v) = d_H(v) = 2 = d_H(u) = d_G(u)$. This contradicts Claim 2.
Suppose to the contrary that a 2-vertex u in H is adjacent to a 3-vertex v.
The case $d_G(v) = 3$ contradicts Claim 4.
If $d_G(v) \geq 4$, then the vertex v is adjacent to $d_G(v) - 3$ leaves in G. This contradicts Claim 3. Hence H cannot contain a 2-vertex adjacent to a 2-vertex or a 3-vertex. ■

Next, we complete the proof by using the discharging method derived by Wang and Wang [17]. We present their method here for the convenience of readers. First of all, define an initial charge function $w(v) = d_H(v)$ for every $v \in V(H)$. Next, rearrange the weights according to the designed rule. When the discharging is finished, we have a new charge w'. However, the sum of all charges is kept fixed. Finally, we want to show that $w'(v) \geq 3$ for all $v \in V(H)$. This leads to the following contradiction:

$$3 = \frac{3|V(H)|}{|V(H)|} \leq \frac{\sum_{v \in V(H)} w'(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w(v)}{|V(H)|} = \frac{2|E(H)|}{|V(H)|} \leq \text{mad}(H) < 3.$$

The discharging rule is defined as follows :

(R). Every vertex v of degree at least 4 gives $\frac{1}{2}$ to each adjacent 2-vertex.

Let $v \in V(H)$. By Claim 7(1), we get $d_H(v) \geq 2$.

Case 1. $d_H(v) = 2$. The vertex v is adjacent to two vertices of degree at least 4 by Claim 7(5). Thus each of vertices sends $\frac{1}{2}$ to v by (R). Hence $w'(v) = d_H(v) + 2(\frac{1}{2}) = 2 + 1 = 3$.

Case 2. $d_H(v) = 3$. We have $w'(v) = w(v) = 3$.

Case 3. $d_H(v) = 4$. Suppose to the contrary that v is adjacent to at least three

2-vertices, say v_1, v_2, and v_3. By Claim 7(3), we get $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$.

Subcase 3.1 $d_G(v) = 4$. Then we have that a 4-vertex v is adjacent to three 2-vertices in G. This contradicts Claim 5.

Subcase 3.2 $d_G(v) = k \geq 5$. Then there exist $k-4$ neighbors of v in G which are leaves. This contradicts Claim 3.

Thus v is adjacent to at most two 2-vertices in H. Hence $w'(v) \geq 4 - 2(\frac{1}{2}) = 3$.

Case 4. $d_H(v) = 5$. Suppose to the contrary that v is adjacent to five 2-vertices, say v_1, v_2, v_3, v_4, and v_5. By Claim 7(3), we get $d_G(v_i) = 2$ for $1 \leq i \leq 5$.

Subcase 4.1 $d_G(v) = 5$. Then we have that a 5-vertex v is adjacent to five 2-vertices in G. This contradicts Claim 6.

Subcase 4.2 $d_G(v) = k \geq 6$. Then there exist $k-5$ neighbors of v in G which are leaves. This contradicts Claim 3.

Thus v is adjacent to at most four 2-vertices. Hence $w'(v) \geq 5 - 4(\frac{1}{2}) = 3$.

Case 5. $d_H(v) \geq 6$. The vertex v is adjacent to at most $d_H(v)$ 2-vertices and hence $w'(v) \geq d_H(v) - \frac{1}{2}d_H(v) = \frac{1}{2}d_H(v) \geq 3$ by **(R)**.

■

Theorem 2.2. *Let G be a graph with $\text{mad}(G) < 3$ and without adjacent vertices of maximum degree. Let $K'(G) = \max\{\Delta(G)+1, 6\}$. Then $Ch''_a(G) \leq K'(G)$.*

Proof. The proof is proceeded by contradiction. Assume that G is a minimum counterexample. Let $|L(v)| \geq K'(G)$ for each vertex v and $|L(e)| \geq K'(G)$ for each edge e in G. With the same argument, we can prove that G satisfies Claims 1, 2, 4, 5, and 6 as in Theorem 2.1.

If G does not satisfy Claim 3, we suppose that v is a k-vertex, $k \geq 4$, with neighbors $v_1, v_2, \ldots, v_k$ such that $d_G(v_1) = 1, d_G(v_i) \leq 2$ for $2 \leq i \leq k-2$, we denote by $u_i \neq v$ the second neighbor of v_i. Let $G' = G - v_1$. Without loss of generality, we may assume that $\phi(v) = 1, \phi(vv_i) = i$ for $i = 2, 3, \ldots, k$.

If $d_G(v) = \Delta(G)$, then $d_G(v_{k-1}) \neq \Delta(G) \neq d_G(v_k)$ by assumption. We can proceed by coloring vv_1 and v_1. Next, we consider the case $d_G(v) = k < \Delta(G)$. Let $a, b \in L(vv_1) \setminus \{1, 2, 3, \ldots, k, a\}$.

If $C(v_{k-1}) \neq \{1, 2, 3, ..., k, a\} \neq C(v_k)$, we color vv_1 with a. Thus $C(v) = \{1, 2, 3, ..., k, a\}$.

If $C(v_{k-1}) \neq \{1, 2, 3, ..., k, b\} \neq C(v_k)$, we color vv_1 with b. Thus $C(v) = \{1, 2, 3, ..., k, b\}$. Hence $C(v) \neq C(v_i)$ for $i = k-1, k$. Assume that $C(v_{k-1}) = \{1, 2, 3, \ldots, k, a\}$ and $C(v_k) = \{1, 2, 3, \ldots, k, b\}$. The remaining proof is similar to that of Claim 3 in Theorem 2.1. Therefore, G satisfies Claim 3.

Similarly, let H be the graph obtained by removing all leaves of G. Then $\text{mad}(H) \leq \text{mad}(G) < 3$ by Lemma 1.2. Using the same initial charge function $w(v) = d_H(v)$ for every $v \in V(H)$ and the same discharging rule (R) as in Theorem 2.1, we can complete the proof by providing a contradiction. ■

2.2. Graphs with Maximum Average Degree Less Than $\frac{8}{3}$

Theorem 2.3. *If G is a graph with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \leq 3$, then $Ch''_a(G) \leq 5$.*

Proof. The proof is proceeded by contradiction. Assume that G is a minimum counterexample. Let $|L(v)| \geq 5$ for each vertex v and $|L(e)| \geq 5$ for each edge e in G. For any proper subgraph G' of G, we always assume that there is an avd-total-$L_{G'}$-coloring ϕ of

G' by minimality of G.

Claim 1. G satisfies the following properties
(a) No 2-vertex is adjacent to a leaf.
(b) No 3-vertex is adjacent to a leaf and another vertex with degree at most 2.
(c) There are no adjacent 2-vertices.

Proof. (a) Suppose to the contrary that G contains a 2-vertex v with neighbors v_1, v_2 such that v_1 is a leaf and u_2 is another neighbor of v_2 if it exists. Let $G' = G - vv_1$. Since $|L(vv_1)| \geq 5$, we can color vv_1 with a color in $L(vv_1) \setminus \{\phi(v), \phi(vv_2), \phi(v_2), \phi(v_2u_2)\}$. Thus $C(v) \neq C(v_2)$. Finally, we can color v_1 from $L(v_1) \setminus \{\phi(v), \phi(vv_1)\}$.
(b) Suppose to the contrary that G contains 3-vertex v with neighbors v_1, v_2, v_3 such that $d_G(v_1) = 1$ and $d_G(v_2) \leq 2$. Let $G' = G - v_1$. Since $|L(vv_1)| = 5$, we can color vv_1 with a color a in $L(vv_1) \setminus \{\phi(v), \phi(vv_2), \phi(vv_3)\}$ to make $C(v) \neq C(v_3)$. Finally, we can color v_1 from $L(v_1) \setminus \{\phi(v), a\}$.
(c) Suppose to the contrary that G contains two adjacent 2-vertices u and v. Let $u_1 \neq v$ be the second neighbor of u, and v_1 be the second neighbor of v. Note that $d_G(v_1)$ and $d_G(u_1) = 2$ or 3 by (a). Then the proof is similar to that of Claim 2 in Theorem 2.1.
■

Claim 2. Suppose that v is a 3-vertex adjacent to a leaf v_1 and two other vertices v_2 and v_3. Let ϕ be an avd-total-L_{G-v_1}-coloring of the subgraph $G - v_1$. Then
(1) $L(vv_1) = \{\phi(v), \phi(v_2), \phi(v_3), \phi(vv_2), \phi(vv_3)\}$, $C(v_2) = \{\phi(v), \phi(v_2), \phi(vv_2), \phi(vv_3)\}$, and $C(v_3) = \{\phi(v), \phi(v_3), \phi(vv_2), \phi(vv_3)\}$,
(2) $|L(v) \setminus \{\phi(v_2), \phi(v_3), \phi(vv_2), \phi(vv_3)\}| = 1$.

Proof. (1) Without loss of generality, we may assume that $\phi(v) = 1$ and $\phi(vv_i) = i$ for $i = 2, 3$. Let $\{a, b\} \subseteq L(vv_1) \setminus \{1, 2, 3\}$. Note that we cannot extend an avd-total-L-coloring ϕ to a counterexample G. If $C(v_2) \neq \{1, 2, 3, a\} \neq C(v_3)$, we color vv_1 with a. If $C(v_2) \neq \{1, 2, 3, b\} \neq C(v_3)$, we color vv_1 with b. Hence $C(v) \neq C(v_i)$ for $i = 2, 3$. Assume that $C(v_2) = \{1, 2, 3, a\}$ and $C(v_3) = \{1, 2, 3, b\}$. Since $\phi(v_2) \neq 1, 2$, we have $\phi(v_2) = 3$ or a. If $\phi(v_2) = 3$, then we recolor $\phi(v)$ with $t \in L(v) \setminus \{1, 2, 3, \phi(v_3)\}$ and color vv_1 with a color in $L(vv_1) \setminus \{1, 2, 3, t\}$. Thus $1 \in C(v_2) \setminus C(v)$ and $1 \in C(v_3) \setminus C(v)$. Hence $C(v) \neq C(v_i)$ for $i = 2, 3$. Moreover, $\phi(v_2) = a$. Similarly, $\phi(v_3) = b$. Therefore, $L(vv_1) = \{1, 2, 3, a, b\} = \{\phi(v), \phi(vv_2), \phi(vv_3), \phi(v_2), \phi(v_3)\}$, $C(v_2) = \{1, 2, 3, a\} = \{\phi(v), \phi(vv_2), \phi(vv_3), \phi(v_2)\}$, and $C(v_3) = \{1, 2, 3, b\} = \{\phi(v), \phi(vv_2), \phi(vv_3), \phi(v_3)\}$.
(2) Claim 2(2) follows Claim 2(1) immediately. ■

Claim 3. There do not exist two adjacent 3-vertices each of which is adjacent to a leaf.

Proof. Suppose to the contrary that G contains two adjacent 3-vertices u and v such that u is adjacent to a leaf u_1 and v is adjacent to a leaf v_1. Let u_2 and v_2 be the third neighbor of u and v respectively. Let $G' = G - u_1$. By Claim 2(1), we may assume that $\phi(u) = 1$, $\phi(uu_2) = 2$, $\phi(uv) = 3$, $\phi(u_2) = 4$, $\phi(v) = 5$, $C(u_2) = \{1, 2, 3, 4\}$, $C(v) = \{1, 2, 3, 5\}$, and $L(uu_1) \setminus \{1, 2, 3\} = \{4, 5\}$. By Claim 2(2), $L(u) = \{1, 2, 3, 4, 5\}$.
Case 1. If $4 \in L(uv)$ and $C(v_2) \neq \{1, 2, 4, 5\}$, then we recolor uv by 4. Now we have $C(v) = \{1, 2, 4, 5\}$. Thus $C(v) \neq C(v_2)$. Since $\{\phi(uu_2), \phi(u_2), \phi(uv), \phi(v)\} = \{2, 4, 5\}$, we have $|L(u) \setminus \{\phi(uu_2), \phi(u_2), \phi(uv), \phi(v)\}| \geq 2$. This contradicts Claim 2(2).
Case 2. If $4 \in L(uv)$ and $C(v_2) = \{1, 2, 4, 5\}$, then we recolor uv by 4 and vv_1 by

$b \in L(vv_1) \setminus \{1,2,4,5\}$. Since $b \in C(v) \setminus C(v_2)$, $C(v) \neq C(v_2)$. Now, ϕ is an avd-total-$L_{G'}$-coloring of G'. Since $\{\phi(uu_2), \phi(u_2), \phi(uv), \phi(v)\} = \{2,4,5\}$, we have $|L(u) \setminus \{\phi(uu_2), \phi(u_2), \phi(uv), \phi(v)\}| \geq 2$. This contradicts Claim 2(2).

Case 3. If $4 \notin L(uv)$, then we color uu_1 by 4 and recolor uv by $c \in L(uv) \setminus \{1,2,3,4,5\}$. Since $|L(vv_1)| \geq 5$, we can choose a color that is not 4 to recolor vv_1 such that $C(v) \neq C(v_2)$. Since $c \in C(u) \setminus C(u_2)$, we get $C(u) \neq C(u_2)$. Since $4 \in C(u) \setminus C(v)$, we get $C(u) \neq C(v)$. Recolor v_1 as needs to complete an avd-total-L-coloring of G. ■

Claim 4. There is no 3-vertex is adjacent to two 3-vertices each of which is adjacent to a leaf.

Proof. Assume that G contains a 3-vertex u with neighbors x, y, z such that y is adjacent to a leaf y_1, and z is adjacent to a leaf z_1. Let y_2 be the remaining neighbor of y and z_2 be the remaining neighbor of z. Let $G' = G \setminus \{y_1, z_1\}$. Let ϕ_1 be an avd-total-$L_{G'}$-coloring of G'. Let a partial total-L-coloring ϕ of G obtained from a coloring ϕ_1 except that we uncolor u, y, z, uy, and uz.

Choose $L'(u) \subseteq L(u) \setminus \{\phi(ux), \phi(x)\}$, $L'(uy) \subseteq L(uy) \setminus \{\phi(ux), \phi(yy_2)\}$, and $L'(uz) \subseteq L(uz) \setminus \{\phi(ux), \phi(zz_2)\}$ such that $|L'(u)| = |L'(uy)| = |L'(uz)| = 3$ with $\phi_1(u) \in L'(u), \phi_1(uy) \in L'(uy)$, and $\phi_1(uz) \in L'(uz)$. Let $A = \{(c_1, c_2, c_3) : c_1 \in L'(uy), c_2 \in L'(u), c_3 \in L'(uz), c_1 \neq c_2 \neq c_3 \neq c_1\}$.

Note that if we extend ϕ to uy, u, uz with $(\phi(uy), \phi(u), \phi(uz)) \in A$, then we have a proper partial total-L-coloring of G. One can see that we can continue extending ϕ to $y, yy_1, y_1, z, zz_1, z_1$ to have a proper total-L-coloring of G. However, the resulting coloring is an avd-total-L-coloring of G if and only if $C(u) \neq C(y) \neq C(y_2)$, and $C(u) \neq C(z) \neq C(z_2)$, and $C(u) \neq C(x)$. Now we attempt to show that there is (c_1, c_2, c_3) in A such that extending ϕ to uy, u, uz with c_1, c_2, c_3, respectively, can lead to an avd-total-L-coloring of G. Let A_x be a set of (c_1, c_2, c_3) in A such that using $(\phi(uy), \phi(u), \phi(uz)) = (c_1, c_2, c_3)$ cannot lead to an avd-total-L-coloring of G in which $C(u) \neq C(x)$. Let A_y be a set of (c_1, c_2, c_3) in A such that using $(\phi(uy), \phi(u), \phi(uz)) = (c_1, c_2, c_3)$ cannot lead to an avd-total-L-coloring of G in which $C(u) \neq C(y) \neq C(y_2)$. The definition of A_z is defined similarly to A_y.

Observation 1. $|A_y| \leq 2$. Assume $(1,2,3) \in A_y$. Without loss of generality, let $\phi(ux) = 4$. Let $\phi(yy_2) = a$ and $\phi(y_2) = b$. By the definition of A_y and Claim 2(2), we have $1, 2, a, b$ are four distinct elements in $L(y)$.

Suppose to the contrary that a is neither 3 nor 4. Choose $\phi(y) \in L(y) \setminus \{1, 2, a, b\}$. Since $L(yy_1) \setminus \{1, a, \phi(y)\}$ has at least two elements, we can choose $\phi(yy_1)$ such that $C(y) \neq C(y_2)$. Since $a \in C(y) \setminus C(u)$, we also have $C(y) \neq C(u)$. This contradicts the definition of A_y. Thus $a = 3$ or 4.

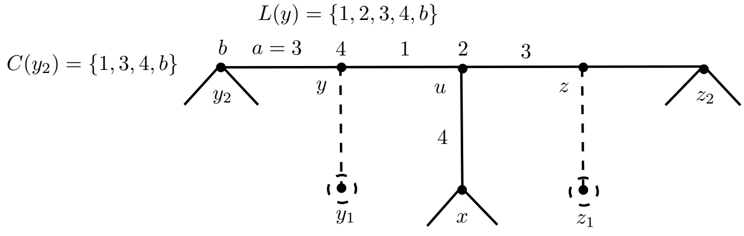

Fig. 1 Case 1. $a = 3$

Case 1. $a = 3$. Let $d \in L(y) \setminus \{1, 2, 3, b\}$. Choose $\phi(y) = d$. Suppose to the contrary that d is not 4. Since $L(yy_1) \setminus \{1, 3, d\}$ has at least two elements, we can choose $\phi(yy_1)$ such that $C(y) \neq C(y_2)$. Since $d \in C(y) \setminus C(u)$, we have $C(y) \neq C(u)$. This contradicts the definition of A_y. Thus $\phi(y) = d = 4$ and $L(y) = \{1, 2, 3, 4, b\}$. By Claim 2(1), $L(yy_1) = \{1, 2, 3, 4, b\}$. Moreover, $C(y_2) = \{1, 3, 4, b\}$ otherwise we can choose $\phi(yy_2) = b$ to make $C(y_2) \neq C(y) \neq C(u)$.

Now we claim that $(2, 1, 3)$ is the only other possible element in A_y. Let $(c_1, c_2, c_3) \in A_y$. Claim 2(2) implies that $\{c_1, c_2\} = \{1, 2\}$. Consequently, we must choose $\phi(y) = 4$. Suppose to the contrary that $c_3 \neq 3$. We can extend $\phi(yy_1)$ to obtain $C(y) \neq C(y_2)$. Since $3 \in C(y) \setminus C(u)$, we also have $C(y) \neq C(u)$ which contradicts the definition of A_y. Thus $c_3 = 3$. Hence $(c_1, c_2, c_3) = (1, 2, 3)$ or $(2, 1, 3)$.

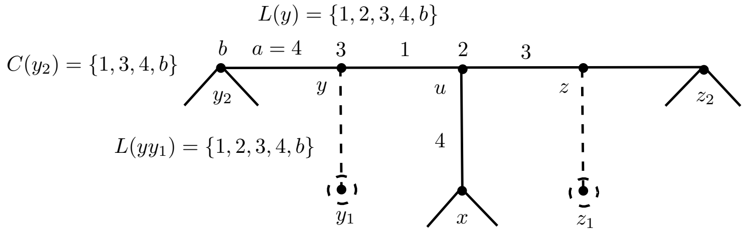

Fig. 2 Case 2. $a = 4$

Case 2. $a = 4$.

Similar to the previous case, one can show that $\phi(y) = 3, L(y) = L(yy_1) = \{1, 2, 3, 4, b\}$, and $C(y_2) = \{1, 3, 4, b\}$. Now we claim that $(3, 2, 1)$ is the only other possible element in A_y. Let $(c_1, c_2, c_3) \in A_y$. Claim 2(2) implies that $\{c_1, c_2\} \subseteq \{1, 2, 3\}$.

Suppose to the contrary that $\{c_1, c_2\} = \{1, 3\}$. Choose $\phi(y) = 2$ for $\{c_1, c_2\} = \{1, 3\}$ and $\phi(yy_1) = b$. Since $2 \in C(y) \setminus C(y_2)$ and $b \in C(y) \setminus C(u)$, we also have $C(y) \neq C(y_2)$ and $C(y) \neq C(u)$ which contradicts the definition of A_y. Thus $\{c_1, c_2\} \neq \{1, 3\}$.

If $(c_1, c_2) = (2, 1)$, then we can choose $\phi(y) = 3$ and $\phi(yy_1) = b$ to obtain $C(u) \neq C(y) \neq C(y_2)$. If $(c_1, c_2) = (2, 3)$, then we can choose $\phi(y) = 1$ and $\phi(yy_1) = b$ to obtain $C(u) \neq C(y) \neq C(y_2)$. Combining with $\{c_1, c_2\} \neq \{1, 3\}$, we have $(c_1, c_2) = (1, 2)$ or $(3, 2)$.

We show that $c_3 \in \{1, 2, 3\}$. If $c_3 = d \notin \{1, 2, 3, 4, b\}$, then we can choose $\phi(yy_1)$ after choosing $\phi(y)$ to obtain $C(y) \neq C(y_2)$. Since $d \in C(u) \setminus C(y)$, we also have $C(y) \neq C(u)$. Thus $c_3 \in \{1, 2, 3, 4, b\}$. Since $\phi(ux) = 4$, we have $c_3 \neq 4$. If $c_3 = b$, we can choose $\phi(y) \neq b \neq \phi(yy_1)$ to obtain $C(u) \neq C(y) \neq C(y_2)$. Thus $c_3 \in \{1, 2, 3\}$. Consequently (c_1, c_2, c_3) is $(1, 2, 3)$ or $(3, 2, 1)$. This completes the proof of the observation.

Next, we consider $|A|$ and $|A_x|$ to complete the proof. Assume $\phi_1(uy) = 1, \phi_1(u) = 2, \phi_1(uz) = 3$. Consider the case that $L'(uy) = L'(u) = L'(uz)$. By the definition of $L'(uy), L'(u)$, and $L'(uz)$, we have $L'(uy) = L'(u) = L'(uz) = \{1, 2, 3\}$. Thus $|A| = 6$. Extending ϕ to uy, u, uz where $(\phi(uy),\phi(u),\phi(uz))$ is in A always results in $\{\phi(uy),\phi(u),\phi(uz)\} = \{1, 2, 3\}$. Thus $C(u) = \{1, 2, 3, 4\}$ which is not equal to $C(x)$ from the beginning. Hence $|A_x| = 0$. Therefore $|A \setminus (A_x \cup A_y \cup A_z)| \geq 6 - (0 + 2 + 2) \geq 2$.

Now assume that at least two of $L'(uy), L'(u), L'(uz)$ are not equal. Recall that each of $L'(uy), L'(u), L'(uz)$ has 3 elements. We show that $|A| \geq 10$. First, consider the case $d \in L'(uy) \cup L'(u) \cup L'(uz)$ is in one set, say $L'(uy)$, but not in other two sets. There are at least 6 sets of $(c_1 = d, c_2, c_3)$ and at least 4 sets of $(c_1 \neq d, c_2, c_3)$ in A. Thus $|A| \geq 10$

in this case. Next, consider the case that each $d \in L'(uy) \cup L'(u) \cup L'(uz)$ is in at least two of $L'(uy), L'(u), L'(uz)$. This happens only if $|L'(uy) \cup L'(u) \cup L'(uz)| = 4$ and only one element appears in $L'(uy) \cap L'(u) \cap L'(uz)$. One can enumerate that $|A| = 11$.

Let $(c_1, c_2, c_3) \in A_x$. Thus $C(x) = \{c_1, c_2, c_3, 4\}$. By the aforementioned property of $L'(uy), L'(u)$, and $L'(uz)$, we may assume $c_3 \notin L'(uy)$. Thus (c_3, c_1, c_2) and (c_3, c_2, c_1) is not in A_x. Hence $|A_x| \leq 4$. Therefore $|A \setminus (A_x \cup A_y \cup A_z)| \geq 10 - (4 + 2 + 2) \geq 2$.

This means we can extend ϕ to uy, u, uz with $(\phi(uy), \phi(u), \phi(uz)) \in A \setminus (A_x \cup A_y \cup A_z)$. By definitions of A_x, A_y, and A_z, we can extend ϕ further to be an avd-total-L-coloring of G ∎

With a similar proof, one can obtain the followings:

Claim 5. There is no 3-vertex v is adjacent to a 2-vertex and a 3-vertex u such that u is adjacent to a leaf.

Claim 6. There is no 3-vertex v is adjacent to two 2-vertices.

Proof. Suppose to the contrary that G contains a 3-vertex v adjacent to two 2-vertices v_1, v_2, and the third vertex v_3. Let $G' = G - \{vv_1, vv_2\}$. Without loss of generality, we may assume that $\phi(v_3) = 1$,$\phi(vv_3) = 2$, and $C(v_3) \subseteq \{1, 2, 3, 4\}$. Since $|L(v)| = 5$, we can choose $a \in L(v) \setminus \{1, 2, 3, 4\}$ to recolor v and then properly color vv_1,vv_2, and recolor v_1,v_2 (if needed). Since $a \in C(v) \setminus C(v_3)$, we get $C(v) \neq C(v_3)$. Moreover, Claim 1(c) yields that v_i is not adjacent to a 2-vertex for each $i = 1, 2$. Consequently, if u_i is the second neighbor of v_i where $i = 1$ or 2, then $C(v_i) \neq C(u_i)$. Thus we obtain a desired coloring. ∎

Let H be the graph obtained by removing all leaves of G. The properties of the graph H are collected in the following Claim 7:

Claim 7.

(1) Each vertex in H has degree at least 2.
(2) There are no adjacent 2-vertices.
(3) Every 3-vertex is adjacent to at most one 2-vertex.

Proof. (1) Suppose to the contrary that a vertex v has $d_H(v) \leq 1$.
If $d_G(v) = 2$, then v is adjacent to a leaf in G which contradicts Claim 1(a).
If $d_G(v) = 3$, then v is adjacent to at least two leaves in G which contradicts Claim 1(b).

(2) Suppose to the contrary that H contains a 2-vertex u adjacent to a 2-vertex v.
The case $d_G(u) = d_G(v) = 2$ contradicts Claim 1(c).
The case $d_G(u) = 3$ and $d_G(v) = 2$ or $d_G(u) = 2$ and $d_G(v) = 3$ contradicts Claim 1(b).
The case $d_G(u) = d_G(v) = 3$, implies there exist two adjacent 3-vertices each of which is adjacent to a leaf. This contradicts Claim 3.

(3) Suppose to the contrary that H contains a 3-vertex u adjacent to two 2-vertices, say x and y.
The case $d_G(x) = d_G(y) = 2$ contradicts Claim 6.
The case $d_G(x) = 3$ and $d_G(y) = 2$ or $d_G(x) = 2$ and $d_G(y) = 3$ contradicts Claim 5.
The case $d_G(x) = d_G(y) = 3$ contradicts Claim 4. ∎

Next, we complete the proof by using the discharging method derived by Wang and Wang [17]. We present their method here for the convenience of readers. We use the same initial charge function $w(v) = d_H(v)$ for all $v \in V(H)$ and define the discharging rule as follows :

(R′). Every 3-vertex gives $\frac{1}{3}$ to its adjacent 2-vertex.

Let $w'(v)$ denote the new charge of a vertex v after the discharging process is finished on H. If v is a 3-vertex, then v is adjacent to at most one 2-vertex by Claim 7(3). Hence we have $w'(v) \geq 3 - \frac{1}{3} = \frac{8}{3}$ by (R$'$). If v is a 2-vertex, then v is not adjacent to any 2-vertex by Claim 7(2). It follows that v is adjacent to two 3-vertices. Hence we have $w'(v) = 2 + \frac{1}{3} + \frac{1}{3} = \frac{8}{3}$ by (R$'$). Therefore, $w'(v) \geq \frac{8}{3}$ for any $v \in V(H)$. However, this leads to the following contradiction :

$$\frac{8}{3} = \frac{\frac{8}{3}|V(H)|}{|V(H)|} \leq \frac{\sum_{v\in V(H)} w'(v)}{|V(H)|} = \frac{\sum_{v\in V(H)} w(v)}{|V(H)|} = \frac{2|E(H)|}{|V(H)|} \leq \text{mad}(H) < \frac{8}{3}.$$

■

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References

[1] P.N. Balister, B. Bollobás, R.H. Schelp, Vertex distinguishing coloring of graphs with $\Delta(G) = 2$, Discrete Math. 252 (2) (2002) 17–29.

[2] C. Bazgan, A. Harkat-Benhamdine, H. Li, On the adjacent vertex-distinguishing proper edge-coloring of graphs, J. Combin. Theory, Ser. B. 75 (1999) 288–301.

[3] A.C. Burris, R.H. Schelp, Vertex-distinguishing proper edge-colorings, J. of Graph Theory 26 (2) (1997) 73–82.

[4] Z. Zhang, L. Liu, J. Wang, Adjacent strong edge coloring of graphs, Appl. Math. Lett. 15 (2002) 623–626.

[5] Z. Zhang, X. Chen, J. Li, B. Yao, X. Lu, J. Wang, On adjacent-vertex-distinguishing total coloring of graphs, Sci. China Ser. A. 48 (3) (2005) 289–299.

[6] H. Wang, On the adjacent vertex-distinguishing total chromatic numbers of the graphs with $\Delta(G) = 3$, J. Comb. Optim. 14 (2007) 87–109.

[7] X. Chen, On the adjacent vertex distinguishing total coloring numbers of graphs with $\Delta = 3$, Discrete Math. 308 (2008) 4003–4007.

[8] J. Hulgan, Concise proofs for adjacent vertex-distinguishing total colorings, Discrete Math. 309 (2009) 2548–2550.

[9] W. Wang, Y. Wang, Adjacent vertex distinguishing total colorings of outerplanar graphs, J. Comb. Optim. 19 (2010) 123–133.

[10] W. Wang, D. Huang, The adjacent vertex distinguishing total coloring of planar graphs, J. Comb. Optim. 27 (2014) 379–396.

[11] A.G. Luiz, C.N. Campos, C.P. de Mello, AVD-total-colouring of complete equipartite graphs, Discrete Appl. Math. 184 (2015) 189–195.

[12] T. Coker, K. Johannson, The adjacent vertex distinguishing total chromatic number, Discrete Math. 312 (2012) 2741–2750.

[13] V. Pedrotti, C.P. de Mello, Adjacent-vertex-distinguishing total coloring of indifference graphs, Mat. Contemp. 39 (2010) 101–110.

[14] X. Chen, Z. Zhang, Y. Sun, Adjacent-vertex-distinguishing total chromatic numbers on mono-cycle graphs and the square of cycles, International Journal of Pure and Applied Mathematics 18 (4) (2005) 481–490.

[15] D. Huang, W. Wang, C. Yan, A note on the adjacent vertex distinguishing total chromatic number of graphs, Discrete Math. 312 (2012) 3544–3546.

[16] A. Papaioannou, C. Raftopoulou, On the AVDTC of 4-regular graphs, Discrete Math. 330 (2014) 20–40.

[17] W. Wang, Y. Wang, Adjacent vertex distinguishing total coloring of graphs with lower average degree, Taiwanese J. Math. 12 (4) (2008) 979–990.

[18] V.G. Vizing, Vertex colorings with given colors, Metody Diskret. Analiz. (in Russian) 29 (1976) 3–10.

[19] P. Erdős, A.L. Rubin, H. Taylor, Choosability in graphs, In Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, Congr. Num. 26 (1979) 125–157.