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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

New Splitting Algorithm for Mixed Equilibrium Problems on Hilbert Spaces

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Abstract In this paper, we introduce a self-adaptive splitting algorithm for solving mixed equilibrium problems on Hilbert Spaces. Prove the convergence of the sequences generated by our algorithm with conditions and used assuming Hölder continuity. We propose some examples for supporting our algorithm. In the last chapter, give the application from signal processing.

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1. Introduction

In 2008, Ceng and Yao [1] studied the mixed equilibrium problem on real Hilbert space H. A subset C of H be a nonempty closed convex and mapping $\phi : C \to \mathbb{R}$ and bifunction $F : C \times C \to \mathbb{R}$ such that $F(x, x) = 0$, for all $x \in C$. Then find $x^* \in C$ such that

$$F(x^*, y) + \phi(y) - \phi(x^*) \geq 0, \quad \forall y \in C, \tag{MEP}$$

is called a *mixed equilibrium problem* on C. We denote by SOL(MEP), the solution set of the mixed equilibrium (MEP) (For more details, one can see [1, 2] and related literature).

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The problem (MEP) is very general in the sense that it includes, as special cases, *minimization problems* and *equilibrium problems*.

Spacial case:

(i) If $F(x,y) = 0$ for all $x, y \in C$, then problem (MEP) is reduced to the following *minimization problem*. Find $x^* \in C$ such that

$$\phi(y) \geq \phi(x^*), \ \forall y \in C. \tag{1.1}$$

(ii) If $\phi = 0$, then (MEP) is equivalent to the following *equilibrium problem* by Blum and Oettli [3]. Find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C. \tag{EP}$$

Defined

$$S = \{x^* \in C : F(x^*, y) \geq 0, \ \forall y \in C\},$$

we called S is the solution set of the problem. The theory of *equilibrium problem* plays a very important role in variational inequality problems, optimization problems and fixed point problems (see, for instance [4, 5]).

The mixed equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, minimization problems, fixed point problems, Nash equilibrium problems in noncooperative games, and others; (see, for instance [6–15]).

Following from the bifunction F, sometimes it is very hard to deal with but it can be seen as a sum of two simpler bifunctions F_1 and F_2. We see that the problem EP becomes the following problem:

$$\text{find } x^* \in C \text{ such that } F_1(x^*, y) + F_2(x^*, y) \geq 0, \ \forall y \in C. \tag{1.2}$$

We will assume that the function F_1 and F_2 verify the following conditions:

(A1) $F(x,x) = 0$ for all $x, y \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$ for any $x, y, z \in C$;
(A4) for each $x \in C, y \longmapsto F(x,y)$ is convex and lower-semicontinuous.

The following notion appears implicitly in [3].

Definition 1.1. [16] The resolvent of a bifunction $F : C \times C \to \mathbb{R}$ is the set-valued operator

$$J_F : H \to 2^C : x \mapsto \{z \in K | (\forall y \in K) F(z,y) + \langle z - x, y - z\rangle \geq 0\}. \tag{1.3}$$

Under these assumptions, for each $r > 0$ and $x \in H$, the resolvent of F is single valued (See, the paper [11]).

The extragradient method proposed by Antipin [17] and developed by Quoc et al. [18] (see also [19, 20]) for solving the problem (EP) as follows.

$$\begin{cases} x_0 \in C \\ y_n = \arg\min\limits_{t \in C} \{\lambda_n F(x_n, t) + \frac{1}{2}\|t - x_n\|^2\}, \\ x_{n+1} = \arg\min\limits_{t \in C} \{\lambda_n F(y_n, t) + \frac{1}{2}\|t - y_n\|^2\}. \end{cases} \tag{1.4}$$

Motivated and inspired by the definition 1.1 and algorithms 1.4, we will introduce a new splitting algorithm for solving (MEP) on Hilbert space. We will prove a sequence generated by our algorithm converges to a solution in non-ergodic sense.

In each part of the paper be divided into the following chapters. Section 2 introduce some definitions, theorems, lemma and results for further investigation. In Section 3, established a non-ergodic splitting algorithms for solving equilibrium problem (MEP) under the privilege of the new assumptions considered for the component functions. In the last Section 4, we give some numerical result.

2. Preliminaries

In this section, we present some properties, theorems and definition for using in our results. Suppose H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We recall that strong convergence and weak convergence of $\{x_n\}$ are defined by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Definition 2.1. Let C be a nonempty closed convex subset of H. By P_C we denote the projection operator on C with the norm $\| \cdot \|$, that is

$$P_C(x) = argmin\{\|y - x\| : y \in C\}, \ \forall x \in H.$$

We said that P_C is the metric projection of H onto C. Since $\| \cdot -x\|^2$ is a strong convex function, therefore $P_C(x)$ is singleton and well defined for every $x \in H$.

Lemma 2.2. [21] *Let H be a real Hilbert space. Then, the following equation hold:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \forall x, y \in H$;

(ii) $\|x + y\|^2 = \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H$;

(iii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \forall \lambda \in (0, 1)$ *and* $x, y \in H$.

Definition 2.3. Let C be a nonempty closed convex. A bifunction F is said to be

(i) *monotone* on C if $F(x, y) + F(y, x) \leq 0, \ \forall x, y \in C$;

(ii) *pseudomonotone* on C if $F(x, y) \geq 0 \Rightarrow F(y, x) \leq 0, \ \forall x, y \in C$.

From the definition, it follows that $(a) \Rightarrow (b)$.

Definition 2.4. [22] A bifunction $F : C \times C \to \mathbb{R}$ is said to be *$\tau-$Hölder continuous* in the first argument (resp. the second argument) if there exists constants $L > 0$ and $\tau \in (0, 1]$ such that

$$\begin{aligned} |F(x, y) - F(z, y)| &\leq L\|x - z\|^\tau, \ \forall x, y, z \in C \\ (\text{ resp. } |F(x, y) - F(x, z)| &\leq L\|y - z\|^\tau, \ \forall x, y, z \in C). \end{aligned}$$

Definition 2.5. [11] Let C be a nonempty closed convex subset of H and $F : C\times C \to \mathbb{R}$. For any $\lambda > 0$, the *resolvent* of F is the set-valued operator $J^F_\lambda : H \to 2^C$ defined by

$$J^F_\lambda(x) = \{z \in C|\lambda F(z, t) - \langle z - x, t - z\rangle \geq 0, \ \forall t \in C\}, \ \forall x \in H.$$

Definition 2.6. [23] Let C be a nonempty subset of H and $\{x_n\}$ be a sequence in H. Then $\{x_n\}$ is *quasi-Fejér monotone* with respect to C, for all $x \in C$ if

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \epsilon_n,$$

for each $n \in \mathbb{N}$ and $\{\epsilon_n\}$ be a sequence in $(0, \infty)$ satisfies $\sum_{n=1}^{\infty} \epsilon_n < +\infty$.

Theorem 2.7. [23] *Let $\{x_n\}$ be a sequence in H and let C be a nonempty subset of H such that $\{x_n\}$ is quasi-Fejér monotone with respect to C. Then the following hold:*

(i) For every $x \in C$, $\|x_n - x\|$ converges.
(ii) $\{x_n\}$ is bounded.
(iii) Suppose that every weak cluster point of $\{x_n\}$ belongs to C. Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.8. [24] *Let C be a closed convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be convex and subdifferentiable on C. Then, x^* is a solution to the following convex problem*

$$\min\{g(x) : x \in C\} \Leftrightarrow 0 \in \partial g(x^*) + N_C(x^*).$$

3. Main Results

In this section, we introduce a splitting algorithm for solving the mixed equilibrium problem. We give some assumptions and properties of function $F(x, y)$. Let Lemma 3.2 for proving the main theorem.

The following assumptions will be used in the subsequent discussions.

(M1) For every $x \in C$, $y \mapsto F(x, y)$ is convex;
(M2) for every $y \in C$, $x \mapsto F(x, y)$ is upper semi-continuous;
(M3) F is monotone;
(M4) ϕ is convex, lower semi-continuous and increasing;
(M5) $\mathrm{SOL(MEP)} \neq \emptyset$.

Proposition 3.1. *Under assumptions* (M1)–(M5), we have

(a) $\mathrm{SOL(MEP)} = \{y \in C : F(x, y) + \phi(y) - \phi(x) \leq 0, \ \forall x \in C\}$,
(b) SOL(MEP) is closed and convex.

Proof. (a) Let $\mathrm{SOL(MEP)}_d := \{y \in C : F(x, y) + \phi(y) - \phi(x) \leq 0, \ \forall x \in C\}$.
($\Rightarrow$) We prove that $\mathrm{SOL(MEP)} \subset \mathrm{SOL(MEP)}_d$. Let $x^* \in \mathrm{SOL(MEP)}$, we have

$$F(x^*, y) + \phi(y) - \phi(x^*) \geq 0, \qquad \forall y \in C.$$

That is,

$$F(x^*, y) \geq \phi(x^*) - \phi(y), \qquad \forall y \in C. \tag{3.1}$$

Adding $F(y, x^*)$ to both side (3.1), we get

$$F(x^*, y) + F(y, x^*) \geq F(y, x^*) + \phi(x^*) - \phi(y). \tag{3.2}$$

From F is monotone, i.e. $F(y, x^*) + \phi(x^*) - \phi(y) \leq 0$. Therefore, $x^* \in \mathrm{SOL(MEP)}_d$.

($\Leftarrow$) We prove that $\mathrm{SOL(MEP)}_d \subset \mathrm{SOL(MEP)}$. Let $x^* \in \mathrm{SOL(MEP)}_d$. Let $\lambda \in (0, 1]$, $y \in C$, and set $z_\lambda = \lambda y + (1 - \lambda)x^*$, we have

$$F(z_\lambda, x^*) + \phi(x^*) - \phi(z_\lambda) \leq 0, \qquad \forall z_\lambda \in C. \tag{3.3}$$

And from ϕ is convex, we have

$$(1 - \lambda)\phi(z_\lambda) + \lambda\phi(z_\lambda) = \phi(z_\lambda) \leq \lambda\phi(y) + (1 - \lambda)\phi(x^*).$$

So, $(1-\lambda)(\phi(z_\lambda)-\phi(x^*)) \leq \lambda(\phi(y)-\phi(z_\lambda))$.
Since $y \mapsto F(x,y)$ is convex, we get

$$0 = F(z_\lambda, z_\lambda) \leq \lambda F(z_\lambda, y) + (1-\lambda)F(z_\lambda, x^*). \tag{3.4}$$

Adding $(1-\lambda)(\phi(z_\lambda)-\phi(x^*))$ to both sides of (3.4),

$$\begin{aligned}(1-\lambda)(\phi(z_\lambda)-\phi(x^*)) &\leq \lambda F(z_\lambda, y) + (1-\lambda)F(z_\lambda, x^*) + (1-\lambda)(\phi(z_\lambda)-\phi(x^*))\\ &\leq \lambda F(z_\lambda, y) + (1-\lambda)F(z_\lambda, x^*) + \lambda(\phi(y)-\phi(z_\lambda)).\end{aligned}$$

Implies that,

$$-[(1-\lambda)(F(z_\lambda, x^*) + \phi(x^*) - \phi(z_\lambda))] \leq \lambda[F(z_\lambda, y) + \phi(y) - \phi(z_\lambda)].$$

From (3.3), hence $F(F(\gamma(t), y) + \phi(y) - \phi(\gamma(t)) \geq 0$ as $\lambda \geq 0$.

Since F is upper semi-continuous of $F(\cdot, y)$ and ϕ is lower semi-continuous, taking $z_\lambda \to 0^+$ we get,

$$0 \leq \limsup_{z_\lambda \to 0^+} F(z_\lambda, y) + \phi(y) - \liminf_{z_\lambda \to 0^+} \phi(x^*) \leq F(x^*, y) + \phi(y) - \phi(x^*).$$

This implies that $x^* \in \mathrm{SOL(MEP)}$.

(b) Consider the sequence satisfying $\{x_n\} \subset \mathrm{SOL(MEP)}$ and $x_n \to x^*$. Since $x \mapsto F(x,y)$ is upper semi-continuous, we have

$$F(x^*, y) \geq \limsup_{n\to\infty} F(x_n, y), \quad \forall y \in C. \tag{3.5}$$

Also, ϕ is lower semi-continuous, we have

$$\liminf_{n\to\infty} \phi(x_n) \geq \phi(x^*). \tag{3.6}$$

Combining (3.5) and (3.6), we obtain that

$$F(x^*, y) - \phi(x^*) \geq F(x^*, y) \geq \limsup_{n\to\infty} F(x_n, y) - \liminf_{n\to\infty} \phi(x_n).$$

Adding $\phi(y)$ for each $y \in C$ to both sides of the last inequality,

$$\begin{aligned}F(x^*, y) + \phi(y) - \phi(x^*) &\geq \limsup_{n\to\infty} F(x_n, y) + \phi(y) - \liminf_{n\to\infty} \phi(x_n)\\ &\geq 0.\end{aligned}$$

Therefore, $x^* \in \mathrm{SOL(MEP)}$, i.e. SOL(MEP) is closed. Next, let $x_1^*, x_2^* \in \mathrm{SOL(MEP)}$, then

$$F(y, x_i^*) + \phi(x_i^*) - \phi(y) \leq 0, \ \forall i = 1, 2. \tag{3.7}$$

Consider

$$\begin{aligned}F(y, \lambda x_1^* + (1-\lambda)x_2^*) + \phi(\lambda x_1^* + (1-\lambda)x_2^*) - \phi(y) &\leq \lambda F(y, x_1^*) + (1-\lambda)(y, x_2^*)\\ &\quad + \lambda\phi(x_1^*) + (1-\lambda)\phi(x_2^*) - \phi(y)\\ &\leq \lambda[F(y, x_1^*) + \phi(x_1^*) - \phi(y)]\\ &\quad + (1-\lambda)[F(y, x_2^*) + \phi(x_2^*) - \phi(y)].\end{aligned}$$

From (3.7), we conclude that

$$F(y, \lambda x_1^* + (1-\lambda)x_2^*) + \phi(\lambda x_1^* + (1-\lambda)x_2^*) - \phi(y) \leq 0.$$

Therefore $\mathrm{SOL(MEP)_d}$ is convex, implies that SOL(MEP) is convex. ∎

Algorithm 1

Initialization: Choose $x_0 \in C$ and a sequence $\{\lambda_n\} \subset (0, \infty)$.
Iterative Step: Given x_n, compute y_n and x_{n+1} by
Step 1. Compute

$$y_n = J^F_{\lambda_n}(x_n).$$

Step 2. Compute

$$x_{n+1} = \arg\min_{t\in C} \left\{ \lambda_n \phi(t) + \frac{1}{2}\|t - y_n\|^2 \right\}.$$

Update $n =: n + 1$ and go back to **Step 1.**

Lemma 3.2. *Let C be a nonempty closed convex subset of H and the condition (i)-(v) hold. Moreover, suppose F is τ-Hölder continuous in the first argument or second argument. Then the sequences $\{x_n\}$, $\{y_n\}$ generated by Algorithm 1 satisfy the following properties.*

(a) There exists $M > 0$ such that

$$\|x_n - y_n\| \leq M\lambda_n^{\frac{1}{2-\tau}}, \qquad \forall n \geq 1.$$

(b) There exists $L > 0$ such that

$$\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 + 2\lambda_n(F(x_n, x) + \phi(x) - \phi(x_n)) + L\lambda_n^{\frac{2}{2-\tau}}, \qquad \forall x \in C. \quad (3.8)$$

Proof. (a) From $y_n = J^F_{\lambda_n}(x_n)$, by the Definition 2.5, we have $y_n \in C$ such that

$$\lambda_n F(y_n, t) - \langle y_n - x_n, t - y_n\rangle \geq 0, \qquad \forall t \in C.$$

That is,

$$\lambda_n F(y_n, t) \geq \langle y_n - x_n, t - y_n\rangle, \qquad \forall t \in C. \quad (3.9)$$

Put $t = x_n \in C$ into (3.9), we get

$$\begin{aligned} \|x_n - y_n\|^2 &\leq \lambda_n F(y_n, x_n) \\ &\leq \lambda_n| - F(y_n - x_n)|. \end{aligned}$$

Since $x_n \in C$, then $F(x_n, x_n) = 0$, we get

$$\|x_n - y_n\|^2 \leq \lambda_n|F(x_n, x_n) - F(y_n, x_n)|. \quad (3.10)$$

Besides, from F is Hölder continuous in the first argument, then there exist $Q > 0$ and $\tau \in (0, 1]$ such that

$$|F(x_n, x_n) - F(y_n, x_n)| \leq Q\|x_n - y_n\|^\tau. \quad (3.11)$$

From (3.10) and (3.11), we have

$$\begin{aligned} \frac{1}{\lambda_n}\|x_n - y_n\|^2 &\leq Q\|x_n - y_n\|^\tau \\ \|x_n - y_n\|^{2-\tau} &\leq \lambda_n Q \\ \|x_n - y_n\| &\leq (\lambda_n Q)^{\frac{1}{2-\tau}}. \end{aligned}$$

Hence, $\|x_n - y_n\| \leq M\lambda_n^{\frac{1}{2-\tau}}$, where $M = Q^{\frac{1}{2-\tau}}$.

(b) Since x_{n+1} solves the convex program

$$\arg\min_{t\in C}\left\{\lambda_n\phi(t)+\frac{1}{2}\|t-y^n\|^2\right\}.$$

if and only if

$$0\in\partial(\lambda_n\phi(\cdot)+\frac{1}{2}\|\cdot-y_n\|^2)(x_{n+1})+N_C(x_{n+1}),$$

where $N_C(x_{n+1})$ is normal cone of C at x_{n+1}. So, there exist $w\in\partial\phi(x_{n+1})$ and $v\in N_C(x_{n+1}):=\{z\in H:\langle z,x-x_{n+1}\rangle\le 0,\ \forall x\in C\}$ such that

$$0=\lambda_n w+x_{n+1}-y_n+v.$$

Therefore $v=y_n-\lambda_n w-x_{n+1}$. By definition of $N_C(x_{n+1})$, we have

$$\langle y_n-\lambda_n w-x_{n+1},x-x_{n+1}\rangle\le 0$$

we get

$$\langle y_n-x_{n+1},x-x_{n+1}\rangle-\langle\lambda_n w,x-x_{n+1}\rangle\le 0$$

and

$$\langle y_n-x_{n+1},x-x_{n+1}\rangle\le\langle\lambda_n w,x-x_{n+1}\rangle,\qquad\forall x\in C.$$

From $w\in\partial\phi(x_{n+1})$, we have

$$\phi(x)-\phi(x_{n+1})\ge\langle w,x-x_{n+1}\rangle,\qquad\forall x\in C.$$

Combining the two last inequalities, we obtain

$$\begin{aligned}\lambda_n(\phi(x)-\phi(x_{n+1}))&\ge\lambda_n\langle w,x-x_{n+1}\rangle\\&\ge\langle y_n-x_{n+1},x-x_{n+1}\rangle,\qquad\forall x\in C.\end{aligned}\tag{3.12}$$

Consider, using properties of Hilbert spaces, we have

$$\begin{aligned}\|y_n-x\|^2&=\|(x_n-x)+(y_n-x_n)\|^2\\&\le\|x_n-x\|^2+2\langle y_n-x_n,y_n-x\rangle,\end{aligned}\tag{3.13}$$

and

$$\|x_{n+1}-x\|^2\le\|y_n-x\|^2+2\langle x_{n+1}-y_n,x_{n+1}-x\rangle.\tag{3.14}$$

By (3.13) with (3.14), and using (3.9) and (3.12), we have

$$\begin{aligned}\|x_{n+1}-x\|^2&\le\|x_n-x\|^2+2\langle x_{n+1}-y_n,x_{n+1}-x\rangle+2\langle y_n-x_n,y_n-x\rangle\\&\le\|x_n-x\|^2+2\lambda_nF(y_n,x)+2\lambda_n(\phi(x)-\phi(x_{n+1}))\\&\le\|x_n-x\|^2+2\lambda_n(F(y_n,x)+\phi(x)-\phi(x_{n+1})).\end{aligned}$$

Since ϕ is increasing, we have $-\phi(x_{n+1})\le-\phi(x_n)$. It implies that

$$\|x_{n+1}-x\|^2\le\|x_n-x\|^2+2\lambda_n(F(y_n,x)+\phi(x)-\phi(x_n)).\tag{3.15}$$

Following (3.15) and using τ- Hödler continuity of F, we obtain

$$\begin{aligned}\|x_{n+1}-x\|^2&\le\|x_n-x\|^2+2\lambda_n(F(y_n,x)+F(x_n,x)-F(x_n,x)+\phi(x)-\phi(x_n))\\&\le\|x_n-x\|^2+2\lambda_n(F(x_n,x)+\phi(x)-\phi(x_n))+2\lambda_n|F(x_n,x)-F(y_n,x)|\\&\le\|x_n-x\|^2+2\lambda_n(F(x_n,x)+\phi(x)-\phi(x_n))+2\lambda_nQ\|x_n-y_n\|^\tau\\&\le\|x_n-x\|^2+2\lambda_n(F(x_n,x)+\phi(x)-\phi(x_n))+2\lambda_nQ((Q\lambda_n)^{\frac{1}{2-\tau}})^\tau\\&\le\|x_n-x\|^2+2\lambda_n(F(x_n,x)+\phi(x)-\phi(x_n))+L\lambda_n^{\frac{2}{2-\tau}},\end{aligned}\tag{3.16}$$

where $L = 2Q^{\frac{2}{2-\tau}} > 0$. ∎

Theorem 3.3. *Let C be a nonempty closed convex subset of H and assume that all the hypothesis of Lemma 3.2. Suppose that the sequence $\{\lambda_n\}$ satisfies*

(i) $\liminf_{n\to\infty} \lambda_n \geq 0$,

(ii) $\sum_{n=1}^{\infty} \lambda_n^{\frac{2}{2-\tau}} < +\infty$.

Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a solution of (MEP).

Proof. Firstly, we prove that the sequence $\{x_n\}$ is bounded. Let $x = x^* \in \mathrm{SOL(MEP)}$. By Proposition 3.1, we have $F(x_n, x^*) + \phi(x^*) - \phi(x_n) \leq 0$ for $n \geq 1$. Hence, we get

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + L\lambda_n^{\frac{2}{2-\tau}}, \quad \forall n \in \mathbb{N}.$$

That is $\{x_n\}$ is quasi Fejér monotone with respect to SOL(MEP). From Theorem 2.7, we conclude that $\{x_n\}$ is bounded. So $\{x_n\}$ is bounded and there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in C$.

Next, we prove that $x^* \in \mathrm{SOL(MEP)}$. Follows from (3.8), implies that

$$\frac{1}{2\lambda_{n_k}}[\|x_{n_k+1} - x\|^2 - \|x_{n_k} - x\|^2] \leq F(x_{n_k}, x) + \phi(x) - \phi(x_{n_k}) + L\lambda_{n_k}^{\frac{\tau}{2-\tau}}, \quad \forall x \in C.$$

Taking the upper limit as $k \to \infty$, since $\liminf_{n\to\infty} \lambda_n \geq 0$, $\lim_{n\to\infty} \lambda_n^{\frac{2}{2-\tau}} = 0$ and $x_{n_k} \rightharpoonup x^*$, we obtain that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} F(x_{n_k}, x) + \phi(x) - \limsup_{k\to\infty} \phi(x_{n_k}) \\ &\leq \limsup_{k\to\infty} F(x_{n_k}, x) + \phi(x) - \liminf_{k\to\infty} \phi(x_{n_k}) \\ &\leq F(x^*, x) + \phi(x) - \phi(x^*) \end{aligned}$$

for all $x \in C$, therefore $x^* \in \mathrm{SOL(MEP)}$.

Finally, From Theorem 2.7, hence the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to SOL(MEP). ∎

4. Numerical Results

In this section, we give some example for support our results. All the numerical examples are implemented in MATLAB R2018b running on a laptop with Intel®Core $3^{TM} - 6006$U 2.0 Ghz 4 GB Ram.

Example 4.1. Given H be a real Hilbert space and let a bifunction $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = \langle Ax, y - x\rangle$, for all $x, y \in \mathbb{R}^2$. where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\phi(x) = \phi(x_1, x_2) = 2\|x_1\|$ for all $x \in \mathbb{R}^2$.

We verify $(M1)-(M5)$ satisfies the assumption.

(M1) For each $x \in \mathbb{R}^2$ we have

$$\begin{aligned} F(x, \lambda\overline{x} + (1-\lambda)y) &= \langle Ax, (\lambda\overline{x} + (1-\lambda)y) - ((1-\lambda)x - \lambda x)\rangle \\ &= \lambda\langle Ax, \overline{x} - x\rangle + (1-\lambda)\langle Ax, y - x\rangle \\ &= \lambda F(x, \overline{x}) + (1-\lambda)F(x, y), \ \forall \overline{x}, y \in \mathbb{R}^2. \end{aligned}$$

That is $F(\cdot, y)$ is convex.

(M2) From continuously of $F(x, y) = \langle Ax, y - x\rangle$, we deduce that its upper semi-continuous.

(M3) For each $x, y \in \mathbb{R}^2$ and $F(x, y) = \langle Ax, y - x\rangle$ we have,

$$\begin{aligned} F(x, y) + F(y, x) &= \langle Ax, y - x\rangle + \langle Ay, x - y\rangle \\ &= (-x_2(y_1 - x_1) + x_1(y_2 - x_2)) + (-y_2(x_1 - y_1) + y_1(x_2 - y_2)) \\ &= 0. \end{aligned}$$

That is $F(x, y) + F(y, x) \leq 0$, therefore, F is monotone.

(M4) ϕ is convex, lower semi-continuous and increasing. Since $\phi(x_1, x_2) = 2\|x_1\|$, easy to see.

(M5) Consider,

$$\begin{aligned} F(x^*, y) + \phi(y) - \phi(x^*) &= \langle Ax^*, y - x^*\rangle + 2\|y_1\| - 2\|x_1^*\|, \ \forall x, y \in \mathbb{R}^2 \\ &= \langle Ax^*, y\rangle + \langle Ax^*, -x^*\rangle + 2\|y_1\| - 2\|x_1^*\|, \ \forall x, y \in \mathbb{R}^2. \end{aligned}$$

Therefore, if $x^* = 0$ then SOL(MEP) $\neq \emptyset$.

Choosing $\lambda_n = \frac{1}{n}$, where $n \in \mathbb{N}$. Hence, all of assumptions are satisfied and our algorithm converge to a solution $x^* \in [0, 1]$.

Example 4.2. Given H be a real Hilbert space and let a bifunction $F : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ defined by $F(x, y) = \langle Ax + By + q, y - x\rangle$ for all $x, y \in \mathbb{R}^5$, where

$$q = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix}, A = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let a function $\phi : \mathbb{R}^5 \to \mathbb{R}$ defined by $\phi(x) = \phi(x_1, x_2, x_3, x_4, x_5) = \|x_1\| = |x_1|$ for all $x \in \mathbb{R}^5$.

As the matrices A and B are positive definite with $F(x, x) = 0$ for all $x \in \mathbb{R}^5$, the assumptions (M1)–(M5) considered are all satisfied. We now use Algorithms 1 to solve MEP, see the table:

Table 1. Iterations of Algorithm 1 in Example 4.2 with starting point $x^0 = (1,2,3,4,5)^T$, choosing $\lambda_n = \dfrac{1}{n+1}$.

Iter(n)	x_1^n	x_2^n	x_3^n	x_4^n	x_5^n	$\|x^{n-1} - x^n\|$
0	1	2	3	4	5	
1	8.0001×10^{-7}	1.3056	1.0432	0.8355	0.9825	2.1106
2	4.8000×10^{-8}	1.5458×10^{-4}	1.5457×10^{-4}	1.5458×10^{-4}	1.5456×10^{-4}	5.0001
3	8.0000×10^{-6}	2.5436	2.4800	2.5184	2.4580	4.8178×10^{-5}
4	2.0000×10^{-6}	2.5436	2.4800	2.5184	2.4580	4.6463×10^{-5}
5	2.4000×10^{-6}	2.5436	2.4800	2.5184	2.4580	3.4521×10^{-5}
6	2.8000×10^{-6}	2.5435	2.4800	2.5184	2.4581	4.1269×10^{-5}

Table 2. Iterations of Algorithm 1 in Example 4.2 with starting point $x^0 = (0,0,0,0,0)^T$, choosing $\lambda_n = \dfrac{1}{n+1}$.

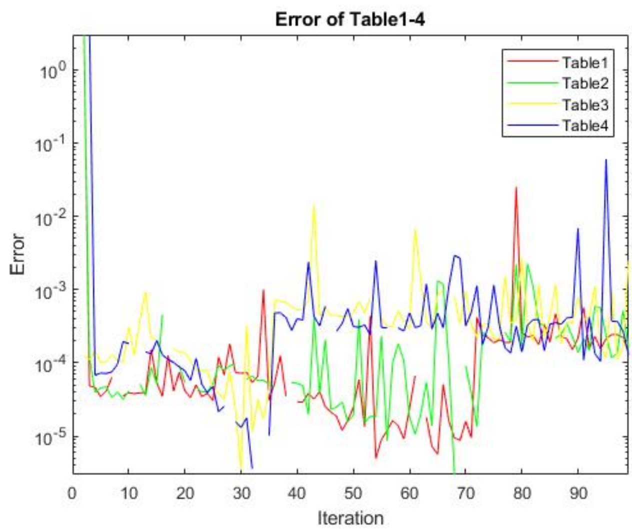
Iter(n)	x_1^n	x_2^n	x_3^n	x_4^n	x_5^n	$\|x^{n-1} - x^n\|$
0	0	0	0	0	0	
1	8.0000×10^{-7}	1.8542	1.8542	1.8542	1.8542	3.7078
2	1.2000×10^{-6}	2.8291×10^{-4}	2.8297×10^{-4}	2.8290×10^{-4}	2.8294×10^{-4}	5.0000
3	1.6000×10^{-6}	2.4721	2.5544	2.4560	2.5175	1.2226×10^{-4}
4	1.0000×10^{-5}	2.4722	2.5543	2.4561	2.5174	3.9269×10^{-5}
5	2.4000×10^{-6}	2.4722	2.5543	2.4561	2.5174	4.5353×10^{-5}
6	1.4000×10^{-5}	2.4722	2.5543	2.4561	2.5174	4.7255×10^{-5}

Table 3. Iterations of Algorithm 1 in Example 4.2 with starting point $x^0 = (1,2,3,4,5)^T$, choosing $\lambda_n = \dfrac{1}{2n}$.

Iter(n)	x_1^n	x_2^n	x_3^n	x_4^n	x_5^n	$\|x^{n-1} - x^n\|$
0	1	2	3	4	5	
1	8.0001×10^{-7}	1.3056	1.0432	0.8355	0.9825	2.9191
2	1.6000×10^{-6}	2.6669	2.5072	2.3579	2.4679	1.2652×10^{-4}
3	2.3999×10^{-6}	2.6668	2.5072	2.3580	2.4679	1.1032×10^{-4}
4	1.6000×10^{-5}	2.6667	2.5072	2.3581	2.4680	1.4540×10^{-4}
5	4.0006×10^{-6}	2.6666	2.5072	2.3582	2.4680	1.0148×10^{-4}
6	2.3999×10^{-5}	2.6666	2.5072	2.3582	2.4680	1.0318×10^{-4}

TABLE 4. Iterations of Algorithm 1 in Example 4.2 with starting point $x^0 = (0,0,0,0,0)^T$, choosing $\lambda_n = \dfrac{1}{2n}$.

Iter(n)	x_1^n	x_2^n	x_3^n	x_4^n	x_5^n	$\|x^{n-1} - x^n\|$
0	0	0	0	0	0	
1	8.0000×10^{-7}	1.8542	1.8542	1.8542	1.8542	3.7078
2	1.2800×10^{-6}	2.6401×10^{-4}	2.6400×10^{-4}	2.6401×10^{-4}	2.6402×10^{-4}	9.9996
3	2.4000×10^{-6}	4.9921	4.9660	5.0094	5.0326	5.0009
4	6.3998×10^{-7}	2.4725	2.4015	2.5250	2.6011	6.7972×10^{-5}
5	4.0002×10^{-6}	2.4725	2.4015	2.5249	2.6010	7.2525×10^{-5}
6	4.7998×10^{-6}	2.4725	2.4016	2.5249	2.6011	7.1630×10^{-5}



From Table 1, Table 2, we consider at two starting point $x^0 = (1,2,3,4,5)^T$, $x^0 = (0,0,0,0,0)^T$ and choosing $\lambda_n = \dfrac{1}{n+1}$. Table 3, Table 4, choosing two starting point $x^0 = (1,2,3,4,5)^T$, $x^0 = (0,0,0,0,0)^T$ with $\lambda_n = \dfrac{1}{2n}$.

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