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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

The Fixed Point Property of a Non-Unital Abelian Banach Algebra Generated by an Element with Infinite Spectrum

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Abstract A Banach space X is said to have the fixed point property if for each nonexpansive mapping $T : E \to E$ on a bounded closed convex subset E of X has a fixed point. Assume that X is an infinite dimensional non-unital Abelian Banach algebra satisfying: (i) condition (A) defined in [W. Fupinwong, S. Dhompongsa, The fixed point property of unital Abelian Banach algebras, Fixed Point Theory and Applications (2020)], (ii) $\|x\| \leq \|y\|$ for each $x, y \in X$ such that $|\tau(x)| \leq |\tau(y)|$ for each $\tau \in \Omega(X)$, (iii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$. We show that there is an element $(x_0, 0)$ in X such that

$$\langle x_0, 0\rangle_{\mathbb{R}} = \overline{\left\{\sum_{i=1}^{k} \alpha_i (x_0, 0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}$$

does not have the fixed point property. This result is a generalization of Theorem 21 in [P. Thongin, W. Fupinwong, The fixed point property of a Banach algebra generated by an element with infinite spectrum, Journal of Function Spaces (2018)]. Moreover, as a consequence of the proof, for each element $(x_0, 0)$ in X with infinite spectrum and $\sigma(x_0, 0) \subset \mathbb{R}$, the Banach algebra $\langle x_0, 0\rangle = \overline{\left\{\sum_{i=1}^{k} \alpha_i (x_0, 0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{C}\right\}}$ generated by $(x_0, 0)$ does not have the fixed point property.

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1. Introduction

A Banach space X is said to *have the fixed point property* if for each nonexpansive mapping $T : E \to E$ on a bounded closed convex subset E of X has a fixed point, to *have the weak fixed point property* if for each nonexpansive mapping $T : E \to E$ on a weakly compact convex subset E of X has a fixed point.

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In 1981, D.E. Alspach [1] proved that there exists an isometry $T : E \to E$ on a weakly compact convex subset E of the Lebesgue space $L_1[0,1]$ without a fixed point. Consequently, $L_1[0,1]$ does not have the weak fixed point property.

In 1983, J. Elton, P.K. Lin, E. Odell, and S. Szarek [2] showed that $C(\alpha, \mathbb{R})$ has the weak fixed point property, if α is a compact ordinal with $\alpha < \omega^\omega$.

In 1997, A.T. Lau, P.F. Mah, and A. Ulger [3] proved the following theorem.

Theorem 1.1. *Let X be a locally compact Haudorff space. If $C_0(X)$ has the weak fixed point propert, then X is dispersed.*

Moreover, by applying Theorem 1.1, they proved the following results.

Corollary 1.2. [3] *Let G be a locally compact group. Then the C^*-algebra $C_0(G)$ has the weak fixed point property if and only if G is discrete.*

Corollary 1.3. [3] *A von Neumann algebra M has the weak fixed point property if and only if M is finite dimensional.*

In 2005, T.D. Benavides and M.A. Japon Pineda [4] studied the concept of ω-almost weak orthogonality in the Banach lattice $C(K)$ and proved the following results.

Theorem 1.4. [4] *Let X be a ω-almost weakly orthogonal closed subspace of $C(K)$, where K is a metrizable compact space. Then X has the weak fixed point property.*

Theorem 1.5. [4] *Let K be a metrizable compact space. Then the following conditions are all equivalent:*
1) $C(K)$ is ω-almost weakly orthogonal.
2) $C(K)$ is ω-weakly orthogonal.
3) $K^{(\omega)} = \emptyset$.

Corollary 1.6. [4] *Let K be a compact set with $K^{(\omega)} = \emptyset$. Then $C(K)$ has the weak fixed point property.*

In 2010, W. Fupinwong and S. Dhompongsa [5] showed that each infinite dimensional real unital Abelian Banach algebra X with $\Omega(X) \neq \emptyset$ satisfying: (i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$ then $\|x\| \leq \|y\|$, (ii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$, does not have the fixed point property. Moreover, they proved the following theorem.

Theorem 1.7. [5] *Let X be an infinite dimensional complex unital Abelian Banach algebra satisfying condition (A) and each of the following:*
(i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$.
(ii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$.
Then X does not have the fixed point property.

In 2010, D. Alimohammadi and S. Moradi [6] used the above result to obtain sufficient conditions to show that some unital unifromly closed subalgebras of $C(X)$, where X is a compact space, do not have the fixed point property.

In 2011, S. Dhompongsa, W. Fupinwong, and W. Lawton [7] showed that a C^*-algebra has the fixed point property if and only if it is finite dimensional.

In 2012, W. Fupinwong [8] showed that the unitality in the results proved in [5] can be omitted.

In 2016, by using Urysohn's lemma and Schauder-Tychonoff fixed point theorem, D. Alimohammadi [9] proved the following result.

Theorem 1.8. [9] *Let Ω be a locally compact Hausdorff space. Then the following statements are equivalent:*
(i) Ω is infinite set.
(ii) $C_0(\Omega)$ is infinite dimensional.
(iii) $C_0(\Omega)$ does not have the fixed point property.

In 2017, J. Daengsaen and W. Fupinwong [10] proved that each infinite dimensional real Abelian Banach algebra X with $\Omega(X) \neq \emptyset$ and satisfying: (i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$, (ii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$, does not have the fixed point property.

Recently, in 2018, P. Thongin and W. Fupinwong [11] proved the following result.

Theorem 1.9. [11] *Let X be an infinite dimensional complex unital Abelian Banach algebra satisfying: (i) condition (A), (ii) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$ then $\|x\| \leq \|y\|$, (iii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$. Then there exists an element x_0 in X such that*

$$\langle x_0 \rangle_{\mathbb{R}} = \overline{\left\{ \sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}$$

does not have the fixed point property.

Furthermore, as a consequence of the proof, for each element x_0 in X with infinite spectrum and $\sigma(x_0) \subset \mathbb{R}$, the Banach algebra $\langle x_0 \rangle_{\mathbb{R}}$ generated by x_0 does not have the fixed point property.

In this paper, we show that the unitality in the result proved in [11] can be omitted.

2. Preliminaries and Lemmas

Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let X be a Banach algebra over $\mathbb{F}$. The unitization $\widetilde{X}$ of X is the Banach algebra $X \bigoplus \mathbb{F}$, where the multiplication on $\widetilde{X}$ is defined by

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu),$$

and the norm on $\widetilde{X}$ is defined by

$$\|(x, \lambda)\| = \|x\| + |\lambda|.$$

It can be seen that $\widetilde{X}$ is a unital Banach algebra over $\mathbb{F}$ with the unit $(0, 1)$. Denote by $\Omega(X)$ the set of all characters on X.

A complex Banach algebra X is said to satisfy condition (A) if, for each $x \in X$, there exists an element $y \in X$ such that $\tau(y) = \overline{\tau(x)}$, for each $\tau \in \Omega(X)$.

Note that

$$\Omega(\widetilde{X}) = \{\widetilde{\tau} : \tau \in \Omega(X)\} \cup \{\tau_\infty\},$$

where $\widetilde{\tau}$ is defined from $\tau \in \Omega(X)$ by

$$\widetilde{\tau}(x, \lambda) = \tau(x) + \lambda,$$

and τ_∞ is the canonical homomorphism defined by

$$\tau_\infty(x, \lambda) = \lambda.$$

It can be seen that, if X satisfies condition (A), then so does the unitization $\widetilde{X}$. Moreover, if X is Abelian, then $\widetilde{X}$ is also Abelian.

Let X be an Abelian Banach algebra over $\mathbb{F}$. The Gelfand representation $\varphi : X \to C(\Omega(X))$ is defined by $x \mapsto \widehat{x}$, where $\widehat{x}$ is defined by

$$\widehat{x}(\tau) = \tau(x),$$

for each $\tau \in \Omega(X)$.

The Jacobson radical $J(X)$ of a Banach algebra X over $\mathbb{F}$ is the intersection of all regular maximal left ideals of X. Note that if X is a unital complex Banach algebra and $x \in J(X)$ then the spectral radius $r(x)$ of x is equal to zero. A Banach algebra X over $\mathbb{F}$ is said to be semi-simple if $J(X) = \{0\}$.

The following lemmas are all very useful for proving our main result.

Lemma 2.1. *Let X be a complex non-unital Banach algebra satisfying*

$$\inf\{\|\widehat{x}\|_\infty : x \in X, \|x\| = 1\} > 0.$$

Then:
(i) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0.$
(ii) $\inf\{\|\widehat{(x,\lambda)}\|_\infty : (x,\lambda) \in \widetilde{X}, \|(x,\lambda)\| = 1\} > 0.$

Proof. Let X be a complex non-unital Banach algebra satisfying

$$\inf\{\|\widehat{x}\|_\infty : x \in X, \|x\| = 1\} > 0.$$

(i) We first note that

$$\{\tau(x) : \tau \in \Omega(X)\} \subset \sigma(x),$$

for each $x \in X$, thus it suffices to prove that $\|\widehat{x}\|_\infty \leq r(x)$, for each $x \in X$ and then

$$0 < \inf\{\|\widehat{x}\|_\infty : x \in X, \|x\| = 1\} \leq \inf\{r(x) : x \in X, \|x\| = 1\}.$$

(ii) Assume to the contrary that

$$\inf\{\|\widehat{(x,\lambda)}\|_\infty : (x,\lambda) \in \widetilde{X}, \|(x,\lambda)\| = 1\} = 0.$$

So there is a sequence $\{(x_n, \lambda_n)\}$ such that $\lim \|\widehat{(x_n,\lambda_n)}\|_\infty = 0$ and $\|(x_n,\lambda_n)\| = 1$, for each $n \in \mathbb{N}$. Since

$$|\lambda_n| \leq \max\left\{ \sup_{\tau\in\Omega(X)} |\tau(x_n) + \lambda_n|,\ \ |\lambda_n| \right\} = \sup_{\omega\in\Omega(\widetilde{X})} |\omega(x_n,\lambda_n)| = \|\widehat{(x_n,\lambda_n)}\|_\infty,$$

for each $n \in \mathbb{N}$, so

$$\lim |\lambda_n| \leq \lim \|\widehat{(x_n,\lambda_n)}\|_\infty = 0.$$

We can conclude only that $\lim |\lambda_n| = 0$. Since $\|\widehat{x_n}\|_\infty \leq \|\widehat{(x_n,\lambda_n)}\|_\infty + |\lambda_n|$, for each $n \in \mathbb{N}$, it follows that

$$\lim \|\widehat{x_n}\|_\infty \leq \lim \|\widehat{(x_n,\lambda_n)}\|_\infty + \lim |\lambda_n| = 0.$$

We may assume by passing through a subsequence that $\{|\lambda_n|\}$ is a decreasing sequence converging to zero and $|\lambda_n| \leq 1/2$, for each $n \in \mathbb{N}$. From

$$1 = \|(x_n,\lambda_n)\| = \|x_n\| + |\lambda_n|,$$

we have $\|x_n\| \geq 1/2$, for each $n \in \mathbb{N}$. Then

$$\lim \left\| \widehat{\left(\frac{x_n}{\|x_n\|} \right)} \right\|_\infty \leq \lim \left\| \widehat{\left(\frac{x_n}{1/2} \right)} \right\|_\infty = 2 \lim \|\widehat{x_n}\|_\infty = 0.$$

Hence $\inf\{\|\widehat{x}\|_\infty : x \in X, \|x\| = 1\} = 0$. This leads to a contradiction. ∎

Lemma 2.2. *Let X be a complex non-unital Banach algebra satisfying*

$$\inf\{r(x) : x \in X, \|x\| = 1\} > 0.$$

Then:
(i) $\inf\{r(x, \lambda) : (x, \lambda) \in \widetilde{X}, \|(x, \lambda)\| = 1\} > 0.$
(ii) X and $\widetilde{X}$ are semi-simple.

Proof. Let X be a complex non-unital Banach algebra satisfying

$$\inf\{r(x) : x \in X, \|x\| = 1\} > 0.$$

(i) Suppose, on the contrary that

$$\inf\{r(x, \lambda) : (x, \lambda) \in \widetilde{X}, \|(x, \lambda)\| = 1\} = 0.$$

There is, of course, a sequence $\{(x_n, \lambda_n)\}$ such that $\lim r(x_n, \lambda_n) = 0$ and $\|(x_n, \lambda_n)\| = 1$, for each $n \in \mathbb{N}$. Since $\lambda_n = \tau_\infty(x_n, \lambda_n) \in \sigma(x_n, \lambda_n)$, for each $n \in \mathbb{N}$, so

$$\lim |\lambda_n| \leq \lim r(x_n, \lambda_n) = 0.$$

We see that $r(x_n, 0) \leq r(x_n, \lambda_n) + |\lambda_n|$, for each $n \in \mathbb{N}$ It follows that

$$\lim r(x_n, 0) \leq \lim r(x_n, \lambda_n) + \lim |\lambda_n| = 0.$$

Thus we may as well assume by passing through a subsequence that $\{|\lambda_n|\}$ is a decreasing sequence converging to zero and $|\lambda_n| \leq 1/2$, for each $n \in \mathbb{N}$. From

$$1 = \|(x_n, \lambda_n)\| = \|(x_n, 0)\| + |\lambda_n|,$$

we obtain $\|(x_n, 0)\| \geq 1/2$, for each $n \in \mathbb{N}$ and thus

$$\lim r\left(\frac{(x_n, 0)}{\|(x_n, 0)\|}\right) \leq \lim r\left(\frac{(x_n, 0)}{1/2}\right) = 2 \lim r(x_n, 0) = 0.$$

We deduce that $\inf\{r(x) : x \in X, |x| = 1\} = 0$. This contradiction shows that $\inf\{r(x, \lambda) : (x, \lambda) \in \widetilde{X}, \|(x, \lambda)\| = 1\} > 0$.

(ii) From (i), it follows that

$$\inf\{r(x, \lambda) : (x, \lambda) \in \widetilde{X}, \|(x, \lambda)\| = 1\} > 0.$$

Thus, it suffices to prove that, for each $(x, \lambda) \in \widetilde{X}$, $r(x, \lambda) = 0$ implies $(x, \lambda) = (0, 0)$. We note that $r(x, \lambda) = 0$, for each $(x, \lambda) \in J(\widetilde{X})$ and then $J(\widetilde{X}) = \{0\}$. Therefore, we can deduce that $\widetilde{X}$ is semi-simple.

Since every ideal in a semi-simple Banach algebra is also semi-simple, so X is semi-simple. This completes the proof. ∎

The following lemma was proved in [12].

Lemma 2.3. [12] *In any infinite dimensional semi-simple complex Banach algebra, there exists an element with an infinite spectrum.*

Lemma 2.4. *Let X be an infinite dimensional complex non-unital Banach algebra with condition (A) and satisfying*

$$\inf\{r(x) : x \in X, \|x\| = 1\} > 0.$$

Then there exists $x_0 \in X$ with infinite spectrum and $\omega(x_0, 0) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$.

Proof. It follows from Lemma 2.2 that X is semi-simple. Applying Lemma 2.3, we see that there exists an element x in X which infinite spectrum. From condition (A), there exists $y \in X$ such that

$$\tau(y) = \overline{\tau(x)},$$

for each $\tau \in \Omega(X)$. Hence

$$\tau(xy) = \tau(x)\tau(y) = \tau(x)\overline{\tau(x)} \in \mathbb{R}.$$

Evidently, $\omega\left((x, 0)(y, 0)\right) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$. ∎

The following lemma was proved in [11].

Lemma 2.5. [11] *Let X be an infinite dimensional complex unital Banach algebra, and let x_0 be an element in X with infinite spectrum. Then $\{x_0^n : n \in \mathbb{N}\}$ is linearly independent.*

Lemma 2.6. *Let X be an infinite dimensional complex non-unital Banach algebra satisfying condition (A), and let $(x_0, 0)$ be an element in $\widetilde{X}$ with infinite spectrum and $\omega(x_0, 0) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$. Define*

$$Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i (x_0, 0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}.$$

Then Z is an infinite dimensional real unital Abelian Banach algebra with $\Omega(Z) \neq \emptyset$.

Proof. From Lemma 2.5, $\{x_0^n : n \in \mathbb{N}\}$ is linearly independent in X, so Z is infinite dimensional.

Let $\omega \in \Omega(\widetilde{X})$, and define $\tau : Z \to \mathbb{R}$ by

$$(x, \lambda) \mapsto \omega(x, \lambda).$$

τ is real-valued since $\omega(x_0, 0) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$. It can be seen that τ is a nonzero homomorphism on Z. So $\Omega(Z) \neq \emptyset$. ∎

Lemma 2.7. *Let X be an infinite dimensional complex non-unital Banach algebra satisfying condition (A), and let $(x_0, 0)$ be an element in $\widetilde{X}$ with infinite spectrum and $\omega(x_0, 0) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$. Define*

$$Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i (x_0, 0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}.$$

If X satisfies

$$\inf\{\|\widehat{x}\|_\infty : x \in X, \|x\| = 1\} > 0,$$

then Z is a real unital Abelian Banach algebra satisfying the following conditions:
(i) The Gelfand representation φ from Z into $C_{\mathbb{R}}(\Omega(Z))$ is a bounded isomorphism.
(ii) The inverse φ^{-1} is also a bounded isomorphism.

Proof. (i) From Lemma 2.1, we have

$$\inf\{\|\widehat{(x,\lambda)}\|_\infty : (x,\lambda) \in \widetilde{X}, \|(x,\lambda)\| = 1\} > 0,$$

it follows that $ker(\varphi) = \{0\}$. So φ is injective. We have $\varphi(Z)$ is a subalgebra of $C_{\mathbb{R}}(\Omega(Z))$. Next, we will now show that $\varphi(Z)$ is complete. We are now in a position to prove that $\{z_n\}$ is a Cauchy sequence. Suppose, on the contrary that $\{z_n\}$ is not Cauchy. Then, there exists, of course, $\varepsilon_0 > 0$ and subsequences $\{z'_n\}$ and $\{z''_n\}$ of $\{z_n\}$ such that

$$\|z'_n - z''_n\| \geq \varepsilon_0,$$

for each $n \in \mathbb{N}$. Letting $y_n = (z'_n - z''_n)/\varepsilon_0$. We have $\|y_n\| \geq 1$, for each $n \in \mathbb{N}$. Since $\{\widehat{z_n}\}$ is Cauchy, it suffices to conclude that $\lim \widehat{y_n} = 0$. It follows that

$$\begin{aligned} 0 &< \inf\{\|\widehat{(x,\lambda)}\|_\infty : (x,\lambda) \in \widetilde{X}, \|(x,\lambda)\| = 1\} \\ &\leq \inf_{n\in\mathbb{N}} r\left(\frac{y_n}{\|y_n\|}\right) = \inf_{n\in\mathbb{N}} \left\|\frac{\widehat{y_n}}{\|y_n\|}\right\|_\infty = 0, \end{aligned}$$

which is a contradiction. So we conclude that $\{z_n\}$ is a Cauchy sequence. Then $\{z_n\}$ is a convergent sequence in Z, say $\lim z_n = z_0 \in Z$. Therefore,

$$\lim \|\widehat{z_n} - \widehat{z_0}\|_\infty = 0.$$

Indeed, for each $n \in \mathbb{N}$, $\|\widehat{z_n} - \widehat{z_0}\|_\infty = \|\varphi(z_n - z_0)\|_\infty \leq \|z_n - z_0\|$. So $\varphi(Z)$ is complete subalgebra of $C_{\mathbb{R}}(\Omega(Z))$ separating the points of $\Omega(Z)$, and annihilating no point of $\Omega(Z)$. It follows from the Stone-Weierstrass theorem that φ is surjective.

(ii) is a consequence of the open mapping theorem. ■

Lemma 2.8. [11] *Let X be a unital Abelian Banach algebra. If there exists an element x in X with infinite spectrum $\sigma(x)$ and $\sigma(x) \subset \mathbb{R}$, then there exists $y \in X$ satisfying the following conditions:*
(i) $1 \in \sigma(y) \subset [0,1]$.
(ii) There exists a strictly decreasing sequence in $\sigma(y)$.

Lemma 2.9. *Let X be an infinite dimensional complex non-unital Banach algebra satisfying condition (A) and*

$$\inf\{\|\widehat{x}\|_\infty : x \in X, \|x\| = 1\} > 0,$$

and let $(x_0, 0)$ be an element in $\widetilde{X}$ with infinite spectrum and $\omega(x_0,0) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$. Define

$$Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i (x_0,0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}.$$

Assume that there exists a bounded sequence $\{(y_n,0)\}$ in Z which contains no convergent subsequences and such that $\{\omega(y_n,0) : \omega \in \Omega(Z)\}$ is finite for each $n \in \mathbb{N}$. Then there exists an element $(z_0,0) \in Z$ such that $\{\omega(z_0,0) : \omega \in \Omega(Z)\}$ is equal to $\{0,1,\frac{1}{2},\frac{2}{3},\frac{3}{4},...\}$ or $\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\}$.

Proof. It follows form Lemma 2.6 and Lemma 2.7 that Z is an infinite dimensional real unital Abelian Banach algebra with $\Omega(Z) \neq \emptyset$ and homeomorphic to $C_{\mathbb{R}}\Omega(Z)$. Suppose that there exists a bounded sequence $\{(y_n, 0)\}$ in Z which contains no convergent subsequences and such that $\{\omega(y_n, 0) : \omega \in \Omega(Z)\}$ is finite, for each $n \in \mathbb{N}$. From the proof of Lemma 2.10 (ii) in [?], we obtain that

$$\Omega(Z) = (\cup_{n\in\mathbb{N}} G_n) \cup F,$$

where F is a closed set in $\Omega(Z)$, G_n is closed and open, for each $n \in \mathbb{N}$, and $\{F, G_1, G_2, ...\}$ is a partition of $\Omega(Z)$. We first note that the restriction $\tau_\infty|_Z$ of the canonical homomorphism $\tau_\infty \in \Omega(\widetilde{X})$ on Z is a character on Z. There are two cases to be considered. If $\tau_\infty|_Z$ is in F, define $\psi : \Omega(Z) \to \mathbb{R}$ by

$$\psi(\tau) = \begin{cases} 1, & \text{if } \tau \in G_1, \\ \frac{1}{n}, & \text{if } \tau \in G_n, n \geq 2, \\ 0, & \text{if } \tau \in F. \end{cases}$$

If $\tau_\infty|_Z$ is in G_{n_0}, for some $n_0 \in \mathbb{N}$, we may assume without loss of generality that $n_0 = 1$, define $\psi : \Omega(Z) \to \mathbb{R}$ by

$$\psi(\tau) = \begin{cases} 0, & \text{if } \tau \in G_1, \\ \frac{n-1}{n}, & \text{if } \tau \in G_n, n \geq 2, \\ 1, & \text{if } \tau \in F. \end{cases}$$

For each case, the inverse image of each closed set in $\psi(\Omega(Z))$ is closed, so $\psi \in C(\Omega(Z))$. Let $\varphi : Z \to C(\Omega(Z))$ be the Gelfand representation. Therefore, $\varphi^{-1}(\psi)$ is an element in Z, say (z_0, λ), such that $\{\omega(z_0, \lambda) : \omega \in \Omega(\widetilde{X})\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. Moreover, $\lambda = 0$ since $\tau_\infty|_Z(z_0, \lambda) = \psi(\tau_\infty|_Z) = 0$. ∎

Lemma 2.10. *Let X be an infinite dimensional complex non-unital Banach algebra satisfying (A) and*

$$\inf\{\|\widehat{x}\|_\infty : x \in X, \|x\| = 1\} > 0,$$

and let $(x_0, 0)$ be an element in $\widetilde{X}$ with infinite spectrum and $\omega(x_0, 0) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$. Define

$$Z = \overline{\left\{ \sum_{i=0}^{k} \alpha_i (x_0, 0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}.$$

Then there exists a sequence $\{(z_n, 0)\}$ in Z such that $\{\tau(z_n, 0) : \tau \in \Omega(Z)\} \subset [0, 1]$, for each $n \in \mathbb{N}$, and $\{\widehat{(z_n, 0)}^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(\widetilde{Z})$.

Proof. From Lemma 2.7 and the proof of Lemma 2.10 (iii) in [5], there exists $(z_1, \lambda_1) \in Z$ such that $\{\omega(z_1, \lambda_1) : \omega \in \Omega(Z)\}$ is infinite. So $\sigma(z_1, \lambda_1)$ is infinite. Using Lemma 2.8, we may assume without generality that (z_1, λ_1) satisfies

$$1 \in \sigma(z_1, \lambda_1) \subset [0, 1]$$

and there exists a strictly decreasing sequence of real number in $\sigma(z_1, \lambda_1)$, say $\{a_n\}$. Moreover, we may as well assume that $a_1 < 1$.

Define a continuous function $g_1 : [0, 1] \to [0, 1]$ by

$$g_1(t) = \begin{cases} \frac{t}{a_1}, & \text{if } t \in [0, a_1], \\ 1 + \frac{(g_1(a_2)-1)(t-a_1)}{2(1-a_1)}, & \text{if } t \in [a_1, 1]. \end{cases}$$

So g_1 is joining the points $(0,0)$ and $(a_1,1)$, and $g_1(1) \in (g_1(a_2),1)$.

Let $\widehat{(z_2,\lambda_2)} = g_1 \circ \widehat{(z_1,\lambda_1)}$, and define a continuous function $g_2 : [0,1] \to [0,1]$ by

$$g_2(t) = \begin{cases} \frac{t}{g_1(a_2)}, & \text{if } t \in [0, g_1(a_2)], \\ 1 + \frac{(g_2(g_1(a_3))-1)(t-g_1(a_2))}{2(1-g_1(a_2))}, & \text{if } t \in [g_1(a_2), 1]. \end{cases}$$

So g_2 is joining the point $(0,0)$ and $(g_1(a_2),1)$ and $g_2(1) \in (g_2(g_1(a_3)),1)$.

Let $\widehat{(z_3,\lambda_3)} = g_2 \circ \widehat{(z_2,\lambda_2)}$. Continuing in this manner, we get a sequence of points $\{(z_n,\lambda_n)\}$ in Z with $1 \in \{\omega(z_n,\lambda_n) : \omega \in \Omega(Z)\} \subset [0,1]$, for each $n \in \mathbb{N}$, and $\{\widehat{(z_n,\lambda_n)}^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(Z)$.

Let $\{(z_{n_k},\lambda_{n_k})\}$ be a subsequence of $\{(z_n,\lambda_n)\}$ such that $\lambda_{n_k} \neq 1$, for each $n_k \in \mathbb{N}$. It can be seen that $\{(z_{n_k},0)\}$ is the sequence in Z such that

$$\{\omega(z_{n_k},0) : \tau \in \Omega(Z)\} \subset [0,1],$$

for each $n_k \in \mathbb{N}$, and $\{\widehat{(z_{n_k},0)}^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(Z)$. Indeed, $\lambda_n = 1$ and $\{\widehat{(z_n,\lambda_n)}^{-1}\{1\}\}$ is singleton implies $\{\widehat{(z_n,\lambda_n)}^{-1}\{1\}\} = \emptyset$. ∎

Lemma 2.11. *Let X be an infinite dimensional complex non-unital Abelian Banach algebra, let*

$$Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i (x_0,0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}},$$

and let $(x,0) \in Z \cap X$ with $\widehat{(x,0)}^{-1}\{1\} \neq \emptyset$, and $0 \leq \omega(x,0) \leq 1$, for each $\omega \in \Omega(Z)$. Define

$$E = \{(z,0) \in Z : 0 \leq \omega(z,0) \leq 1, \text{ for each } \omega \in \Omega(Z), \text{ and } \omega(z,0) = 1 \text{ if } \omega \in A\},$$

where $A = \widehat{(x,0)}^{-1}\{1\}$, and define $T : E \to E$ by

$$(z,0) \mapsto (xz,0).$$

Assume that X satisfies the following condition:
If $x,y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$.
Then E is a nonempty bounded closed convex subset of $Z \cap X$ and $T : E \to E$ is a nonexpansive mapping.

Proof. It is easy to see that E is closed and convex. We can deduce that E is nonempty since $(x,0) \in E$.

Let $(z,0) \in E$. It follows that

$$|\omega(z,0)| \leq 1 = |\omega(0,1)|,$$

for each $\omega \in \Omega(Z)$. Therefore,

$$\|(z,0)\| \leq \|(0,1)\| = 1.$$

Thus, it suffices to conclude that E is bounded.

Let $\omega \in \Omega(X)$ and let $(z,0),(z',0) \in E$. We have

$$\begin{aligned} |\omega(T(z,0) - T(z',0))| &= |\omega((x,0)(z,0) - (x,0)(z',0))|, \\ &= |\omega(x,0)||\omega((z,0) - (z',0))|, \\ &\leq |\omega((z,0) - (z',0))|. \end{aligned}$$

Then

$$\|T(z,0)-T(z',0)\| \leq \|(z,0)-(z',0)\|.$$

So T is nonexpansive. ∎

3. Main Result

Now, we prove the main result in this paper.

Theorem 3.1. *Let X be an infinite dimensional complex non-unital Abelian Banach algebra satisfying condition (A) and the following conditions:*
(i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$,
(ii) $\inf\{\|\hat{x}\|_\infty : x \in X, \| x \| = 1\} > 0$. Then there exists an element $(x_0, 0)$ in X such that

$$\langle x_0, 0\rangle_{\mathbb{R}} = \overline{\left\{\sum_{i=1}^{k} \alpha_i (x_0,0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}$$

does not have the fixed point property.

Proof. Let X be an infinite dimensional complex non-unital Abelian Banach algebra satisfying (i), (ii), and condition (A). It follows from Lemma 2.1 and 2.4 that there is an element $(x_0, 0)$ in $\widetilde{X}$ with infinite spectrum and $\omega(x_0, 0) \in \mathbb{R}$, for each $\omega \in \Omega(\widetilde{X})$. Let

$$Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i (x_0,0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}.$$

Applying Lemma 2.6, Z is an infinite dimensional real unital Abelian Banach algebra with $\Omega(Z) \neq \emptyset$. From Lemma 2.10, it follows that there exists a sequence $\{(z_n, 0)\}$ in Z such that $\{\omega(z_n, 0) : \omega \in \Omega(Z)\} \subset [0, 1]$, for each $n \in \mathbb{N}$, and $\widehat{(z_1,0)}^{-1}\{1\}$, $\widehat{(z_2,0)}^{-1}\{1\}$, $\widehat{(z_3,0)}^{-1}\{1\}, \ldots$ are pairwise disjoint.

Write $A_n = \widehat{(z_n,0)}^{-1}\{1\}$, define

$$E_n = \{(z,0) \in Z : 0 \leq \omega(z,0) \leq 1, \text{ for each } \omega \in \Omega(Z), \text{ and } \omega(z,0) = 1 \text{ if } \omega \in A_n\},$$

and define $T_n : E_n \to E_n$ by

$$(z,0) \mapsto (z_n z, 0).$$

Using Lemma 2.11, E_n is a bounded closed convex subset in Z and T_n is nonexpansive, for each $n \in \mathbb{N}$.

Suppose, on the contrary that $\langle x_0, 0\rangle_{\mathbb{R}}$ has fixed point property. For each $n \in \mathbb{N}$, since E_n is also a bounded closed convex subset in X, so T_n has a fixed point in E_n, say $(y_n, 0)$. Since $(y_n, 0)$ is a fixed point of T_n, so $(y_n, 0) = (z_n y_n, 0)$. Then $\widehat{(y_n,0)} = \widehat{(z_n,0)}\widehat{(y_n,0)}$, and then

$$\widehat{(y_n,0)}(\omega) = \begin{cases} 0, & \text{if } \omega \text{ is not in } A_n, \\ 1, & \text{if } \omega \text{ is in } A_n, \end{cases}$$

for each $n \in \mathbb{N}$. Since $A_1, A_2, A_3, \ldots$ are pairwise disjoint, so $\|\widehat{(y_m,0)} - \widehat{(y_n,0)}\| = 1$, if $m \neq n$. Thus $\{\widehat{(y_n,0)}\}$ has no convergent subsequences. Since Z and $C_{\mathbb{R}}(\Omega(Z))$ are homeomorphic, so $\{(y_n, 0)\}$ has no convergent subsequences. From Lemma 2.9, there is an element $(z_0, 0)$ in Z such that $\{\omega(z_0, 0) : \omega \in \Omega(Z)\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

Write $A_0 = \widehat{(z_0,0)}^{-1}\{1\}$, define

$$E_0 = \{(z,0) \in Z : 0 \le \omega(z,0) \le 1, \text{ for each } \omega \in \Omega(Z), \text{ and } \omega(z,0) = 1 \text{ if } \omega \in A_0\},$$

and define $T_0 : E_0 \to E_0$ by

$$(z,0) \mapsto (z_0 z, 0).$$

It follows from Lemma 2.11 that T_0 is a nonexpansive mapping on a bounded closed convex subset E_0 in X. So T_0 has a fixed point in E_0, say $(y_0,0)$. There are two cases to be considered.

Case(1) $\{\omega(z_0,0) : \omega \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$:

Hence $\widehat{(y_0,0)} = \widehat{(z_0,0)}\widehat{(y_0,0)}$. Then

$$\widehat{(y_0,0)}(\omega) = \begin{cases} 0, & \text{if } \omega \text{ is not in } A_0, \\ 1, & \text{if } \omega \text{ is in } A_0. \end{cases}$$

So

$$A_0 = \widehat{(y_0,0)}^{-1}\{1\} = \widehat{(z_0,0)}^{-1}\{1\}$$

and

$$\Omega(Z)\backslash A_0 = \widehat{(y_0,0)}^{-1}\{0\} = \bigcup_{n=0}^{\infty} \left(\widehat{(z_0,0)}^{-1}\{\frac{n}{n+1}\} \right).$$

It follows from

$$\{\omega(z_0,0) : \omega \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$$

that $\left\{ \widehat{(z_0,0)}^{-1}\{\frac{n}{n+1}\} : n \in \mathbb{N} \right\} \bigcup \left\{ \widehat{(z_0,0)}^{-1}\{0\} \right\}$ is a pairwise disjoint open covering of the compact set $\Omega(Z)\backslash A_0$, which is a contradiction.

Case(2) $\{\omega(z_0,0) : \omega \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$:

$$E = \{(z,1) \in Z : 0 \le \omega(z,1) \le 1, \text{ for each } \omega \in \Omega(Z), \text{ and } \omega(z,1) = 1 \text{ if } \omega \in A\},$$

where $A = \widehat{(-z_0,1)}^{-1}\{1\}$.
It can be seen that E is a bounded closed convex subset of Z.

Define $T : E \to E$ by

$$(z,1) \mapsto (-z_0,1)(z,1),$$

for each $(z,1) \in E$. We have

$$\{\omega(-z_0,1) : \omega \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}.$$

Define $S : Z \to Z$ by

$$(z,\lambda) \mapsto (-z, 1-\lambda).$$

It follows from (i) that $STS : S(E) \to S(E)$ is a nonexpansive mapping on a bounded closed convex subset $S(E)$ of $\langle x_0, 0\rangle_{\mathbb{R}}$. So then $T = S(STS)S$ is a nonexpansive mapping on E.

STS has a fixed point, since $\langle x_0, 0\rangle_{\mathbb{R}}$ has the fixed point property. It follows that T has a fixed point, say $(y_0,1)$. Then

$$\widehat{(y_0,1)}(\omega) = \begin{cases} 0, & \text{if } \omega \text{ is not in } A, \\ 1, & \text{if } \omega \text{ is in } A, \end{cases}$$

and

$$\widehat{(y_0,1)}^{-1}\{1\} = \widehat{(-z_0,1)}^{-1}\{1\} = A.$$

So

$$\Omega(Z)\backslash A = \widehat{(y_0,\lambda_0)}^{-1}\{0\} = \bigcup_{n=0}^{\infty}\left(\widehat{(-z_0,1)}^{-1}\{\frac{n}{n+1}\}\right).$$

It follows from

$$\{\omega(-z_0,1) : \omega \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$$

that $\left\{\widehat{(-z_0,1)}^{-1}\{\frac{n}{n+1}\} : n \in \mathbb{N}\right\} \bigcup \left\{\widehat{(-z_0,1)}^{-1}\{0\}\right\}$ is a pairwise disjoint open covering of the compact set $\Omega(Z)\backslash A$, which is a contradiction. So we can dedude that $\langle x_0, 0\rangle_{\mathbb{R}}$ does not have the fixed point property. ■

From the proof of the above theorem, we can show the following corollary.

Corollary 3.2. *Let X be an infinite dimensional complex non-unital Abelian Banach algebra satisfying condition (A) and the following conditions:*
(i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$,
(ii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$.
If $(x_0, 0)$ is an element in X with infinite spectrum and $\sigma(x_0, 0) \subset \mathbb{R}$, then the Banach algebra

$$\langle x_0, 0\rangle = \overline{\left\{\sum_{i=1}^{k} \alpha_i (x_0, 0)^i : k \in \mathbb{N}, \alpha_i \in \mathbb{C}\right\}}$$

generated by $(x_0, 0)$ does not have the fixed point property.

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