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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Global Minimization of Common Best Proximity Points for Generalized Cyclic φ-Contractions in Metric Spaces

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Abstract This paper proves the existence of a common best proximity point of generalized cyclic φ-contraction mappings in a complete metric space with property UC. Moreover, some example and numerical experiments are also given to support our main result.

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Keywords: global minimization; common best proximity point; generalized cyclic φ-contraction; property UC

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1. Introduction

Fixed point theory has been of great interest among mathematicians. One of the most famous work of Stefan Banach, known as Banach contraction principle, has influenced many researchers to develop the theory in different aspects. One may study the theory in a more general setting or even adjust the behavior of the mappings in order to establish fixed point results; see [1], [2] and [3], for example. Research trends have not only focused on the existence of fixed points, but also shifted to related notions such as common fixed points, coincidence points, and best proximity points, to mention but a few; see [4], [5] and [6]. It is also known that theory regarding best proximity points, in particular, can be applied in economics; see [7] and [8]. Pirbavafa and Vaezpour, [8], have recently shown that the existence of equilibrium pairs in free abstract economies can be guaranteed by best proximity points.

Given a system of nonlinear equations of the form $Sx = x$ and $Tx = x$, a solution to the system may not be necessarily achieved, when S and T are not self-mappings. That is, in a metric space with metric d, such an inconsistent system results $d(x, Sx) > 0$ or

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$d(x, Tx) > 0$ for all x. It is then natural to seek a distinguished point x at which the errors $d(x, Sx)$ and $d(x, Tx)$ are minimized. Let us now assume that S and T are mappings between nonempty subsets A and B. Then, it is obvious that $d(A, B) \leq d(x, Sx)$ and $d(A, B) \leq d(x, Tx)$ for all $x \in A$, where $d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$. To this end, one can obtain the common minimum if there exists an $x_0 \in A$ such that

$$d(x_0, Sx_0) = \min_{x \in A} d(x, Sx) = d(A, B) = \min_{x \in A} d(x, Tx) = d(x_0, Tx_0).$$

Such x_0 is called a *common best proximity point* of S and T. In other words, $d(x_0, Sx_0)$ and $d(x_0, Tx_0)$ are globally minimized at $x = x_0$.

Existence of common best proximity points, in general, depends on conditions on the mappings and the structure of the studied spaces. A number of publications are devoted for the existence of common best proximity points through various classes of generalized contractions. In 2009, Al-Thagafi and Shahzad [9] introduced a new class of mappings, called cyclic φ-contractions, and proved best proximity point theorems in a complete metric space. In 2016, Sadiq Basha and Shahzad [10] introduced the notion of a generalized cyclic contraction and proved common best proximity point theorems for the aforesaid mappings in a complete metric space with property UC.

Motivated and inspired by [9] and [10], this paper aims to present another class of mappings, generalized cyclic φ-contractions, and prove the existence of a common best proximity point in a complete metric space with property UC.

It is worth mentioning that not only has existence of common best proximity points been studied theoretically, their convergence has also drawn attention of many authors in optimization theory. The reader may be referred to [11–15] for more details.

2. Mathematical Preliminaries

Given two nonempty subsets A and B of a metric space, the following notions and notations are used in the sequel.

Definition 2.1. An element $x^* \in A$ is said to be a *common best proximity point* of the non-self-mappings $S, T : A \to B$ if

$$d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*).$$

Definition 2.2. A Banach space X is said to be *strictly convex* if

$$\|(x + y)/2\| < 1$$

for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

Definition 2.3. A Banach space X is said to be *uniformly convex* if for any $\varepsilon, 0 < \varepsilon \leq 2$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the inequalities $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply

$$\|(x + y)/2\| \leq 1 - \delta.$$

Definition 2.4. [16] The pair (A, B) is said to satisfy the property UC if

$$\left.\begin{array}{l} d(x_n, y_n) \to d(A, B) \\ d(x'_n, y_n) \to d(A, B) \end{array}\right\} \Longrightarrow d(x_n, x'_n) \to 0$$

for all sequences $\{x_n\}$ and $\{x'_n\}$ in A and for every sequence $\{y_n\}$ in B.

The following examples provide some scenarios where the property UC holds.

Example 2.5. [16] Let A and B be nonempty subsets of a metric space such that $d(A, B) = 0$. Then (A, B) satisfies the property UC.

Example 2.6. [16] Let A, A', B and B' be nonempty subsets of a metric space such that $A \subseteq A'$, $B \subseteq B'$ and $d(A, B) = d(A', B')$. If (A', B') satisfies the property UC, so does (A, B).

Example 2.7. [16] Let A and B be nonempty subsets of a strictly convex Banach space. Assume that A is convex and relatively compact, and the closure of B is weakly compact. Then (A, B) has the property UC.

Example 2.8. [17] Let A and B be nonempty subsets of a uniformly convex Banach space. Assume that A is convex. Then (A, B) has the property UC.

3. Result for Generalized Cyclic φ-Contractions

We recall some basic notions which will be used in our main result.

Definition 3.1. A mapping $T : A \cup B \to A \cup B$ is said to be a *cyclic mapping* if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Definition 3.2. [18] A cyclic mapping $T : A \cup B \to A \cup B$ is called a *cyclic contraction* if there exists a nonnegative real number $\alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B)$$

for all $x \in A$ and $y \in B$.

Definition 3.3. [9] A cyclic mapping $T : A \cup B \to A \cup B$ is called a *cyclic φ-contraction* if $\varphi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing mapping and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$.

Definition 3.4. [10] Let $S, T : A \cup B \to A \cup B$ be cyclic mappings. Then S is said to be a *T-cyclic contraction* (or a *generalized cyclic contraction*) if there exists a nonnegative real number $\alpha < 1$ such that

$$d(Sx, Sy) \leq \alpha d(Tx, Ty) + (1 - \alpha)d(A, B)$$

for all $x \in A$ and $y \in B$.

Next, we introduce the concept of a generalized cyclic φ-contraction.

Definition 3.5. Let $S, T : A \cup B \to A \cup B$ be cyclic mappings. Then S is said to be a *T-cyclic φ-contraction* (or a *generalized cyclic φ-contraction*) if $\varphi : [0, +\infty) \to [0, +\infty)$ is strictly increasing and

$$d(Sx, Sy) \leq d(Tx, Ty) - \varphi(d(Tx, Ty)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$.

Remark 3.6. A generalized cyclic contraction is a generalized cyclic φ-contraction with $\varphi(x) = (1 - \alpha)x$ for $x \geq 0$ and $0 \leq \alpha < 1$.

The following lemma will be used in the proof of our main result.

Lemma 3.7. [16] *Let A and B be nonempty subsets of a metric space. Assume that (A, B) has the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_n) = d(A, B) \quad \textit{or} \quad \lim_{n\to\infty} \sup_{m\geq n} d(x_m, y_n) = d(A, B).$$

holds. Then $\{x_n\}$ is a Cauchy sequence.

We first prove some useful property of common best proximity points of two cyclic mappings.

Proposition 3.8. *Let A and B be nonempty subsets of a metric space such that (A, B) and (B, A) satisfy the property UC. Let $S, T : A \cup B \to A \cup B$ be cyclic mappings. Suppose that S is a T-cyclic φ-contraction. If $x \in A$ and $y \in B$ are common best proximity points of S and T, then*

$$d(x, y) = d(A, B).$$

Proof. Since x is a common best proximity point of S and T, we get

$$d(x, Sx) = d(x, Tx) = d(A, B).$$

As (B, A) satisfies the property UC, we have $Sx = Tx$. Similarly, we also get $Sy = Ty$. Because the mapping S is a T-cyclic φ-contraction, we have

$$\begin{aligned} d(Sx, Sy) &\leq d(Tx, Ty) - \varphi(d(Tx, Ty)) + \varphi(d(A, B)) \\ &= d(Sx, Sy) - \varphi(d(Sx, Sy)) + \varphi(d(A, B)) \end{aligned}$$

which yields that $\varphi(d(Sx, Sy)) \leq \varphi(d(A, B))$. Since φ is strictly increasing, it follows that

$$d(Sx, Sy) = d(A, B).$$

Because (A, B) satisfies the property UC, we get $Sy = x$. Similarly, it can be shown that $Sx = y$. Therefore, we can conclude that

$$d(x, y) = d(A, B).$$

This completes the proof. ∎

The following result is a key result on the existence of a common best proximity point for a generalized cyclic φ-contraction.

Theorem 3.9. *Let A and B be nonempty subsets of a complete metric space such that B is closed. Assume that (A, B) and (B, A) satisfy the property UC. Let $S, T : A \cup B \to A \cup B$ be cyclic mappings satisfying the following conditions:*
(i) $S(A) \subseteq T(A)$ and $S(B) \subseteq T(B)$;
(ii) S is a T-cyclic φ-contraction;
(iii) S and T commute, that is, $ST = TS$;
(iv) T is continuous;
(v) φ is lower semi-continuons.
Then there exist a unique element $x \in A$ and a unique element $y \in B$ such that

$$d(x, Sx) = d(x, Tx) = d(A, B) = d(y, Sy) = d(y, Ty).$$

Moreover, such best proximity points x and y satisfy the condition that

$$d(x, y) = d(A, B).$$

Proof. Let x_0 be a fixed element in A. Since $S(A) \subseteq T(A)$, then there exists an element $x_1 \in A$ such that $Sx_0 = Tx_1$. Given $x_n \in A$, it is possible to choose an element $x_{n+1} \in A$ such that $Sx_n = Tx_{n+1}$, since $S(A) \subseteq T(A)$. Because the mapping S is a T-cyclic φ-contraction, we have

$$d(Sx_n, SSx_n) \leq d(Tx_n, TSx_n) - \varphi(d(Tx_n, TSx_n)) + \varphi(d(A,B)).$$

As S and T commute, we have $d(Tx_n, TSx_n) = d(Sx_{n-1}, SSx_{n-1})$. Thus, we get

$$d(Sx_n, SSx_n) \leq d(Sx_{n-1}, SSx_{n-1}) - \varphi(d(Sx_{n-1}, SSx_{n-1})) + \varphi(d(A,B)), \quad (3.1)$$

so, $d(Sx_n, SSx_n) + \varphi(d(Sx_{n-1}, SSx_{n-1})) \leq d(Sx_{n-1}, SSx_{n-1}) + \varphi(d(A,B))$.
Since $\varphi(d(A,B)) \leq \varphi(d(Sx_{n-1}, SSx_{n-1}))$, and above inequality implies that $d(Sx_n, SSx_n) \leq d(Sx_{n-1}, SSx_{n-1})$. Therefore, $\lim_{n\to\infty} d(Sx_n, SSx_n)$ exists. We also have that $\{\varphi(d(Sx_n, SSx_n))\}_{n=1}^{\infty}$ is non-increasing, so $\lim_{n\to\infty} \varphi(d(Sx_n, SSx_n))$ exists. Taking $n \to \infty$ in (3.1), we can conclude that $\lim_{n\to\infty} \varphi(d(Sx_n, SSx_n)) = \varphi(d(A,B))$.
Since φ is strictly increasing, it follows that

$$d(Sx_n, SSx_n) \to d(A,B) \quad \text{as} \quad n \to \infty.$$

Using the same proof as above, we can show that

$$d(Sx_{n+1}, SSx_n) \to d(A,B) \quad \text{as} \quad n \to \infty.$$

As (B, A) satisfies the property UC, we have

$$d(Sx_m, Sx_{m+1}) \to 0 \quad \text{as} \quad m \to \infty.$$

Further, given $\varepsilon > 0$. Choose a positive integer m_0 such that

$$d(Sx_m, SSx_m) \leq d(A,B) + \frac{\varepsilon}{2} \qquad \text{for} \quad m \geq m_0,$$

$$d(Sx_m, Sx_{m+1}) \leq \frac{\varepsilon}{2} \qquad \text{for} \quad m \geq m_0.$$

Fix m such that $m \geq m_0$. It can be established easily by induction on n that

$$d(Sx_m, SSx_n) \leq d(A,B) + \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$ with $n \geq m$. It is obvious that the preceding condition holds when $n = m$. Assume that the condition holds for some $n \geq m$. Then, we have

$$\begin{aligned} d(Sx_m, SSx_{n+1}) &\leq d(Sx_m, Sx_{m+1}) + d(Sx_{m+1}, SSx_{n+1}) \\ &\leq d(Sx_m, Sx_{m+1}) + d(Tx_{m+1}, TSx_{n+1}) \\ &- \varphi(d(Tx_{m+1}, TSx_{n+1})) + \varphi(d(A,B)) \\ &= d(Sx_m, Sx_{m+1}) + d(Sx_m, SSx_n) - \varphi(d(Sx_m, SSx_n)) \\ &+ \varphi(d(A,B)) \\ &\leq \frac{\varepsilon}{2} + d(A,B) + \frac{\varepsilon}{2} - \varphi(d(Sx_m, SSx_n)) + \varphi(d(A,B)) \\ &\leq d(A,B) + \varepsilon. \end{aligned}$$

Hence, the aforesaid condition holds for $n + 1$. Therefore,

$$\lim_{m\to\infty} \sup_{n\geq m} d(Sx_m, SSx_n) = d(A,B).$$

By Lemma 3.7, $\{Sx_n\}$ is a Cauchy sequence, so it converges to some $y \in B$. Thus, $Sx_n \to y$ and whence $Tx_n \to y$. By the continuity of T, we get $TSx_n \to Ty$ and $TTx_n \to Ty$. Since S and T commute, we have $STx_n \to Ty$. By (ii), we have

$$d(STx_n, Sx_n) \leq d(TTx_n, Tx_n) - \varphi(d(TTx_n, Tx_n)) + \varphi(d(A, B))$$

which implies

$$d(STx_n, Sx_n) + \varphi(d(TTx_n, Tx_n)) \leq d(TTx_n, Tx_n) + \varphi(d(A, B)).$$

Taking the limit inferior yields

$$d(Ty, y) + \liminf_{n\to\infty} \varphi(d(TTx_n, Tx_n)) \leq d(Ty, y) + \varphi(d(A, B)).$$

Thus, $\liminf_{n\to\infty} \varphi(d(TTx_n, Tx_n)) \leq \varphi(d(A, B))$. By the lower semi-continuity of φ, we get

$$\varphi(d(y, Ty)) \leq \liminf_{n\to\infty} \varphi(d(TTx_n, Tx_n)).$$

It follows that $\varphi(d(y, Ty)) = \varphi(d(A, B))$. Since φ is strictly increasing, we get

$$d(y, Ty) = d(A, B).$$

On the other hand,

$$\begin{aligned} d(Sy, Sx_n) &\leq d(Ty, Tx_n) - \varphi(d(Ty, Tx_n)) + \varphi(d(A, B)) \\ &\leq d(Ty, Tx_n). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$d(y, Sy) = d(A, B).$$

As (A, B) satisfies the property UC, we obtain $Sy = Ty$. Put $x := Sy = Ty$. Because the mapping S is a T-cyclic φ-contraction, we get

$$d(x, Sx) = d(Sy, Sx) \leq d(Ty, Tx) - \varphi(d(Ty, Tx)) + \varphi(d(A, B)).$$

In view of the fact that S and T commute, we have

$$d(Ty, Tx) = d(Ty, STy) = d(x, Sx).$$

This implies by above inequality that $\varphi(d(x, Sx)) \leq \varphi(d(A, B))$. Since φ is strictly increasing, it follows that

$$d(x, Sx) = d(A, B),$$

and hence we have

$$d(x, Tx) = d(Ty, Tx) = d(A, B).$$

To show the uniqueness, let $z \in B$ be a common best proximity point of S and T. Then

$$d(z, Sz) = d(z, Tz) = d(A, B).$$

As (A, B) satisfies the property UC, we have $Sz = Tz$. Similarly, it can be shown that $Sx = Tx$. Because the mapping S is a T-cyclic φ-contraction, we get

$$\begin{aligned} d(Sx, Sz) &\leq d(Tx, Tz) - \varphi(d(Tx, Tz)) + \varphi(d(A, B)) \\ &= d(Sx, Sz) - \varphi(d(Sx, Sz)) + \varphi(d(A, B)). \end{aligned}$$

It follows that

$$d(Sx, Sz) = d(A, B).$$

Since (B, A) satisfies the property UC, it follows that $Sx = z$. Similarly, we also obtain $Sx = y$. Therefore, y and z are identical. That is, y is a unique common best proximity point of S and T in B. Similarly, it can be shown that x is a unique common best proximity point of S and T in A. Moreover, by Proposition 3.8, we can conclude that

$$d(x, y) = d(A, B).$$

This completes the proof. ∎

The following result is directly obtained by Theorem 3.9 and Remark 3.6.

Corollary 3.10. [10] *Let A and B be nonempty subsets of a complete metric space such that B is closed. Assume that (A, B) and (B, A) satisfy the property UC. Let $S, T : A \cup B \to A \cup B$ be cyclic mappings satisfying the following conditions:*
(i) $S(A) \subseteq T(A)$ and $S(B) \subseteq T(B)$;
(ii) S is a T-cyclic contraction;
(iii) S and T commute;
(iv) T is continuous.
Then there exist a unique element $x \in A$ and a unique element $y \in B$ such that

$$d(x, Sx) = d(x, Tx) = d(A, B) = d(y, Sy) = d(y, Ty).$$

Moreover, such best proximity points x and y satisfy the condition that

$$d(x, y) = d(A, B).$$

As a consequence of Theorem 3.9, we obtain the following common fixed point theorem.

Corollary 3.11. *Let A be nonempty closed subsets of a complete metric space. Let $S, T : A \to A$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing and lower semi-continuons mapping satisfying the following conditions:*
(i) $S(A) \subseteq T(A)$;
(ii) S and T commute;
(iii) T is continuous;
(iv) there is an $\alpha \in [0, 1)$ such that

$$d(Sx, Sy) \leq d(Tx, Ty) - \varphi(d(Tx, Ty))$$

for all $x, y \in A$.
Then S and T have a unique common fixed point.

The following example illustrates the previous theorem.

Example 3.12. Consider $\mathbb{R}^2$ with the metric

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Let $A = (-\infty, -1] \times (-\infty, 0]$ and $B = [1, \infty) \times (-\infty, 0]$. Let $S, T : A \cup B \to A \cup B$ be defined as

$$S((x, y)) = (-\sqrt[3]{x}, e^y - 1) \text{ and } T((x, y)) = (-x^3, y).$$

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined as

$$\varphi(x) = \begin{cases} \frac{x}{5}, & \text{if } x \in [0, 3], \\ \sqrt{x}, & \text{if } x \in (3, \infty). \end{cases}$$

It is obvious that (A, B) and (B, A) satisfy the property UC and above functions satisfy $(i) - (v)$ in our main theorem. Choose $x_0 = (-10, -10)$. Let $\{x_n\}$ be a sequence in A generated by $y_n \equiv Sx_n = Tx_{n+1}$. We obtain the following numerical experiments for common best proximity points of S and T.

n	x_n	y_n	$d(x_n, y_n)$	$d(x_n, (-1, 0))$	$d(y_n, (1, 0))$
0	(-10.00000,-10.00000)	(2.15443,-0.99995)	15.12386	13.45362	1.52729
1	(-1.29155,-0.99995)	(1.08902,-0.63210)	2.40883	1.04159	0.63834
2	(-1.02883,-0.63210)	(1.00952,-0.46853)	2.04491	0.63276	0.46862
3	(-1.00316,-0.46853)	(1.00105,-0.37408)	2.00644	0.46854	0.37408
4	(-1.00035,-0.37408)	(1.00012,-0.31208)	2.00143	0.37408	0.31208
⋮	⋮	⋮	⋮	⋮	⋮
300	(-1.00000,-0.00661)	(1.00000,-0.00659)	2.00000	0.00661	0.00659

TABLE 1. Numerical experiments of Example 3.12.

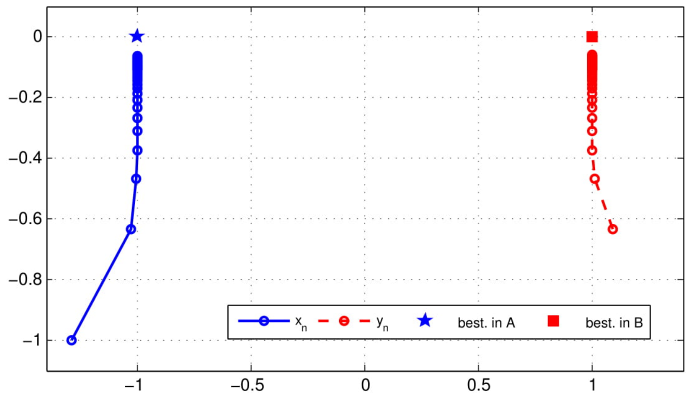

FIGURE 1. $\{x_n\}$ and $\{y_n\}$ of Example 3.12.

Observe that $(-1, 0)$ is a unique common best proximity point of S and T in A and $(1, 0)$ is a unique common best proximity point of S and T in B, and $d(A, B) = 2$.

In the next theorem, we improve the conditions of the previous theorem by replacing the continuity assumption with other type of continuity.

Definition 3.13. Let A and B be nonempty subsets of a metric space. A cyclic mapping $T : A \cup B \to A \cup B$ is said to be *relatively continuous* at a point $x \in A$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < d(A, B) + \delta \Longrightarrow d(Tx, Ty) < d(A, B) + \varepsilon$$

for all $y \in B$. Similarly, one can define relative continuity at a point in B. If T is relatively continuous at each and every point of its domain, then it is simply said to be relatively continuous.

Theorem 3.14. *Let A and B be nonempty closed subsets of a complete metric space. Assume that (A, B) and (B, A) satisfy the property UC. Let $S, T : A \cup B \to A \cup B$ be cyclic mappings satisfying the following conditions:*
(i) $S(A) \subseteq T(A)$ and $S(B) \subseteq T(B)$;
(ii) S is a T-cyclic φ-contraction;
(iii) S and T commute;
(iv) T is relatively continuous;

(v) φ is lower semi-continuons.
Then there exist a unique element $x \in A$ and a unique element $y \in B$ such that

$$d(x, Sx) = d(x, Tx) = d(A, B) = d(y, Sy) = d(y, Ty).$$

Moreover, such best proximity points x and y satisfy the condition that

$$d(x, y) = d(A, B).$$

Proof. Let $\{x_n\}$ be a sequence defined as in Theorem 3.9, that is, $Sx_n = Tx_{n+1}$ for all n. The same as in Theorem 3.9, we have that

$$Sx_n \to y \text{ and } Tx_n \to y$$

for some $y \in B$. Similarly, one can define a sequence $\{y_n\}$ in B such that $Sy_n = Ty_{n+1}$ for all n and

$$Sy_n \to x \text{ and } Ty_n \to x$$

for some $x \in A$. Because the mapping S is a T-cyclic φ-contraction, we obtain

$$d(Sx_n, Sy_n) \leq d(Tx_n, Ty_n) - \varphi(d(Tx_n, Ty_n)) + \varphi(d(A, B)).$$

Taking the limit inferior yields

$$\liminf_{n\to\infty} \varphi(d(Tx_n, Ty_n)) \leq \varphi(d(A, B)).$$

By the lower semi-continuity of φ, we have

$$\varphi(d(y, x)) \leq \liminf_{n\to\infty} \varphi(d(Tx_n, Ty_n)).$$

It follows that $\varphi(d(y, x)) = \varphi(d(A, B))$. Since φ is strictly increasing, we get

$$d(x, y) = d(A, B).$$

In view of the facts that T is relatively continuous and S is a T-cyclic φ-contraction, it follows that S is also relatively continuous. Since $d(x, Tx_n) \to d(A, B)$ and S is relatively continuous, we have

$$d(Sx, STx_n) \to d(A, B).$$

Because T is relatively continuous and $d(x, Sx_n) \to d(A, B)$, we have $d(Tx, TSx_n) \to d(A, B)$. As S and T commute, we get

$$d(Tx, STx_n) \to d(A, B).$$

Since (A, B) satisfies the property UC, it can be concluded that $Sx = Tx$. Because the mapping S is a T-cyclic φ-contraction, we obtain

$$d(Sx, Sy_n) \leq d(Tx, Ty_n) - \varphi(d(Tx, Ty_n)) + \varphi(d(A, B)).$$

Taking the limit inferior yields

$$\liminf_{n\to\infty} \varphi(d(Tx, Ty_n)) \leq \varphi(d(A, B)).$$

This implies by the lower semi-continuity of φ that $\varphi(d(Tx, x)) = \varphi(d(A, B))$, hence

$$d(Sx, x) = d(Tx, x) = d(A, B),$$

and so x is a common best proximity point of S and T in A. Using the same proof as above, we can show that y is also a common best proximity point of S and T in B and

$$d(x, y) = d(A, B).$$

Proceeding as in the proof of Theorem 3.9, it can be shown that the best proximity point is unique. This completes the proof. ■

The following result of Sadiq Basha and Shahzad [10] is a consequence of Theorem 3.14.

Corollary 3.15. [10] *Let A and B be nonempty closed subsets of a complete metric space. Assume that (A, B) and (B, A) satisfy the property UC. Let $S, T : A \cup B \to A \cup B$ be cyclic mappings satisfying the following conditions:*
(i) $S(A) \subseteq T(A)$ and $S(B) \subseteq T(B)$;
(ii) S is a T-cyclic contraction;
(iii) S and T commute;
(iv) T is relatively continuous.
Then there exist a unique element $x \in A$ and a unique element $y \in B$ such that

$$d(x, Sx) = d(x, Tx) = d(A, B) = d(y, Sy) = d(y, Ty).$$

Moreover, such best proximity points x and y satisfy the condition

$$d(x, y) = d(A, B).$$

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