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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Best Proximity Coincidence Point Theorem for G-Proximal Generalized Geraghty Mapping in a Metric Space with Graph G

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Abstract In this work, we present a result on the existence of a best proximity coincidence point of a pair of mappings that is a G-proximal generalized Geraghty mapping in a complete metric space endowed with a directed graph G. Furthermore, if any pair of the two best proximity coincidence points is an edge of the graph G, then the best proximity coincidence point is unique. In addition, an example is given to support the main theorem. Finally, we provide some consequences of the theorem for the special cases of the mapping.

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1. Introduction

In the study of the best proximity point theorems, several sufficient conditions have been suggested to guarantee the existence of an approximate optimal solution to the best proximity point problem. This has been done in a very different approach, for example, see [1–7]. Many results were inspired by the Banach contraction principle for the existence theorem of a fixed point for a self mapping in a metric space. To extend this concept, Geraghty [8] defined a contractive mapping based on the class of mappings $\theta : [0,\infty) \to [0,1)$ such that

$$\lim_{n\to\infty} \theta(t_n) = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} t_n = 0.$$

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In 2013, Biligili, Karapinar, and Sadarangani [9] were interested to find the best proximity point for a pair (A, B) of subsets of a metric space for Geraghty-contractions with the P-property, which is first introduced by Raj [10]. After that, Komal, Kumam, Khammahawong and Sitthithakerngkiet [11] considered the existence theorem for the best proximity coincidence point for α-Geraghty contraction mappings in a complete metric space with the P-property. Later, Zhang and Su [12] introduced the use of the weak P-property and improved the work of Geraghty for the non-self contractions. Moreover, motivated by Geraghty, Ayari [13] defined a class of mappings $\beta : [0, \infty) \to [0, 1]$ such that

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} t_n = 0.$$

With this class, Ayari accomplished the existence and uniqueness results on a best proximity point for α-proximal Geraghty non-self mappings on a closed subset of a complete metric space.

In addition, there are some research for other mappings. For instance, Bunlue and Suantai [14], and Sarnmeta and Suantai [15] investigated the existence of a best proximity point theorem for some nonexpansive mappings. Furthermore, Bunlue and Suantai [16] constructed the hybrid algorithms for common best proximity points of some nonexpansive mappings. On top of that, there are more work on solving global minimization problems involving best proximity points in Hilbert spaces, e.g. [17, 18].

Besides, there are another approach applying graph theory to fixed point theory. Jachymski [19] investigated this concept for the fixed point problems on a metric space. To be more specific, he proposed the Banach contraction principle for mappings on a metric space endowed with a graph. This led to numerous work on fixed points theorems for mappings defined on some space with a graph, for instance, see [20–23].

Recently, Klanarong and Suantai [24] defined a G-proximal generalized contraction in a complete metric space endowed with a graph G and obtained the best proximity point theorems for such contraction. Inspired by above work, we are interested in finding a best proximity coincidence point for a G-proximal generalized Geraghty mapping defined on a closed subset of a complete metric space endowed with a directed graph G.

2. Preliminaries and Definitions

Let us begin with some definitions, which will be used for the rest of the paper. Assume that (X, d) is a metric space and A, B are nonempty subsets of X. Define the following notations.

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\};$$
$$A_0 := \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\};$$
$$B_0 := \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

Then the concept of best proximity is given below.

Definition 2.1. [1, 7] Let $T : A \to B$ and $g : A \to A$ be mappings.

(1) An element $x^* \in A$ is said to be a **best proximity point** of T if

$$d(x^*, Tx^*) = d(A, B).$$

(2) An element $x^* \in A$ is said to be a **best proximity coincidence point** of the pair (T, g) if

$$d(gx^*, Tx^*) = d(A, B).$$

Definition 2.2. [12] Assume that A_0 is nonempty. The pair (A, B) is said to have the **weak P-property** if

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \text{ implies } d(x_1, x_2) \leq d(y_1, y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Next, the idea of a metric space endowed with a directed graph is defined as follows.

Definition 2.3. [19] Let Δ be the diagonal of $X \times X$. A metric space (X, d) is said to be **endowed with a directed graph** G if $G = (V(G), E(G))$ is a directed graph such that the vertex set $V(G)$ consists of all elements in X and the edge set $E(G)$ contains the diagonal Δ of $X \times X$.

In this work, we assume that $E(G)$ contains no parallel edges.

Definition 2.4. [19] Let (X, d) be a metric space endowed with a directed graph G.

(1) A function $f : X \to X$ is said to be **G-continuous at** $x \in X$ if for any sequence $\{x_n\} \in X$ such that $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$,

$$x_n \to x \text{ implies } fx_n \to fx.$$

In addition, f is said to be **G-continuous** if G-continuous at every $x \in X$.

(2) The edge set $E(G)$ is said to have the **transitivity property** if

$$(x, z), (z, y) \in E(G) \text{ implies } (x, y) \in E(G),$$

where $x, y, z \in X$.

3. Main Results

In this section, the main theorem is stated. First, the following definitions are introduced.

Definition 3.1. Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$. Given that $T : A \to B$ and $g : A \to A$ are mappings, T is said to be **G-proximal edge preserving with respect to** g if for any $x_1, x_2, u_1, u_2 \in A$ such that $(x_1, x_2) \in E(G)$,

$$d(gu_1, Tx_1) = d(gu_2, Tx_2) = d(A, B) \text{ implies } (u_1, u_2) \in E(G).$$

Definition 3.2. Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$. Given that $T : A \to B$ and $g : A \to A$ are mappings, the pair (T, g) is said to be a **G-proximal generalized Geraghty mapping** if the following statements hold.

(1) T is G-proximal edge preserving with respect to g.

(2) There exists $\beta \in \mathcal{B}$ such that for any $x, y \in A$ with $(x, y) \in E(G)$,

$$d(Tx, Ty) \leq \beta(d(gx, gy))d(gx, gy).$$

Theorem 3.3. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G)$ has a transitive property. Assume that (A, B) is a pair of nonempty closed subsets of X that has the weak P-property. Define $T : A \to B$ and $g : A \to A$ be mappings such that g is an isometry and the pair (T, g) is a G-proximal generalized Geraghty mapping. Suppose that all of the following hold.*

(i) $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.

(ii) There exist $x, y \in A_0$ such that $d(gx, Ty) = d(A, B)$ and $(y, x) \in E(G)$.

(iii) Either (a) or (b) is true;

(a) T is G-continuous on A.

(b) For any sequence $\{x_n\}$ in A such that $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, if $x_n \to x^*$ for some $x^* \in A$, then there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)}, x^*) \in E(G)$ for each $k \in \mathbb{N}$.

Then (T, g) has a best proximity coincidence point. Moreover, if $(x^*, y^*) \in E(G)$ for any best proximity coincidence points $x^*, y^* \in A$, then (T, g) has a unique best proximity coincidence point.

Proof. From assumption (ii), there are $x_0, x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B) \text{ and } (x_0, x_1) \in E(G).$$

Then, by assumption (i), there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B) \text{ and } (x_1, x_2) \in E(G)$$

since T is G-proximal edge preserving with respect to g. Continuing this procedure, we obtain a sequence $\{x_n\}$ in A_0 such that for each $n \in \mathbb{N}$,

$$d(gx_n, Tx_{n-1}) = d(A, B) \text{ and } (x_{n-1}, x_n) \in E(G). \tag{3.1}$$

According to the fact that (A, B) has the weak P-property, we can conclude that

$$d(gx_n, gx_{n+1}) \leq d(Tx_{n-1}, Tx_n) \text{ for each } n \in \mathbb{N}.$$

Next, we claim that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

Since g is an isometry and (T, g) is a G-proximal generalized Geraghty mapping, for each $n \in \mathbb{N}$, we have that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(gx_n, gx_{n+1}) \\ &\leq d(Tx_{n-1}, Tx_n) \\ &\leq \beta(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n) \\ &= \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n). \end{aligned} \tag{3.2}$$

This is, the sequence $\{d(x_{n-1}, x_n)\}$ is nonincreasing. As a consequence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = r.$$

Suppose that $r > 0$. From (3.2), we obtain the following inequalities by taking the limit as n approaches ∞.

$$1 \leq \lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) \leq 1.$$

Thus $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$. Then, by the definition of β, the limit $\lim_{n\to\infty} d(x_{n-1}, x_n)$ must be zero. This is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{3.3}$$

Now, we will show that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon_0 > 0$ such that there are subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geq k$ for each $k \in \mathbb{N}$, we have that

$$d(x_{n(k)}, x_{m(k)}) \geq \epsilon_0. \tag{3.4}$$

Additionally, we can choose the smallest $m(k)$ satisfying (3.4) for each $k \in \mathbb{N}$ so that

$$d(x_{n(k)}, x_{m(k)-1}) < \epsilon_0.$$

By the triangle inequality, for each $k \in \mathbb{N}$, we get that

$$\begin{aligned}\epsilon_0 &\leq d(x_{n(k)}, x_{m(k)})\\ &\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})\\ &< \epsilon_0 + d(x_{m(k)-1}, x_{m(k)}).\end{aligned}$$

From (3.3), by taking the limit as k approaches ∞, we obtain that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon_0. \tag{3.5}$$

Since $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ are subsequences of $\{x_n\}$, by (3.1), for each $k \in \mathbb{N}$,

$$d(gx_{n(k)+1}, Tx_{n(k)}) = d(A, B) \text{ and } d(gx_{m(k)+1}, Tx_{m(k)}) = d(A, B).$$

Applying the weak P-property of (A, B), we have that

$$d(gx_{n(k)+1}, gx_{m(k)+1}) \leq d(Tx_{n(k)}, Tx_{m(k)}).$$

From (3.1), $(x_{n(k)}, x_{n(k)+1}) \in E(G)$ for each $k \in \mathbb{N}$. By the transitivity property of $E(G)$, it is easy to see that $(x_{n(k)}, x_{m(k)}) \in E(G)$. Since (T, g) is a G-proximal generalized Geraghty mapping, consider that

$$\begin{aligned}d(x_{n(k)+1}, x_{m(k)+1}) &= d(gx_{n(k)+1}, gx_{m(k)+1})\\ &\leq d(Tx_{n(k)}, Tx_{m(k)})\\ &\leq \beta(d(gx_{n(k)}, gx_{m(k)}))d(gx_{n(k)}, gx_{m(k)})\\ &= \beta(d(x_{n(k)}, x_{m(k)}))d(x_{n(k)}, x_{m(k)})\\ &\leq d(x_{n(k)}, x_{m(k)}).\end{aligned}$$

Similarly, by (3.5), we can conclude that

$$1 \leq \lim_{k\to\infty} \beta(d(x_{n(k)}, x_{m(k)})) \leq 1,$$

that is,

$$\lim_{k\to\infty} \beta(d(x_{n(k)}, x_{m(k)})) = 1.$$

Consequently,

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = 0.$$

This contradicts (3.5) because ϵ_0 is positive. Thus, it follows that $\{x_n\}$ is a Cauchy sequence in the closed subset A of the complete metric space (X, d). Then there exists $x^* \in A$ such that $\lim_{n\to\infty} x_n = x^*$.

Now suppose that the condition (a) holds. Since $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, by the G-continuity of T and the continuity of g on A, we can conclude that

$$\lim_{n\to\infty} Tx_n = Tx^* \text{ and } \lim_{n\to\infty} gx_n = gx^*.$$

Therefore,

$$\lim_{n\to\infty} d(gx_{n+1}, Tx_n) = d(gx^*, Tx^*).$$

From (3.1), we have that

$$\lim_{n\to\infty} d(gx_{n+1}, Tx_n) = d(A, B).$$

Thus $d(gx^*, Tx^*) = d(A, B)$ because the limit must be unique.

On the other hand, suppose that the condition (b) holds. Then there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)}, x^*) \in E(G)$ for all $k \in \mathbb{N}$. Since (T, g) is a G-proximal generalized Geraghty mapping, we have that

$$\begin{aligned} d(Tx_{n(k)}, Tx^*) &\leq \beta(d(gx_{n(k)}, gx^*))d(gx_{n(k)}, gx^*) \\ &= \beta(d(x_{n(k)}, x^*))d(x_{n(k)}, x^*) \\ &\leq d(x_{n(k)}, x^*). \end{aligned} \tag{3.6}$$

By the triangular inequality, consider that

$$\begin{aligned} d(gx^*, Tx^*) &\leq d(gx^*, gx_{n(k)+1}) + d(gx_{n(k)+1}, Tx_{n(k)}) + d(Tx_{n(k)}, Tx^*) \\ &= d(x^*, x_{n(k)+1}) + d(gx_{n(k)+1}, Tx_{n(k)}) + d(Tx_{n(k)}, Tx^*). \end{aligned}$$

Then

$$d(gx^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(gx_{n(k)+1}, Tx_{n(k)}) \leq d(Tx_{n(k)}, Tx^*).$$

From (3.6), it follows that

$$d(gx^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(gx_{n(k)+1}, Tx_{n(k)}) \leq d(x_{n(k)}, x^*).$$

Taking the limit as k approaches ∞, by (3.1), we obtain

$$d(gx^*, Tx^*) - d(A, B) \leq 0 \text{ and so, } d(gx^*, Tx^*) \leq d(A, B).$$

Note that since $gx^* \in A$ and $Tx^* \in B$, we have that $d(A, B) \leq d(gx^*, Tx^*)$. Then we can conclude that $d(gx^*, Tx^*) = d(A, B)$.

Moreover, let x^* and y^* be any two best proximity coincidence points of (T, g) such that $(x^*, y^*) \in E(G)$. That is,

$$d(gx^*, Tx^*) = d(gy^*, Ty^*) = d(A, B).$$

By the weak P-property of (A, B), we have that

$$d(gx^*, gy^*) \leq d(Tx^*, Ty^*).$$

Since g is an isometry and (T, g) is a G-proximal generalized Geraghty mapping, it follows that

$$\begin{aligned} d(x^*, y^*) &= d(gx^*, gy^*) \\ &\leq d(Tx^*, Ty^*) \\ &\leq \beta(d(gx^*, gy^*))d(gx^*, gy^*) \\ &= \beta(d(x^*, y^*))d(x^*, y^*) \\ &\leq d(x^*, y^*). \end{aligned}$$

If $d(x^*, y^*) > 0$, then $\beta(d(x^*, y^*)) = 1$, and, by the definition of β, $d(x^*, y^*) = 0$, a contradiction. Therefore, $d(x^*, y^*) = 0$ which implies that $x^* = y^*$. Hence, (T, g) has a unique best proximity coincidence point if $(x^*, y^*) \in E(G)$ for any best proximity coincidence points $x^*, y^* \in A$. ∎

Example 3.4. Let $X = \mathbb{R}^3$ be a complete metric space with the metric d given by

$$d((x, y, z), (u, v, w)) = \sqrt{(x-u)^2 + (y-v)^2 + (z-w)^2},$$

where $(x, y, z), (u, v, w) \in \mathbb{R}^3$. Let

$$A = \{(3, 4, z) : 0 \le z \le 6\} \text{ and } B = \{(-3, -4, z) : 0 \le z \le 3\}$$

be the closed subsets of $\mathbb{R}^3$. It is straightforward to show that the pair (A, B) has the weak P-property with $d(A, B) = 10$. Then

$$A_0 = \{(3, 4, z) : 0 \le z \le 3\} \text{ and } B_0 = \{(-3, -4, z) : 0 \le z \le 3\}.$$

Define $T : A \to B$ to be a mapping such that

$$T(3, 4, z) = (-3, -4, \ln(z + 1)) \text{ for } 0 \le z \le 6,$$

and define g to be the identity mapping on A. Then

$$T(A_0) = \{(-3, -4, z) : 0 \le z \le \ln 4\} \subseteq B_0.$$

Next, let $G = (V(G), E(G))$ be a directed graph with $V(G) = X$ and

$$E(G) = \{((x, y, z), (u, v, w)) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \le u, y \ge v \text{ and } z \ge w\}.$$

It is easy to see that $E(G)$ has the transitive property. Moreover, we have that T is G-continuous on A. In addition, some tedious manipulation yields that the condition (ii) in Theorem 3.3 is satisfied.

Now it remains to show that (T, g) is a G-proximal generalized Geraghty mapping. First, we need to prove that T is G-proximal edge preserving with respect to g. Let $(3, 4, x), (3, 4, y), (3, 4, u), (3, 4, v) \in A$ such that $((3, 4, x), (3, 4, y)) \in E(G)$ and

$$d(g(3, 4, u), T(3, 4, x)) = d(g(3, 4, v), T(3, 4, y)) = d(A, B).$$

That is,

$$d((3, 4, u), (-3, -4, \ln(x + 1))) = d((3, 4, v), (-3, -4, \ln(y + 1))).$$

Then

$$u = \ln(x + 1) \text{ and } v = \ln(y + 1).$$

Since $x \ge y$, it implies that $u \ge v$, and so $((3, 4, u), (3, 4, v)) \in E(G)$. Thus we can conclude that T is G-proximal edge preserving with respect to g. Next, define a mapping $\beta : [0, \infty) \to [0, 1]$ to be $\beta(t) = \dfrac{\ln(1 + t)}{t}$ for $t \ne 0$ and $\beta(0) = 1$. Let $(3, 4, x), (3, 4, y) \in A$ such that $((3, 4, x), (3, 4, y)) \in E(G)$, i.e., $x \ge y$. If $x = y$, then we are done. Suppose that $x > y$. Consider that

$$\begin{aligned} d(T(3,4,x),T(3,4,y)) &= d((-3,-4,\ln(x+1)),(-3,-4,\ln(y+1))) \\ &= |\ln(x+1) - \ln(y+1)| \\ &= \left|\ln\left(\frac{x+1}{y+1}\right)\right| \\ &= \left|\ln\left(1+\frac{x-y}{y+1}\right)\right| \\ &\leq \ln(1+|x-y|) \\ &= \frac{\ln(1+|x-y|)}{|x-y|}|x-y| \\ &= \beta(d(g(3,4,x),g(3,4,y)))d(g(3,4,x),g(3,4,y)). \end{aligned}$$

Therefore, T is a G-proximal generalized Geraghty mapping. By Theorem 3.3, (T,g) has a best proximity coincidence point in A. In fact, $(3,4,0)$ is a best proximity coincidence point of (T,g).

4. Consequence

As a result of our main theorem, we are able to obtain some corollaries. First, we give some definitions.

Definition 4.1. Let (X,d) be a metric space endowed with a directed graph G such that $V(G) = X$. Given that $T : A \to B$ and $g : A \to A$ are mappings, the pair (T,g) is said to be a **G-proximal generalized mapping** if the following statements hold.

(1) T is G-proximal edge preserving with respect to g.
(2) There exists $k \in [0,1)$ such that for any $x,y \in A$ with $(x,y) \in E(G)$,
$$d(Tx,Ty) \leq kd(gx,gy).$$

By applying Theorem 3.3 with $\beta(t) = k$ for $k \in [0,1)$, we obtain the first corollary as follows.

Corollary 4.2. *Let (X,d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G)$ has a transitive property. Assume that (A,B) is a pair of nonempty closed subsets of X that has the weak P-property. Define $T : A \to B$ and $g : A \to A$ be mappings such that g is an isometry and the pair (T,g) is a G-proximal generalized mapping. Suppose that all of the following hold.*

(i) $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.
(ii) There exist $x,y \in A_0$ such that $d(gx,Ty) = d(A,B)$ and $(y,x) \in E(G)$.
(iii) Either (a) or (b) is true;
(a) T is G-continuous on A.
(b) For any sequence $\{x_n\}$ in A such that $(x_n,x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, if $x_n \to x^*$ for some $x^* \in A$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x^*) \in E(G)$ for each $k \in \mathbb{N}$.

Then (T,g) has a best proximity coincidence point. Moreover, if $(x^*,y^*) \in E(G)$ for any best proximity coincidence points $x^*,y^* \in A$, then (T,g) has a unique best proximity coincidence point.

Definition 4.3. Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$. Given that $T : A \to B$ and $g : A \to A$ are mappings, the pair (T, g) is said to be a ***G*-proximal type R mapping** if the following statements hold.

(1) T is G-proximal edge preserving with respect to g.
(2) For any $x, y \in A$ with $(x, y) \in E(G)$,

$$d(Tx, Ty) \leq \frac{d(gx, gy)}{d(gx, gy) + 1}.$$

By applying Theorem 3.3 with $\beta(t) = \dfrac{1}{t+1}$ for $t \in [0, \infty)$, we obtain the second corollary as follows.

Corollary 4.4. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G)$ has a transitive property. Assume that (A, B) is a pair of nonempty closed subsets of X that has the weak P-property. Define $T : A \to B$ and $g : A \to A$ be mappings such that g is an isometry and the pair (T, g) is a G-proximal type R mapping. Suppose that all of the following hold.*

(i) $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.
(ii) There exist $x, y \in A_0$ such that $d(gx, Ty) = d(A, B)$ and $(y, x) \in E(G)$.
(iii) Either (a) or (b) is true;
  (a) T is G-continuous on A.
  (b) For any sequence $\{x_n\}$ in A such that $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, if $x_n \to x^*$ for some $x^* \in A$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x^*) \in E(G)$ for each $k \in \mathbb{N}$.

Then (T, g) has a best proximity coincidence point. Moreover, if $(x^*, y^*) \in E(G)$ for any best proximity coincidence points $x^*, y^* \in A$, then (T, g) has a unique best proximity coincidence point.

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