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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Approximation of Fixed Points for a Class of Generalized Nonexpansive Mappings in Banach Spaces

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Abstract Recently, Patir et al. [B. Patir, N. Goswami, V.N. Mishra, Some results on fixed point theory for a class of generalized nonexpansive mappings, Fixed Point Theory Appl. (2018)] introduced new class of generalized nonexpansive mappings which is a new condition on mappings called condition $B_{\gamma,\mu}$. They studied some existences and convergence theorems for such class of mappings. This new class of mappings is important because it contains the class of Suzuki mappings and hence the class of nonexpansive mappings. In this paper, we further studied this new class of mappings and as a result some new convergence theorems are established using up-to-date iteration process of Hussain et al. [N. Hussain, K. Ullah, M. Arshad, Fixed point approximation of Suzuki generalized nonexpansive mappings via new faster iteration process, J. Nonlinear Convex Anal. 19 (8) (2018) 1383–1393].

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1. Introduction

A mapping T on a subset D of a Banach space E is called contraction if and only if there is a real number $r \in [0,1)$ such that

$$||Tu - Tv|| \leq r||u - v||, \text{ for all } u, v \in D. \tag{1.1}$$

If (1.1) is hold at $r = 1$, then T is called nonexpansive. A point $p \in D$ is called a fixed point for T if and only if $Tp = p$. Throughtout the work, the notation $fix(T)$ will

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represent the set $\{p \in D : Tp = p\}$. The mapping T is called quasi-nonexpansive if and only if for each $p \in fix(T)$ and $u \in D$, we have

$$||Tu - Tp|| \leq ||u - p||.$$

It is now well-known that the set $fix(T)$ is nonempty if T acting on nonempty closed bounded convex subset of a uniformly convex Banach space (see, Browder [1], Gohde [2] and Kirk [3]).

In [4] Suzuki introduced a new class of mappings (which is the direct extension of the class of nonexpansive mappings). A mapping $T : D \to D$ is said to be Suzuki mapping (or satisfy condition (C)) if and only if

$$\frac{1}{2}||u - Tu|| \leq ||u - v|| \;\Rightarrow\; ||Tu - Tv|| \leq ||u - v||,$$

for each $u, v \in D$.

Recently in 2018, Patir et al. [5] introduced new condition on mappings called condition $B_{\gamma,\mu}$. A mapping $T : D \to D$ is said to satisfy condition $B_{\gamma,\mu}$ (or Patir mapping) if and only if there exists $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ satisfying $2\mu \leq \gamma$ such that for each $u, v \in D$,

$$\gamma||u - Tu|| \leq ||u - v|| + \mu||v - Tv||$$
$$\text{implies } ||Tu - Tv|| \leq (1 - \gamma)||u - v|| + \mu(||u - Tv|| + ||v - Tu||).$$

They also showed that, this new class of mappings is larger than the class of Suzuki mappings.

Example 1.1. [5] Let $D = [0, 2]$. Set T as follow:

$$Tu = \begin{cases} 0 & \text{if } \; u \neq 2 \\ 1 & \text{if } \; u = 2. \end{cases}$$

Here T satisfies condition $B_{\gamma,\mu}$, but does not condition (C).

For fixed points investigation of contraction, nonexpansive and generalized nonexpansive mappings, we often use the well-known Picard [6], Mann [7], Ishikawa [8], S [9], Noor [10], Abbas [11], SP [12], S* [13], CR [14], Normal-S [15], Picard-Mann hybrid [16], Picard-S [17] and Thakur et al. [18] iterative processes. For more details and some recent literature on iteration processes, we refer the reader to [19–25]

The Picard iteration process [6] is defined as follow:

$$\left.\begin{array}{l} u_1 \in D, \\ u_{n+1} = Tu_n, n \geq 1, \end{array}\right\} \tag{1.2}$$

The Mann iteration process [7] reads as follow:

$$\left.\begin{array}{l} u_1 \in D, \\ u_{n+1} = (1 - a_n)u_n + a_nTu_n, n \geq 1, \end{array}\right\} \tag{1.3}$$

where $a_n \in (0, 1)$.

The Ishikawa iterative process [8] is the extension of Mann iterative process [7] from one-step to two-steps:

$$\left.\begin{array}{l} u_1 \in D, \\ v_n = (1 - b_n)u_n + b_nTu_n, \\ u_{n+1} = (1 - a_n)u_n + a_nTv_n, n \geq 1, \end{array}\right\} \tag{1.4}$$

where $a_n, b_n \in (0, 1)$.

In 2007, Agarwal et al. [9] introduced the following new two-step iteration process known as S iteration process, which converges faster than Picard, Mann and Ishikawa iterations:

$$\left.\begin{aligned} &u_1 \in D,\\ &v_n = (1-b_n)u_n + b_n T u_n,\\ &u_{n+1} = (1-a_n)Tu_n + a_n T v_n, n \geq 1, \end{aligned}\right\} \tag{1.5}$$

where $a_n, b_n \in (0,1)$.

In 2014, Gursoy and Karakaya [17] introduced Picard-S hybrid iteration process, which converges faster than all Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas and Normal-S iteration process:

$$\left.\begin{aligned} &u_1 \in D,\\ &w_n = (1-b_n)u_n + b_n T u_n,\\ &v_n = (1-a_n)Tu_n + a_n T w_n,\\ &u_{n+1} = T v_n, n \geq 1, \end{aligned}\right\} \tag{1.6}$$

where $a_n, b_n \in (0,1)$.

In 2016, Thakur et al. [18] used a new iteration process. With the help of an numerical example, they proved that this new process is faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration processes:

$$\left.\begin{aligned} &u_1 \in D,\\ &w_n = (1-b_n)u_n + b_n T u_n,\\ &v_n = T\left((1-a_n)u_n + a_n w_n\right),\\ &u_{n+1} = T v_n, n \geq 1, \end{aligned}\right\} \tag{1.7}$$

where $a_n, b_n \in (0,1)$.

Remark 1.2. The rate of convergence of iteration process (1.6) and (1.7) is almost same.

Recently in 2018, Hussain et al. [26] proposed the following new iteration process so-called K iteration process:

$$\left.\begin{aligned} &u_1 \in D,\\ &w_n = (1-b_n)u_n + b_n T u_n,\\ &v_n = T((1-a_n)Tu_n + a_n T w_n),\\ &u_{n+1} = T v_n, n \geq 1, \end{aligned}\right\} \tag{1.8}$$

where $a_n, b_n \in (0,1)$.

They proved some weak and strong convergence results of K iteration process for the class of Suzuki generalized nonexpansive mappings. Also, they proved numerically that K iteration process is better than the leading three-step Picard-S and leading two-step S iteration process.

In this article, we extend their results to the more general formulation of Patir generalized nonexpansive mappings (mappings with $B_{\gamma,\mu}$ condition).

2. Preliminaries

Let D be a nonempty subset of a Banach space E and $\{x_n\}$ a bounded sequence in E. For each $u \in E$ define:

(1) asymptotic radius of $\{u_n\}$ at u by $A_r(u, \{u_n\}) := \limsup_{n\to\infty} ||u - u_n||$;
(2) asymptotic radius of $\{u_n\}$ relative to D by $A_r(D, \{u_n\}) = \inf\{A_r(u, \{u_n\}) : u \in D\}$;
(3) asymptotic center of $\{u_n\}$ relative to D by $A_c(D, \{u_n\}) = \{u \in D : A_r(u, \{u_n\}) = A_r(D, \{u_n\})\}$.

When the space E is uniformly convex [27], then the set $A_c(D, \{u_n\})$ is always singleton. Notice also, the set $A_c(D, \{u_n\})$ is convex as well as nonempty provided that D is weakly compact convex, (see e.g., [28, 29]).

We say that, a Banach space E has the Opial's property if and only if for all $\{u_n\}$ in E which weakly converges to $u \in E$ and for every $v \in E$-$\{u\}$, one has $\limsup_{n\to\infty} ||u_n - u|| < \limsup_{n\to\infty} ||u_n - v||$.

Proposition 2.1. [5] *Let D be a nonempty subset of a Banach space E having Opial property. Let $T : D \to D$ satisfies the condition $B_{\gamma,\mu}$. If p is a fixed point of $T : D \to D$, then for each $u \in D$*

$$||Tp - Tu|| \leq ||p - u||.$$

From Proposition 2.1, we obtain the following facts.

Lemma 2.2. *Let D be a nonempty subset of a Banach space E. Let $T : D \to D$ satisfies the condition $B_{\gamma,\mu}$. Then the set $fix(T)$ is closed. Moreover, if E is strictly convex and D is convex then $fix(T)$ is also convex.*

Theorem 2.3. [5] *Let D be a nonempty subset of a Banach space E having Opial property. Let $T : D \to D$ satisfies the condition $B_{\gamma,\mu}$. If $\{u_n\}$ is sequence in E such that*

(i) $\{u_n\}$ converges weakly to h,
(ii) $\lim_{n\to\infty} ||Tu_n - u_n|| = 0$,
then $Th = h$.

Proposition 2.4. [5] *Let D be a nonempty subset of a Banach space E. Let $T : D \to D$ satisfies the condition $B_{\gamma,\mu}$. Then, for all $u, v \in D$ and $c \in [0, 1]$,*

(i) $||Tu - T^2u|| \leq ||u - Tu||$,
(ii) at least one of the following ((a) and (b)) holds:
(a) $\frac{c}{2}||u - Tu|| \leq ||u - v||$
(b) $\frac{c}{2}||Tu - T^2u|| \leq ||Tu - v||$.
The condition (a) implies $||Tu-Tv|| \leq (1-\frac{c}{2})||u-v||+\mu(||u-Tv||+||v-Tu||)$ and condition (b) implies $||T^2u-Tv|| \leq (1-\frac{c}{2})||Tu-v||+\mu(||Tu-Tv||+||v-T^2u||)$.
(iii) $||u - Tv|| \leq (3 - c)||u - Tu|| + \left(1 - \frac{c}{2}\right) ||u - v|| + \mu(2||u - Tu|| + ||u - Tv|| + ||v - Tu|| + 2||Tu - T^2u||)$.

The following facts are in [30].

Lemma 2.5. *Let E be a UCBS and $0 < p \leq \xi_n \leq q < 1$ for every $n \geq 1$. If $\{u_n\}$ and $\{v_n\}$ are two sequences in E such that $\limsup_{n\to\infty} ||u_n|| \leq l$, $\limsup_{n\to\infty} ||v_n|| \leq l$ and $\lim_{n\to\infty} ||\xi_n u_n + (1 - \xi_n)v_n|| = l$ for some $a \geq 0$ then, $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

3. Main Results

Lemma 3.1. *Let D be a nonempty closed convex subset of a UCBS E and $T : D \to D$ be a mapping satisfying condition $B_{\gamma,\mu}$ with $fix(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence generated by (1.8), then $\lim_{n\to\infty} ||u_n - p||$ exists for all $p \in fix(T)$.*

Proof. Let $p \in fix(T)$. By Proposition 2.1, we have

$$\begin{aligned} ||w_n - p|| &= ||(1-b_n)u_n + b_n Tu_n - p|| \\ &\leq (1-b_n)||u_n - p|| + b_n||Tu_n - p|| \\ &\leq (1-b_n)||u_n - p|| + b_n||u_n - p|| \\ &\leq ||u_n - p||, \end{aligned}$$

and

$$\begin{aligned} ||v_n - p|| &= ||T((1-a_n)Tu_n + a_n Tw_n) - p|| \\ &\leq ||(1-a_n)Tu_n + a_n Tw_n - p|| \\ &\leq (1-a_n)||Tu_n - p|| + a_n||Tw_n - p|| \\ &\leq (1-a_n)||u_n - p|| + a_n||w_n - p||. \end{aligned}$$

They imply that

$$\begin{aligned} ||u_{n+1} - p|| &= ||Tv_n - p|| \leq ||v_n - p|| \\ &\leq (1-a_n)||v_n - p|| + a_n||w_n - p|| \\ &\leq (1-a_n)||u_n - p|| + a_n||u_n - p|| \\ &\leq ||u_n - p||. \end{aligned}$$

Thus $\{||u_n - p||\}$ is non-increasing and bounded, which implies that $\lim_{n\to\infty} ||u_n - p||$ exists for all $p \in fix(T)$. ∎

Theorem 3.2. *Let D be a nonempty closed convex subset of a UCBS E and $T : D \to D$ be a mapping satisfying condition $B_{\gamma,\mu}$. Let $\{u_n\}$ be a sequence generated by (1.8). Then, $fix(T) \neq \emptyset$ if and only if $\{u_n\}$ is bounded and $\lim_{n\to\infty} ||Tu_n - u_n|| = 0$.*

Proof. Let $fix(T) \neq \emptyset$ and $p \in fix(T)$. By Lemma 3.1, $\lim_{n\to\infty} ||u_n - p||$ exists and $\{u_n\}$ is bounded. Suppose

$$\lim_{n\to\infty} ||u_n - p|| = l. \tag{3.1}$$

By the proof of Lemma 3.1 together with (3.1), we have

$$\limsup_{n\to\infty} ||w_n - p|| \leq \limsup_{n\to\infty} ||u_n - p|| = l. \tag{3.2}$$

By Proposition 2.1, we have

$$\limsup_{n\to\infty} ||Tu_n - p|| \leq \limsup_{n\to\infty} ||u_n - p|| = l. \tag{3.3}$$

Again by the proof of Lemma 3.1, we have

$$||u_{n+1} - p|| \leq (1-a_n)||u_n - p|| + a_n||w_n - p||.$$

It follows that

$$||u_{n+1}-p||-||u_n-p|| \leq \frac{||u_{n+1}-p||-||u_n-p||}{a_n} \leq ||w_n-p||-||u_n-p||.$$

So, we can get $||u_{n+1}-p|| \leq ||w_n-p||$. Therefore,

$$l \leq \liminf_{n\to\infty} ||w_n-p||. \tag{3.4}$$

From (3.2) and (3.4), we obtain

$$l = \lim_{n\to\infty} ||w_n-p||. \tag{3.5}$$

From (3.5), we have

$$\begin{aligned} r &= \lim_{n\to\infty} ||w_n-p|| \\ &= \lim_{n\to\infty} ||(1-b_n)u_n + b_nTu_n - p|| \\ &= \lim_{n\to\infty} ||(1-b_n)(u_n-p) + b_n(Tu_n-p)||. \end{aligned}$$

Hence,

$$l = \lim_{n\to\infty} ||(1-b_n)(u_n-p) + b_n(Tu_n-p)||. \tag{3.6}$$

Now from (3.1), (3.3) and (3.6) together with Lemma 2.5, we obtain

$$\lim_{n\to\infty} ||Tu_n-u_n|| = 0.$$

Conversely, let $p \in A_c(D,\{u_n\})$. By Proposition 2.4(iii), for $\gamma = \frac{c}{2}$, $c \in [0,1]$,

$$\begin{aligned} ||u_n-Tp|| &\leq (3-c)||u_n-Tu_n|| + \left(1-\frac{c}{2}\right)||u_n-p|| + \mu(2||u_n-Tu_n|| \\ &+ ||u_n-Tp|| + ||p-Tu_n|| + 2||Tu_n-T^2u_n||) \\ &\leq (3-c)||u_n-Tu_n|| + \left(1-\frac{c}{2}\right)||u_n-p|| + \mu(2||u_n-Tu_n|| \\ &+ ||u_n-Tp|| + ||u_n-p|| + ||u_n-Tu_n|| + 2||u_n-Tu_n||) \end{aligned}$$

(by Proposition 2.4(i))

$$\Rightarrow (1-\mu)\limsup_{n\to\infty} ||u_n-Tp|| \leq (1-\frac{c}{2}+\mu)\limsup_{n\to\infty} ||u_n-p||$$

$$\begin{aligned} \Rightarrow \limsup_{n\to\infty} ||u_n-Tp|| &\leq \left(\frac{1-\frac{c}{2}+\mu}{1-\mu}\right)\limsup_{n\to\infty} ||u_n-p|| \\ &\leq \limsup_{n\to\infty} ||u_n-p|| \end{aligned}$$

$$\left(\text{as } \frac{1-\frac{c}{2}+\mu}{1-\mu} \leq 1, \text{ for } 2\mu \leq \gamma = \frac{c}{2}\right)$$

$$\Rightarrow A_r(Tp,\{u_n\}) \leq A_r(p,\{u_n\}).$$

So $Tp \in A_c(D,\{u_n\})$. Since E is uniformly convex Banach space, $A_c(D,\{u_n\}$ is singleton. Hence $Tp = p$. ∎

Theorem 3.3. *Let D a nonempty closed convex subset of a UCBS E having the Opial property and $T : D \to D$ be a mapping satisfying condition $B_{\gamma,\mu}$ with $fix(T) \neq \emptyset$. Then, $\{u_n\}$ generated by (1.8) converges weakly to an element of $fix(T)$.*

Proof. By Theorem 3.2, $\{u_n\}$ is bounded and $\lim_{n\to\infty} ||Tu_n - u_n|| = 0$. Since B is UCBS, B is reflexive. Thus, we can find a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\{u_{n_j}\}$ converges weakly to some $h_1 \in D$. By Theorem 2.3, we obtain $h_1 \in fix(T)$. It is suffice to prove that $\{u_n\}$ converges weakly to h_1. Indeed, if $\{u_n\}$ does not converge weakly to h_1. Then, we can find a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $h_2 \in D$ such that $\{u_{n_k}\}$ converges weakly to h_2 and $h_2 \neq h_1$. Hence, $h_2 \in fix(T)$ by Theorem 2.3. Opial condition and Lemma 3.1, gives us

$$\begin{aligned}\lim_{n\to\infty} ||u_n - h_1|| = \lim_{j\to\infty} ||u_{n_j} - h_1|| &< \lim_{j\to\infty} ||u_{n_j} - h_2|| \\ = \lim_{n\to\infty} ||u_n - h_2|| &= \lim_{k\to\infty} ||u_{n_k} - h_2|| \\ < \lim_{k\to\infty} ||u_{n_k} - h_1|| &= \lim_{n\to\infty} ||u_n - h_1||.\end{aligned}$$

This is a contradiction. So, $h_1 = h_2$. ∎

Theorem 3.4. *Let D be a nonempty closed convex subset of a UCBS E and $T : D \to D$ be a mapping satisfying condition $B_{\gamma,\mu}$. If $fix(T) \neq \emptyset$ and $\liminf_{n\to\infty} dist(u_n, fix(T)) = 0$. Let $\{u_n\}$ be the sequence generated by (1.8). Then $\{u_n\}$ converges strongly to fixed point of T.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} ||u_n - p||$ exists, for all $p \in fix(T)$. So, $\lim_{n\to\infty} dist(u_n, fix(T))$ exists, thus

$$\lim_{n\to\infty} dist(u_n, fix(T)) = 0.$$

Thus, there exists subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $\{\omega_k\}$ in $fix(T)$ with $||u_{n_k} - \omega_k|| \leq \frac{1}{2^k}$. $k \geq 1$. Moreover, $\{u_n\}$ is nonincreasing by the proof of Lemma 3.1. Hence

$$||u_{n_{k+1}} - \omega_k|| \leq ||u_{n_k} - \omega_k|| \leq \frac{1}{2^k}.$$

We prove that $\{\omega_k\}$ is a Cauchy sequence in $fix(T)$.

$$\begin{aligned}||\omega_{k+1} - \omega_k|| &\leq ||\omega_{k+1} - u_{n_{k+1}}|| + ||u_{n_{k+1}} - \omega_k|| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \leq \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty.\end{aligned}$$

This shows that the sequence $\{\omega_k\}$ is Cauchy in $fix(T)$. Since, $fix(T)$ is closed by Lemma 2.2. Therefore, $\omega_n \to p$ for some $p \in fix(T)$. By Lemma 3.1, $\lim_{n\to\infty} ||u_n - p||$ exists. So the proof is finished. ∎

Finally we prove the following strong of $\{u_n\}$ with the help of condition (I). Recall that a self mapping T on a subset D of a Banach space is said to satisfy the condition (I) if and only if there exists a nondecreasing function $\pi : [0, \infty) \to [0, \infty)$ satisfying $\pi 0 = 0$ and $\pi t > 0$ for every $t > 0$ such that

$$||Tu - u|| \geq \pi dist(u, fix(T)) \text{ for all } u \in D.$$

Theorem 3.5. *Let D be a nonempty closed convex subset of a UCBS E and $T : D \to D$ be a mapping satisfying condition $B_{\gamma,\mu}$ with $fix(T) \neq \emptyset$. Let $\{u_n\}$ be the sequence generated by (1.8). If T satisfies condition (I) then $\{u_n\}$ converges strongly to the fixed point of T.*

Proof. From Theorem 3.2, it follows that

$$\liminf_{n\to\infty} ||Tu_n - u_n|| = 0.$$

By condition (I), we have

$$\liminf_{n\to\infty}(\text{dist}(u_n, fix(T))) = 0.$$

The conclusion follows from Theorem 3.4. ∎

4. Example

In this section, we construct a new example of Patir mapping which is not Suzuki.

Example 4.1. We consider the subset $D = [0, \infty)$ of real line and set T on D as follow:

$$Tu = \begin{cases} 0 & \text{if } u < \frac{1}{400} \\ \frac{u}{2} & \text{if } u \geq \frac{1}{400}. \end{cases}$$

Choose $u = \frac{1}{700}$ and $v = \frac{1}{400}$. We see that, $\frac{1}{2}|u - Tu| = \frac{1}{1400} < \frac{3}{2800} = |u - v|$ but $|Tu - Tv| = \frac{1}{800} > \frac{3}{2800} = |u - v|$. Thus T is not Suzuki mapping. Choose $\gamma = 1$ and $\mu = \frac{1}{2}$, we have

Case I: For $u, v < \frac{1}{400}$, we have

$$(1-\gamma)|u - v| + \mu(|u - Tv| + |v - Tu|) \geq 0 = |Tu - Tv|.$$

Case II: For $u, v \geq \frac{1}{400}$, we have

$$\begin{aligned}(1-\gamma)|v|+\mu(|u-Tv|+|v-Tu|) &= \frac{1}{2}(|u - Tv| + |v - Tu|) \\ &= \frac{1}{2}\left(\left|u - \frac{v}{2}\right| + \left|v - \frac{u}{2}\right|\right) \\ &\geq \frac{1}{2}\left(\left|\frac{3u}{2} - \frac{3v}{2}\right|\right) \\ &= \frac{3}{4}|u - v| \geq \frac{1}{2}|u - v| = |Tu - Tv|.\end{aligned}$$

Case III: For $u \geq \frac{1}{400}$ and $v < \frac{1}{400}$, we have

$$\begin{aligned}(1-\gamma)|u-v|+\mu(|u-Tv|+|v-Tu|) &= \frac{1}{2}(|u - Tv| + |v - Tu|) \\ &= \frac{1}{2}\left(|u| + \left|v - \frac{u}{2}\right|\right) \\ &= \frac{1}{2}|u| + \frac{1}{2}\left|v - \frac{u}{2}\right| \\ &\geq \frac{1}{2}|u| = |Tu - Tv|.\end{aligned}$$

Hence, T satisfies condition $B_{1,\frac{1}{2}}$. Moreover, $fix(T) = \{0\}$.

Take $\alpha_n = \frac{4}{5}$ and $\beta_n = \frac{1}{5}$ for $n \geq 1$. By choosing $u_1 = 4$, we may observe in the Table 1 as well as in Figure 1 the behavior of K iterates with the leadings three-step Picard-S and leading two step S iterates.

Table 1. Sequences generated by K, Picard-S and S iteration processes

	K	**Picard-S**	**S**
u_1	4	4	4
u_2	0.46000000000000	0.92000000000000	1.84000000000000
u_3	0.05290000000000	0.21160000000000	0.84640000000000
u_4	0.00608350000000	0.04866800000000	0.38934400000000
u_5	0	0.01119364000000	0.17909824000000
u_6	0	0.00257453720000	0.08238519040000
u_7	0	0	0.03789718758400
u_8	0	0	0.01743270628864
u_9	0	0	0.00801904489277
u_{10}	0	0	0.00368876065067
u_{11}	0	0	0.00169682989931
u_{12}	0	0	0
u_{13}	0	0	0
u_{14}	0	0	0

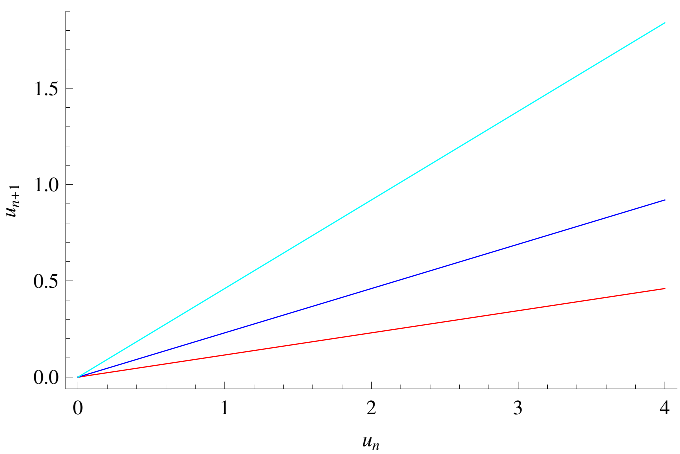

Figure 1. Convergence behavior of K, Picard-S and S iterates for mapping T defined in Example 4.1 where $u_1 = 4$.

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