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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Bound on Poisson Approximation for the Street Light Problem via Stein-Chen Method

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Abstract In this paper, we were, consequently, interested in the problem of the number of bright sections of road and provided the error estimation on poisson approximation of the number of bright sections of road with arbitrary probability by using Stein-Chen coupling method.

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Keywords: poisson approximation; the street light problem; stein-chen method

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1. Introduction

Street lighting has number of important benefits such as promoting security in urban areas and increasing the quality of life by providing lights to extend the working hours [1]. It is well known from the review of literatures that the road lighting has significant safety benefits for drivers, riders, and pedestrians. Regarding the NZTA Economic Evaluation Manual (EEM), street lights can reduce 35% of crashes as the effect of upgrading or improving lighting where lighting is poor [2]. Moreover, street lights is also as likely to reduce crime in these neighborhoods because of a diffusion of benefits [3].

To consider $n \times n$ square grid at each intersection, o refers to a street light. Each of the n^2 lights is broken with the probability of p. So, there are $2n(n-1)$ sections of roads $(-and|)$ bounded on both sides by the lights, see Figure 1 for a $n \times n$ square grid. The section of roads will be dark if the lights to both ends are broken. It might be said that each section of road is dark with probability of p^2.

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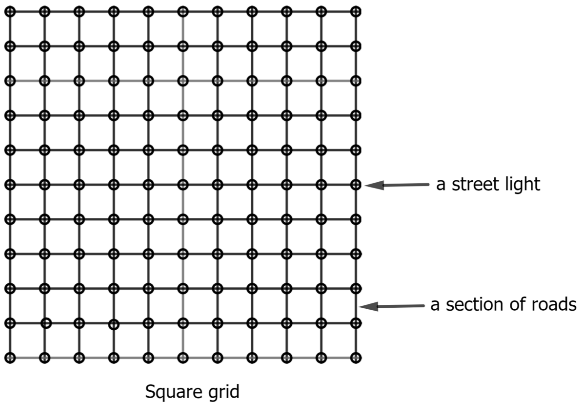

Figure 1. A $n \times n$ square grid

For instance, if the 6×6 square grid (Figure 2) $\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots \mathrm{L}_1$ refer to lights, we, then, have 36 lights and 60 sections of roads. The R_1 road will be dark if the lights A and B are broken. We, consequently, are interested in the following problem: What is the number of bright sections of road ?

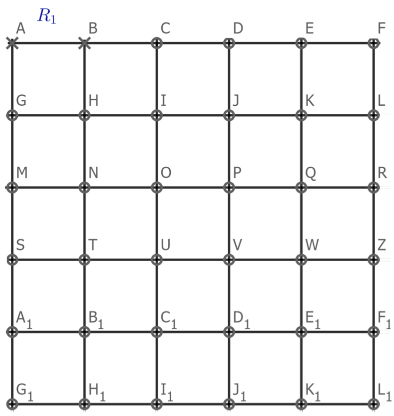

Figure 2. A 6×6 square grid

We can construct the random variable to obtain the problem by the following, let

$$W_n = \sum_{i=1}^{2n(n-1)} X_i, \tag{1.1}$$

be the total number of bright sections of road, where

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ section of road is bright} \\ 0 & \text{otherwise,} \end{cases}$$

and $p_i{}^2$ be the probability of section of road is dark with $i \in \{1, 2, \ldots, 2n(n-1)\}$. For n beginning sufficiently large, it is logical to approximate the distribution of W_n by Poisson distribution with mean $\lambda = \mathrm{E}W_n = [2n(n-1)-1]P(X_i = 1) = (2n^2 - 2n - 1)(1 - p_i{}^2)^{2n(n-1)}$.

In this study, we derived a uniform bound for the error on $\left|P(\tilde{W}_n \in A) - Poi_\lambda(A)\right|$, where W_n be the total number of bright sections of road. The tool for giving our main results consisted of the so-called poisson approximation and Stein-Chen coupling method, which we mentioned them in Section 2. The following theorem is our main results.

Theorem 1.1. *Let W_n be the total number of bright sections of road. Then we have*

1. $\left|P(\tilde{W}_n \in A) - Poi_\lambda(A)\right| \leq (2n^2 - 2n - 1)C_{\lambda,n^2,A}\left(\frac{p^4}{q-p^4}\right)$
2. $\left|P(\tilde{W}_n \in A) - Poi_\lambda(A)\right| \leq (1 - e^{-\lambda})(2n^2 - 2n - 1)\left(\frac{p^4}{q-p^4}\right)$

where $\lambda = (2n^2 - 2n - 1)(1 - p_i{}^2)^{2n(n-1)}$, $C_{\lambda,n^2,A} = \max\left\{\binom{n^2}{\frac{n^2-1}{2}}, \binom{n^2}{\frac{n^2}{2}}\right\}\min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$,

$$\triangle(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\ 2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

when $C_w = \{0, 1, 2, \ldots, w\}$

2. Poisson Approximation via Stein-Chen Method

The Stein-Chen Method of Poisson Approximation provides a powerful technique for computing an error bound when approximating probabilities by a Poisson distribution.

Stein [4] introduced a new powerful technique for the obtaining the rate of convergence to standard normal distribution. Chen [5] applied Stein's idea to obtain approximation results for the Poisson distribution. Our starting point is the Stein equation for Poisson distribution, which gives,

$$I_A(j) - Poi_\lambda(A) = \lambda g_{\lambda,A}(j+1) - jg_{\lambda,A}(j) \tag{2.1}$$

$\lambda > 0$, $j \in \mathbb{N} \cup \{0\}$, $A \subseteq \mathbb{N} \cup \{0\}$ and $I_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$I_A(w) = \begin{cases} 1 & ; w \in A, \\ 0 & ; w \notin A. \end{cases}$$

The solution $g_{\lambda,A}$ of (2.1) is the form

$$g_{\lambda,A}(w) = \begin{cases} (w-1)!\lambda^{-w}e^\lambda[\mathcal{P}_\lambda(I_{A\cap C_{w-1}}) - \mathcal{P}_\lambda(I_A)\mathcal{P}_\lambda(I_{C_{w-1}})] & ; w \geq 1, \\ 0 & ; w = 0 \end{cases}$$

where

$$\mathcal{P}_\lambda(I_A) = e^{-\lambda}\sum_{l=0}^{\infty} I_A(l)\frac{\lambda^l}{l!}$$

and

$$C_{w-1} = \{0, 1, \ldots, w-1\}.$$

By substituting j and λ in (2.1) by any integer-valued random variable W and $\lambda = \mathrm{E}W$, we have

$$\mathrm{P}(W_n \in A) - \mathcal{P}oi_\lambda A = \mathrm{E}(\lambda g_{\lambda,A}(W_n+1)) - \mathrm{E}(W_n g_{\lambda,A}(W_n)). \tag{2.2}$$

In the case where the dependence between the instances of X_i is global, there is an alternative approach to approximating the distribution of W_n. This approach is referred to as The Coupling approach, which was first proposed by Barbour ([6] 1982). This approach is particularly useful when it is possible to construct a random variable. $W_{n,i}$, for each a i on a common probability space with W_n such that $W_{n,i}$ is distributed as $W_n - X_i$ conditional on the event $X_i = 1$.

There have been a number of successful applications of this method, Barbour ([6] 1982), Janson ([7] 1994), Lange ([8] 2003).

Theorem 2.1. *If W_n and $W_{n,i}$ are defined as above, $p_i = \mathrm{E}(X_i) = P(X_i = 1)$, $\lambda = \mathrm{E}(W_n)$, then*

$$|\mathrm{P}(W_n \in A) - \mathcal{P}oi_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i \mathrm{E}|W_n - W_{n,i}| \tag{2.3}$$

where $\| g_{\lambda,A} \| := \sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)]$.

Many authors would like to determine a bound of $\| g_{\lambda,A} \|$. For $A \subseteq \mathbb{N} \cup \{0\}$, Chen ([5], 1975) prove that

$$\| g_{\lambda,A} \| \leq \min\{1, \lambda^{-1}\}$$

and Janson ([7], 1994) showed that

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}). \tag{2.4}$$

In case of non-uniform bound, Neammanee ([9], 2003) showed that

$$\| g_{\lambda,A} \| \leq \min\left\{\frac{1}{w_0}, \lambda^{-1}\right\}$$

and Teerapabolarn and Neammanee ([10], 2005) gave bound of $\| g_{\lambda,A} \|$ where $A = \{0, 1, \ldots, w_0\}$ in the terms of

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0 + 1}\right\}.$$

In general case for any subset A of $\{0, 1, \ldots, n\}$, Santiwipanont and Teerapabolarn ([11], 2006) gave a bound in the form of

$$\| g_{\lambda,A} \| \leq \lambda^{-1} \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A + 1}\right\} \tag{2.5}$$

where

$$\triangle(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\ 2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

The difficult part in applying Theorem 2.1 is to find $W_{n,i}$ which make $\mathrm{E}|W_n - W_{n,i}|$ small enough. So, there has been no solution in general. For the case of $X_1, \ldots, X_n$ are independent, we let $W_{n,i} = W_n - X_i$. Then $\mathrm{E}|W_n - W_{n,i}| = p_i$ and from (2.3), we have

$$|\mathrm{P}(W_n \in A) - \mathcal{P}oi_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i^2.$$

The problem of the construction of $W_{n,i}$ is difficult in the case of dependent indicator summand.

In next section, we will use Theorem 2.1 to prove our main result by constructing the random variable $W_{n,i}$ which make $\mathrm{E}|W_n - W_{n,i}|$ small.

3. Proof of Thorem 1.1

Proof. For each $i \in \{0, 1, 2, \ldots, 2n(n-1)\}$, we defined X_{ij} as following

$$X_{ij} = \begin{cases} 1 & \text{if the } j^{th} \text{ section of road is bright after removing} \\ & \text{the } i^{th} \text{ section of road which is bright,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $W_{n,i} = \sum_{i=1, j\neq i}^{n} X_{ij}$ be the total number of bright sections of road after we removed the i^{th} section which was bright.

Suppose that $\{j_s | s = 1, 2, 3, \ldots, w_0\}$ be the set of w_0 bright sections of road, $w_0 < 2n(n-1)$, so for each $w_0 \in \{0, 1, 2, \ldots, 2n^2 - 2n\}$, we got

$$\mathrm{P}(W_{n,i} = w_0) = (1 - \frac{\sum_{s=1}^{w_0} p_{j_s}^2}{1 - p_i^2})^{2n^2-2n}$$

and

$$\begin{aligned} \mathrm{P}(W_n - X_i = w_0 \mid X_i = 1) &= \frac{\mathrm{P}(W_n - X_i = w_0, X_i = 1)}{\mathrm{P}(X_i = 1)} \\ &= \frac{\mathrm{P}(W_n = w_0 + 1, X_i = 1)}{\mathrm{P}(X_i = 1)} \\ &= \frac{\left\{1 - (p_i^2 + \sum_{s=1}^{w_0} p_{j_2}^2)\right\}^{2n^2-2n}}{(1 - p_i^2)^{2n^2-2n}} \\ &= \frac{(1 - p_i^2 - \sum_{s=1}^{w_0} p_{j_s}^2)^{2n^2-2n}}{(1 - p_i^2)^{2n^2-2n}} \\ &= (1 - \frac{\sum_{s=1}^{w_0} p_{j_s}^2}{1 - p_i^2})^{2n^2-2n} \end{aligned}$$

Then $W_{n,i}$ had the same distribution as $W_n - X_i$ conditional on $X_i = 1$. In order to bound $\mathrm{E}|W_n - W_{n,i}|$, we observed that

- In case $X_i = 1$, we had the i^{th} section of road was bright. Thus the number of bright sections of road were bright after removing the i^{th} section of road, equal to the number of the bright sections of road minus 1, that was

$$W_{n,i} = W_n - 1. \tag{3.1}$$

• In case $X_i = 0$, we had the total number of bright sections of road after we removed the i^{th} section of road and we tested them again as defined, equal to the number of the bright sections of road minus the sum of number of the j^{th} sections of road, where $j \neq i$, was bright sections of road in the first-test and they were dark after we tested them again, let

$$W_{n,i} = W_n - \sum_{i,j=1,i\neq j}^{2n(n-1)} X_j Y_{ij}. \tag{3.2}$$

For each $j \in \{0, 1, 2, \ldots, 2n(n-1)\}$, such that $j \neq i$, we defined the indicator random variable Y_{ij} as follow:

$$Y_{ij} = \begin{cases} 1 & \text{if the } j^{th} \text{ section of road is dark after we test} \\ & \text{ the sections of road again, in which} \\ 0 & \text{otherwise.} \end{cases}$$

We knew that

$$\mathrm{E} \mid W_n - W_{n,i} \mid = \mathrm{E}(W_n - W_{n,i})^+ + \mathrm{E}(W_n - W_{n,i})^-.$$

Where

$$(W_n - W_{n,i})^+ = \max\{W_n - W_{n,i}, 0\},$$

and

$$(W_n - W_{n,i})^- = -\min\{W_n - W_{n,i}, 0\}.$$

Form (3.1) and (3.2).

• In case $X_i = 1$, we had $(W_n - W_{n,i})^+ = 1$ and $(W_n - W_{n,i})^- = 0$

• In case $X_i = 0$, we had $(W_n - W_{n,i})^+ = \sum_{i,j=1,i\neq j}^{n} X_i Y_{ij}$ and $(W_n - W_{n,i})^- = 0$.

Therefore,
$(W_n - W_{n,i})^+ = \sum_{i,j=1,i\neq j}^{n} X_iY_{ij}$ and $(W_n - W_{n,i})^- = 0$.

$$
\begin{aligned}
\mathrm{E(W_n - W_{n,i})^+} &\le \mathrm{E}\{\sum_{\mathrm{i,j=1,i\neq j}}^{\mathrm{2n^2-2n}} \mathrm{X_iY_{ij}}\} \\
&= \sum_{i,j=1,i\neq j}^{2n^2-2n} \mathrm{E\{X_iY_{ij}\}} \\
&= \sum_{i,j=1,i\neq j}^{2n^2-2n} \mathrm{P(X_i = 1, Y_{ij} = 1)} \\
&= \sum_{i,j=1,i\neq j}^{2n^2-2n} \mathrm{P(X_i = 1)P(Y_{ij} = 1)} \\
&= \sum_{i,j=1,i\neq j}^{2n^2-2n} (1-p_j^2)^{2n^2-2n}\Big\{1-(1-\frac{p_i^2p_j^2}{(1-p_j^2)})^{b_i}\Big\} \\
&\le \sum_{i,j=1,i\neq j}^{2n^2-2n} \Big\{1-(1-\frac{p_i^2p_j^2}{1-p_j^2})^{b_i}\Big\} \\
&\le \sum_{i,j=1,i\neq j}^{2n^2-2n} \Big\{1-(1-\frac{p_i^2p_j^2}{1-p_j^2})^{n^2}\Big\} \\
&= \sum_{i,j=1,i\neq j}^{2n^2-2n} \Big\{1-(\frac{1-p_j^2-p_i^2p_j^2}{1-p_j^2})^{n^2}\Big\}, \qquad (3.3)
\end{aligned}
$$

where b_i be the number of the lights was bright in the i^{th} section of road.
Suppose that $p = \max_{1\le i\le n^2} p_i$, we had

$$
\begin{aligned}
\mathrm{E|W_n - W_{n,i}|} &\le \sum_{i,j=1,i\neq j}^{2n^2-2n} \Big\{1-(\frac{1-p_j^2-p_i^2p_j^2}{1-p_j^2})^{n^2}\Big\} \\
&= (2n^2-2n-1)\Big\{1-(\frac{1-p_j^2-p_i^2p_j^2}{1-p_j^2})^{n^2}\Big\} \\
&\le (2n^2-2n-1)\Big\{1-(\frac{1-p^2-p^4}{1-p^2})^{n^2}\Big\} \\
&= (2n^2-2n-1)\Big\{1-(\frac{1-p^2-p^4}{q})^{n^2}\Big\} \\
&= (2n^2-2n-1)\Big\{1-\sum_{k=0}^{n^2}\binom{n^2}{k}(1-p^2)^{n^2-k}(-p^4)^k q^{-n^2}\Big\}
\end{aligned}
$$

$$\begin{aligned}
\mathrm{E}|\mathrm{W_n} - \mathrm{W_{n,i}}| &\leq (2n^2-2n-1)\Big\{1 - \sum_{k=0}^{n^2}\binom{n^2}{k}(q)^{n^2-k}(-p^4)^k q^{-n^2}\Big\} \\
&= (2n^2-2n-1)\Big\{1 - \sum_{k=0}^{n^2}\binom{n^2}{k}(q)^{n^2-k}(-p^4)^k q^{-n^2}\Big\} \\
&= (2n^2-2n-1)\Big\{1 - \sum_{k=0}^{n^2}\binom{n^2}{k}(q)^{-k}(-p^4)^k\Big\} \\
&= (2n^2-2n-1)\Big\{\sum_{k=1}^{n^2}\binom{n^2}{k}(-1)^k(q)^{-k}(p^4)^k\Big\} \\
&\leq \alpha_{n^2}(2n^2-2n-1)\sum_{k=1}^{n^2}(\frac{p^4}{q})^k \\
&= \alpha_{n^2}(2n^2-2n-1)\Big\{\frac{\frac{p^4}{q} - (\frac{p^4}{q})^{n^2+1}}{1-\frac{p^4}{q}}\Big\} \\
&\leq \alpha_{n^2}(2n^2-2n-1)\Big(\frac{p^4}{q-p^4}\Big), \qquad (3.4)
\end{aligned}$$

where $\alpha_{n^2} = \max\Big\{\binom{n^2}{\frac{n^2-1}{2}}, \binom{n^2}{\frac{n^2}{2}}\Big\}$ and $q = 1-p^2$.
Hence, by (2.3), (2.4), (2.5) and (3.4), we had

$$|P(W_n \in A) - Poi_\lambda(A)| \leq (2n^2-2n-1)C_{\lambda,n^2,A}\Big(\frac{p^4}{q-p^4}\Big)$$

and

$$|P(W_n \in A) - Poi_\lambda(A)| \leq (1-e^{-\lambda})(2n^2-2n-1)\Big(\frac{p^4}{q-p^4}\Big)$$

where $\lambda = (2n^2-2n-1)(1-{p_i}^2)^{2n(n-1)}$
and $C_{\lambda,n^2,A} = \max\Big\{\binom{n^2}{\frac{n^2-1}{2}}, \binom{n^2}{\frac{n^2}{2}}\Big\}\min\Big\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\Big\}$. ■

Remark 3.1. It was observed from Theorem 1.1, for n beginning sufficiently large, we could approximate the total number of bright sections of road by using Poisson distribution with mean $\lambda = (2n^2-2n-1)(1-{p_i}^2)^{2n(n-1)}$.

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