



Dedicated to Prof. Suthep Suantai on the occasion of his 60<sup>th</sup> anniversary

# Fixed Point and Endpoint Theorems for $(\alpha, \beta)$ -Meir-Keeler Contraction on the Partial Hausdorff Metric

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**Abstract** The purpose of this work is to introduce the notion of a multivalued strictly  $(\alpha, \beta)$ -admissible mappings and a multivalued  $(\alpha, \beta)$ -Meir-Keeler contractions with respect to the partial Hausdorff metric  $\mathcal{H}_p$  in the framework of partial metric spaces. In addition, we present fixed points and endpoints results for a multivalued  $(\alpha, \beta)$ -Meir-Keeler contraction mappings in the framework of the complete partial metric spaces. The results obtained in this work provides extension as well as substantial generalizations and improvements of several well-known results on fixed point theory and its applications.

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## 1. INTRODUCTION

The theory of fixed point plays an important role in nonlinear functional analysis. It has a lot of significant applications and the concept of fixed point has been applied in almost all area of sciences. In particular, the concept of fixed point has been very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Furthermore, this concepts has been established as an important tool in fields such as Economics, Optimal control, Biology, Chemistry, Engineering, Physics, Game Theory and so on. Banach [1] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. The Banach contraction principle is the most cited and applied theorem in this area

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of mathematics. Due to its importance and fruitful applications, many authors have generalized this result by considering classes of nonlinear mappings which are more general than contraction mappings and metric spaces. More so, over the years, researchers have also develop different iterative algorithms for solving fixed point problems for different nonlinear mappings in different abstract spaces, (see [2–7] and the references therein). In this research paper, we will give a brief over view of some nonlinear mappings that are relevant to our work. For example, in 1969, Meir and Keeler [8] introduced the notion of Meir-Keeler contraction in the frame work of the metric space.

**Definition 1.1.** [8] Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$  be a mapping satisfying the following, for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\epsilon \leq d(x, y) \leq \epsilon + \delta \Rightarrow d(Tx, Ty) < \epsilon, \quad (1.1)$$

for all  $x, y \in X$ . Then  $T$  is a called a Meir-Keeler contraction.

**Remark 1.2.** If  $T$  is a Meir-Keeler contraction then

$$d(Tx, Ty) < d(x, y),$$

for all  $x, y \in X$  with  $x \neq y$ . If  $x = y$ , we then have  $d(Tx, Ty) = 0$ , as such

$$d(Tx, Ty) \leq d(x, y).$$

**Theorem 1.3.** [8] Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be a Meir-Keeler contraction mapping. Then  $T$  has a unique fixed point say  $x^*$ . Moreover, for any  $x \in X$ ,  $\lim_{n \rightarrow \infty} T^n x = x^*$ .

In addition, Suzuki [9] introduced the concept of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for such classes of mappings.

**Definition 1.4.** Let  $(X, d)$  be a metric space. A mapping  $T : X \rightarrow X$  is said to satisfy condition (C) if for all  $x, y \in X$ ,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

**Theorem 1.5.** Let  $(X, d)$  be a compact metric space and  $T : X \rightarrow X$  be a mapping satisfying condition (C) for all  $x, y \in X$ . Then  $T$  has a unique fixed point.

Samet et al. [10] introduced another class of mappings called the  $\alpha$ -admissible mappings and obtained some fixed point results for this classes of mappings.

**Definition 1.6.** [10] Let  $\alpha : X \times X \rightarrow [0, \infty)$  be a function. We say that a self mapping  $T : X \rightarrow X$  is  $\alpha$ -admissible if for all  $x, y \in X$ ,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

**Definition 1.7.** [11] Let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$  be mappings. We say that  $T$  is a triangular  $\alpha$ -admissible if

- (1)  $T$  is  $\alpha$ -admissible and
- (2)  $\alpha(x, y) \geq 1$  and  $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$  for all  $x, y, z \in X$ .

In 2016, Chandok et al. [12] introduced another class of mappings called the TAC-contractive mapping and established some fixed point results in the framework of the complete metric spaces.

**Definition 1.8.** Let  $T : X \rightarrow X$  be a mapping and let  $\alpha, \beta : X \rightarrow \mathbb{R}^+$  be two functions. Then  $T$  is called a cyclic  $(\alpha, \beta)$ -admissible mapping, if

- (1)  $\alpha(x) \geq 1$  for some  $x \in X$  implies that  $\beta(Tx) \geq 1$ ,
- (2)  $\beta(x) \geq 1$  for some  $x \in X$  implies that  $\alpha(Tx) \geq 1$ .

In 2019, Mebawondu et al. [13, 14] generalized the concept of an  $\alpha$ -admissible mapping by introducing the notion of an  $(\alpha, \beta)$ -cyclic admissible mapping.

**Definition 1.9.** [13] Let  $X$  be a nonempty set,  $T : X \rightarrow X$  be a mapping and  $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+$  be two functions. We say that  $T$  is an  $(\alpha, \beta)$ -cyclic admissible mapping, if for all  $x, y \in X$

- (1)  $\alpha(x, y) \geq 1 \Rightarrow \beta(Tx, Ty) \geq 1$ ,
- (2)  $\beta(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$ .

**Remark 1.10.** It is easy to see that if  $\alpha(x, y) = \beta(x, y)$ , we obtain Definition 1.6.

It is worth mentioning that the concepts of  $\alpha$ -admissible mappings, cyclic  $(\alpha, \beta)$ -admissible mappings, and  $(\alpha, \beta)$ -cyclic admissible mappings has been used by researchers in this area to extend, generalize and modify some well-known nonlinear mappings in different abstract spaces. In addition, these concepts have been used to generalize the notion of the Meir-Keeler contractions. These contractions have been extended and generalized by researchers in this area (see [7, 15, 16] and the references therein).

In 1969, Nadler [17] extended the Banach contraction principle from a single-valued mapping to a multivalued mapping by proving the fixed point theorem for multivalued contractions. Let  $(X, d)$  be a metric space,  $N(X)$  denote the collection of nonempty subsets of  $X$  and let  $CB(X)$  denote the collection of all nonempty, closed and bounded subsets of  $X$ . For  $A, B \in CB(X)$ , we define

$$\mathcal{H}(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where  $d(x, B) := \inf\{d(x, b) : b \in B\}$  and it is well-known that  $\mathcal{H}$  is called the Hausdorff metric induced by the metric  $d$ . An element  $x \in X$  is called a fixed point of

$$T : X \rightarrow CB(X) \text{ if } x \in Tx.$$

We denote the set of all fixed points of  $T$  by  $F(T)$ . Nadler established the following result.

**Theorem 1.11.** [17] Let  $(X, d)$  be a complete metric space and  $T$  be a mapping from  $X$  into  $CB(X)$ . Suppose that there exists  $k \in [0, 1)$  such that

$$\mathcal{H}(Tx, Ty) \leq kd(x, y)$$

for all  $x, y \in X$ . Then  $T$  has a fixed point in  $X$ .

The theory of multivalued mappings continues to attract a lot of researchers attention. It has numerous real-world applications in game theory, constrained optimization, differential inclusions, optimal control problems, energy management problems, image reconstruction and so on. Over the years the results of Nadler [17] have been extended and generalized in terms of spaces and nonlinear mappings (see [18, 19] and the references therein).

Rus [20] introduced the notion of endpoints and proved some results on endpoint multivalued operators. An element  $x \in X$  is called an endpoint of  $T : X \rightarrow N(X)$  if  $Tx = \{x\}$ . We denote  $E(T)$  the set of all endpoints of  $T$ . He established the following result.

**Theorem 1.12.** [20] Let  $(X, d)$  be a complete metric space and  $T$  be a mapping from  $X$  into  $CB(X)$ . Suppose that

- (1)  $x \in Tx$  for all  $x \in X$ ,
- (2) there exists a comparison function  $\phi : [0, \infty) \rightarrow [0, \infty)$  and a Picard sequence  $x_{n+1} \in Tx_n$ , for all  $n \in \mathbb{N}$  such that  $\mathcal{D}(Tx_{n+1}) \leq \phi(\mathcal{D}(Tx_n))$  for all  $n \in \mathbb{N}$ , where  $\mathcal{D}(A) := \sup\{d(x, y) : x, y \in A\}$ .

Then  $T$  has a unique end point.

**Remark 1.13.**

- (1) It is worth noting that the concept of fixed points and endpoints are equivalent if  $T$  is a single-valued mapping.
- (2) Clearly,  $E(T) \subseteq F(T)$ . Therefore the concept of endpoints tends to be more challenging compared to the concept of fixed points.

Motivated by the research works mentioned above and the recent interest in this direction of research, we introduce the notion of a multivalued strictly  $(\alpha, \beta)$ -admissible mappings and a multivalued  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to partial Hausdorff metric  $\mathcal{H}_p$  in the framework of the partial metric spaces. In addition, we present fixed points and endpoints results for the multi-valued  $(\alpha, \beta)$ -Meir-Keeler contraction mappings in the framework of the complete partial metric spaces. The results obtained in this work provide extensions as well as substantial generalizations and improvements of several well-known results on fixed point theory and its applications.

## 2. PRELIMINARIES

In this section we introduce some concepts and present some results that will be needed in the sequel.

**Definition 2.1.** [21] A partial metric on a nonempty set  $X$  is a function  $p : X \times X \rightarrow \mathbb{R}^+$  such that for all  $x, y, z \in X$

- (1)  $x = y$  if and only if  $p(x, x) = p(y, y) = p(x, y)$ ;
- (2)  $p(x, x) \leq p(x, y)$ ;
- (3)  $p(x, y) = p(y, x)$ ;
- (4)  $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ .

The pair  $(X, p)$  is called a partial metric space.

**Remark 2.2.** It is easy to see that if  $p(x, y) = 0$ , then using (1) and (2), we have  $x = y$ . On the other hand if  $x = y$ , the expression  $p(x, y)$  is not necessarily 0.

The concept of the partial metric spaces was introduced and studied by Matthews [21]. He provided solutions to some problems of computer science, for example in domain theory and semantics, by transferring the structure of the metric space.

**Example 2.3.** [21] Let  $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$  and define

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}.$$

Then  $(X, p)$  is a partial metric space.

All partial metrics  $p$  on  $X$  generates a  $\mathcal{T}_0$  topology  $\tau_p$  on  $X$ , which has a base the family of the open  $p$ -ball  $\{B_p(x, \alpha) : x \in X, \alpha > 0\}$ , where  $B_p(x, \alpha) = \{y \in X : p(x, y) < p(x, x) + \alpha\}$  for all  $x \in X$  and  $\alpha > 0$ . For a partial metric  $p$  on  $X$  and a function  $d_p : X \times X \rightarrow \mathbb{R}^+$  given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on  $X$ .

**Definition 2.4.** [21] Let  $(X, p)$  be a partial metric space. Then

- (1) a sequence  $\{x_n\}$  in  $(X, p)$  converges to  $x \in X$  if and only if  $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$ ;
- (2) a sequence  $\{x_n\}$  in  $(X, p)$  is said to be a Cauchy sequence if and only if  $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$  exists;
- (3)  $(X, p)$  is said to be complete if every Cauchy sequence  $\{x_n\}$  in  $X$  converges, with to  $\tau_p$ , to a point  $x \in X$  such that  $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$ ;
- (4) a subset  $Y$  of  $(X, p)$  is closed if whenever  $\{x_n\}$  is a sequence in  $Y$  such that  $\{x_n\}$  converges to some  $x \in X$ , then  $x \in Y$ .

**Lemma 2.5.** [16] Let  $(X, p)$  be a partial metric space. Then

- (1) if  $p(x, y) = 0$ , then  $x = y$ ;
- (2) if  $x \neq y$ , then  $p(x, y) > 0$ .

**Lemma 2.6.** [21] Let  $(X, p)$  be a partial metric space. Then

- (1)  $\{x_n\}$  is a Cauchy sequence in  $(X, p)$  if and only if it is a Cauchy sequence in the metric space  $(X, d_p)$ ;
- (2)  $(X, p)$  is said to be complete if and only if the metric space  $(X, d_p)$  is complete. Furthermore,  $\lim_{n \rightarrow \infty} d_p(x_n, x) = 0$  if and only if  $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$ .

In 2012 Aydi et al. [15] introduced and studied the notion of the partial Hausdorff metric  $\mathcal{H}_p$  induced by the partial metric  $p$ . Let  $(X, p)$  be a partial metric space and suppose that  $CB_p(X)$  is the collection of all nonempty, closed and bounded subsets of the partial metric space  $(X, p)$ . For  $A, B \in CB_p(X)$  and  $x \in X$ , define

$$\begin{aligned} p(x, A) &:= \inf\{p(x, a) : a \in A\}; \\ \delta_p(A, B) &:= \sup\{p(a, B) : a \in A\}; \\ \delta_p(B, A) &:= \sup\{p(b, A) : b \in B\}; \\ \mathcal{H}_p(A, B) &= \max\{\delta_p(A, B), \delta_p(B, A)\}. \end{aligned}$$

**Proposition 2.7.** [15] Let  $(X, p)$  be a partial metric space. For  $A, B \in CB_p(X)$  the following hold:

- (1)  $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$ ;
- (2)  $\delta_p(A, A) \leq \delta_p(A, B)$ ;
- (3)  $\delta_p(A, B) = 0$  implies that  $A \subset B$ ;
- (4)  $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$ .

**Proposition 2.8.** [15] Let  $(X, p)$  be a partial metric space. For  $A, B \in CB_p(X)$  the following hold:

- (1)  $\mathcal{H}_p(A, A) \leq \mathcal{H}_p(A, B)$ ;
- (2)  $\mathcal{H}_p(A, B) = \mathcal{H}_p(B, A)$ ;

- (3)  $\mathcal{H}_p(A, B) = 0 \Rightarrow A = B;$
- (4)  $\mathcal{H}_p(A, B) \leq \mathcal{H}_p(A, C) + \mathcal{H}_p(C, B) - \inf_{c \in C} p(c, c).$

**Remark 2.9.** [15] Let  $(X, p)$  be a partial metric space and suppose that  $A$  is a nonempty subset of  $X$ . Then

$$a \in \overline{A} \text{ if and only if } p(a, A) = p(a, a), \tag{2.1}$$

where  $\overline{A}$  denotes the closure of  $A$  with respect to the partial metric  $p$ . Note that  $A$  is closed in  $(X, p)$  if and only if  $A = \overline{A}$ .

Aydi et al. [15] established the following result.

**Lemma 2.10.** [15] Let  $(X, p)$  be a partial metric space with  $A, B \in CB_p(X)$  and  $h > 1$ . Then for any  $a \in A$ , there exists  $b = b(a) \in B$  such that

$$p(a, b) \leq h\mathcal{H}_p(A, B).$$

### 3. MAIN RESULT

In this section we introduce a new class of mappings and prove the existence theorems for fixed points of these class of mappings.

**Definition 3.1.** Let  $X$  be a nonempty set,  $T : X \rightarrow CB_p(X)$  and  $\alpha, \beta : X \times X \rightarrow (0, \infty)$  be three functions. We say that  $T$  is strictly  $(\alpha, \beta)$ -cyclic admissible mapping, if for all  $x, y \in X$  and  $\hat{x} \in Tx, \hat{y} \in Ty$  with

- (1)  $\alpha(x, y) > 1 \Rightarrow \beta(\hat{x}, \hat{y}) > 1,$
- (2)  $\beta(x, y) > 1 \Rightarrow \alpha(\hat{x}, \hat{y}) > 1.$

**Remark 3.2.** Clearly if  $\beta(x, y) = \alpha(x, y)$ , we have  $\alpha(x, y) > 1 \Rightarrow \beta(\hat{x}, \hat{y}) > 1$ , which is the multivalued version of Definition 1.6.

**Lemma 3.3.** Let  $X$  be a nonempty set and  $T : X \rightarrow CB_p(X)$  be a strictly  $(\alpha, \beta)$ -cyclic admissible mapping. Suppose that there exists  $x_0 \in X$  such that  $\alpha(x_0, x_1) > 1$  and  $\beta(x_0, x_1) > 1$ , where  $x_1 \in Tx_0$ . Define the sequence  $\{x_n\}$  by  $x_{n+1} \in Tx_n$ , then  $\alpha(x_m, x_{m+1}) > 1$  implies that  $\beta(x_n, x_{n+1}) > 1$  and  $\beta(x_m, x_{m+1}) > 1$  implies that  $\alpha(x_n, x_{n+1}) > 1$ , for all  $n, m \in \mathbb{N} \cup \{0\}$  with  $m < n$ .

*Proof.* Using our hypothesis and the fact that  $T$  is a strictly  $(\alpha, \beta)$ -cyclic admissible mapping, we have that there exists  $x_0 \in X$  such that

$$\alpha(x_0, x_1) > 1 \Rightarrow \beta(x_1, x_2) > 1$$

and

$$\beta(x_1, x_2) > 1 \Rightarrow \alpha(x_2, x_3) > 1.$$

Continuing in this way we obtain

$$\alpha(x_{2n}, x_{2n+1}) > 1 \text{ and } \beta(x_{2n+1}, x_{2n+2}) > 1, \forall n \in \mathbb{N} \cup \{0\}.$$

Using a similar approach we obtain

$$\beta(x_{2n}, x_{2n+1}) > 1 \text{ and } \alpha(x_{2n+1}, x_{2n+2}) > 1, \forall n \in \mathbb{N} \cup \{0\}.$$

In a similar sense we obtain the same result for all  $m \in \mathbb{N}$ . That is

$$\alpha(x_{2m}, x_{2m+1}) > 1 \text{ and } \beta(x_{2m+1}, x_{2m+2}) > 1$$

and

$$\beta(x_{2m}, x_{2m+1}) > 1 \text{ and } \alpha(x_{2m+1}, x_{2m+2}) > 1, \forall m \in \mathbb{N} \cup \{0\}.$$

In addition since

$$\alpha(x_m, x_{m+1}) > 1 \Rightarrow \beta(x_{m+1}, x_{m+2}) > 1 \Rightarrow \alpha(x_{m+2}, x_{m+3}) > 1 \Rightarrow \dots$$

with  $m < n$ , we deduce that

$$\alpha(x_m, x_{m+1}) > 1 \Rightarrow \beta(x_n, x_{n+1}) > 1.$$

Using a similar approach we have

$$\beta(x_m, x_{m+1}) > 1 \Rightarrow \alpha(x_n, x_{n+1}) > 1.$$

■

**Definition 3.4.** Let  $(X, p)$  be a partial metric space and  $\alpha, \beta : X \times X \rightarrow (0, \infty)$  be two functions. We say that  $T : X \rightarrow CB_p(X)$  is an  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ , if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\epsilon \leq p(x, y) < \epsilon + \delta \Rightarrow \alpha(x, y)\beta(x, y)\mathcal{H}_p(Tx, Ty) < \epsilon, \tag{3.1}$$

for all  $x, y \in X$ .

**Remark 3.5.** If  $T : X \rightarrow CB_p(X)$  is an  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ , then we have

$$\alpha(x, y)\beta(x, y)\mathcal{H}_p(Tx, Ty) < p(x, y), \tag{3.2}$$

for all  $x, y \in X$  when  $p(x, y) > 0$ . On the other hand, observe that if  $p(x, y) = 0$ , we clearly have that  $\mathcal{H}_p(Tx, Ty) = 0$  and using Proposition 2.8 we obtain that  $Tx = Ty$ . Thus for all  $x, y \in X$ , we get that

$$\alpha(x, y)\beta(x, y)\mathcal{H}_p(Tx, Ty) \leq p(x, y). \tag{3.3}$$

**Remark 3.6.** It is also easy to see that if  $\alpha(x, y)\beta(x, y) = 1$ . Consequently we have obtained the multivalued version of Meir-Keeler type contractions in the framework of partial metric spaces.

**Theorem 3.7.** Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow CB_p(X)$  be an  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ . Suppose the following conditions hold:

- (1)  $T$  is strictly  $(\alpha, \beta)$ -cyclic admissible mapping,
- (2) there exists  $x_0 \in X$  and  $x_1 \in Tx_0$  such that  $\alpha(x_0, x_1) > 1$  and  $\beta(x_0, x_1) > 1$ ,
- (3) If  $\{x_n\}$  is a sequence such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  and  $\alpha(x_n, x_{n+1}) > 1, \beta(x_n, x_{n+1}) > 1$ , then  $\alpha(x_n, x) > 1, \beta(x_n, x) > 1$  for all  $n \in \mathbb{N}$ .

Then  $T$  has a fixed point.

*Proof.* We define a sequence  $\{x_n\}$  by  $x_{n+1} \in Tx_n$  for all  $n \in \mathbb{N} \cup \{0\}$ . If we suppose that  $x_0 \in Tx_0$ , then, we obtain the desired result. Now suppose that  $x_0 \notin Tx_0$  for all  $n \in \mathbb{N} \cup \{0\}$ . Since  $T$  is an  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$  and by Remark 3.11, we obtain

$$\alpha(x_0, x_1)\beta(x_0, x_1)\mathcal{H}_p(Tx_0, Tx_1) \leq p(x_0, x_1). \tag{3.4}$$

It is easy to see from condition (2) and Lemma 3.3 that  $\alpha(x_0, x_1)\beta(x_0, x_1) > 1$ . Suppose that  $\alpha(x_0, x_1)\beta(x_0, x_1) = k > 1$ . From Lemma 2.10 and taking  $h = \sqrt{k}$ , we obtain

$$p(x_1, x_2) \leq \sqrt{k}\mathcal{H}_p(Tx_0, Tx_1). \quad (3.5)$$

Using (3.4) and (3.5) we obtain

$$\begin{aligned} p(x_1, x_2) &\leq \sqrt{k}\mathcal{H}_p(Tx_0, Tx_1) \\ &= \sqrt{k^2k^{-1}}\mathcal{H}_p(Tx_0, Tx_1) \\ &= \frac{1}{\sqrt{k}}k\mathcal{H}_p(Tx_0, Tx_1) \\ &\leq \frac{1}{\sqrt{k}}p(x_0, x_1). \end{aligned} \quad (3.6)$$

From Lemma 3.3 we have that  $\alpha(x_n, x_{n+1})\beta(x_n, x_{n+1}) > 1$  and now suppose that

$$\alpha(x_n, x_{n+1})\beta(x_n, x_{n+1}) = k > 1.$$

From (3.6) and inductively we have that

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \frac{1}{\sqrt{k}}p(x_{n-1}, x_n) \\ &\leq \frac{1}{\sqrt{k}} \times \frac{1}{\sqrt{k}}p(x_{n-2}, x_{n-1}) \\ &\leq \frac{1}{\sqrt{k}} \times \frac{1}{\sqrt{k}} \times \frac{1}{\sqrt{k}}p(x_{n-3}, x_{n-2}) \\ &\vdots \\ &\leq \left(\frac{1}{\sqrt{k}}\right)^n p(x_0, x_1). \end{aligned} \quad (3.7)$$

Note that  $k > 1$ ,  $\sqrt{k} > 1$  and thus  $\frac{1}{\sqrt{k}} < 1$ . Therefore we have

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0. \quad (3.8)$$

We now establish that  $\{x_n\}$  is a Cauchy sequence in  $(X, d_p)$ . Using the property (2) of the partial metric space ( $p(x, x) \leq p(x, y)$ ) we deduce that  $p(x_n, x_n) \leq p(x_n, x_{n+1})$ . By taking the limit as  $n \rightarrow \infty$  and using (3.8) we obtain

$$\lim_{n \rightarrow \infty} p(x_n, x_n) = 0. \quad (3.9)$$



Now using property (4) of the partial metric space  $(p(x, y) \leq p(x, z) + p(z, y) - p(z, z))$ , with  $n, m \in \mathbb{N}$ , we have

$$\begin{aligned}
 p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+m}) - p(x_{n+1}, x_{n+1}) \\
 &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+m}) \\
 &\quad - p(x_{n+2}, x_{n+2}) - p(x_{n+1}, x_{n+1}) \\
 &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) \\
 &\quad + p(x_{n+3}, x_{n+m}) - p(x_{n+3}, x_{n+3}) - p(x_{n+2}, x_{n+2}) \\
 &\quad - p(x_{n+1}, x_{n+1}) \\
 &\vdots \\
 &\leq \sum_{i=1}^m \left(\frac{1}{\sqrt{k}}\right)^{n+i-1} p(x_0, x_1) - \sum_{i=1}^{m-1} p(x_{n+i}, x_{n+i}) \\
 &\leq \left(\frac{1}{\sqrt{k}}\right)^n \left(\frac{\sqrt{k}}{\sqrt{k}-1}\right) p(x_0, x_1) - \sum_{i=1}^{m-1} p(x_{n+i}, x_{n+i}).
 \end{aligned} \tag{3.10}$$

Using (3.8) and (3.9) and by taking limit as  $n \rightarrow \infty$  we get

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+m}) = 0. \tag{3.11}$$

Since  $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ , we obtain that  $d_p(x_n, x_{n+m}) = 2p(x_n, x_{n+m}) - p(x_n, x_n) - p(x_{n+m}, x_{n+m})$ , using (3.8), (3.9), (3.11) and by taking limit as  $n, m \rightarrow \infty$ , we get

$$\lim_{n, m \rightarrow \infty} d_p(x_n, x_{n+m}) = 0. \tag{3.12}$$

Thus  $\{x_n\}$  is a Cauchy sequence in  $(X, d_p)$ . Since  $(X, p)$  is complete and using Lemma 2.6,  $(X, d_p)$  we obtain that is a complete metric space. Therefore  $\{x_n\}$  converges to some  $x \in X$  with respect to the metric  $d_p$  and we also have

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0. \tag{3.13}$$

Using condition (3) and since  $T$  is an  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ . By Remark 3.11 we get

$$\alpha(x_n, x)\beta(x_n, x)\mathcal{H}_p(Tx_n, Tx) \leq p(x_n, x) \tag{3.14}$$

and using (3.13) we have

$$\lim_{n \rightarrow \infty} \mathcal{H}_p(Tx_n, Tx) = 0. \tag{3.15}$$

Considering the way the sequence  $\{x_n\}$  is defined we have

$$\lim_{n \rightarrow \infty} p(x_{n+1}, Tx) \leq \lim_{n \rightarrow \infty} \delta_p(Tx_n, Tx) \leq \lim_{n \rightarrow \infty} \mathcal{H}_p(Tx_n, Tx) = 0. \tag{3.16}$$

Now using property (4) of partial metric space  $(p(x, y) \leq p(x, z) + p(z, y) - p(z, z))$ , with  $n \in \mathbb{N}$  we have

$$p(x, Tx) \leq \lim_{n \rightarrow \infty} p(x, x_{n+1}) + \lim_{n \rightarrow \infty} p(x_{n+1}, Tx) - \lim_{n \rightarrow \infty} p(x_{n+1}, x_{n+1}) \tag{3.17}$$

using (3.9), (3.13) and (3.16) we have

$$p(x, Tx) = 0. \tag{3.18}$$

Using (3.13) we get

$$p(x, Tx) = 0 = p(x, x), \tag{3.19}$$

and using Remark 2.9 we obtain that  $x \in Tx$ . ■

**Definition 3.8.** Let  $T : X \rightarrow CB_p(X)$  be a multivalued mapping on a partial metric space  $(X, p)$ .

- (1) An element  $x \in X$  is called an endpoint of  $T$  if  $Tx = \{x\}$ . It is clear that an endpoint of  $T$  is also a fixed point of  $T$ .
- (2)  $T$  has the approximate endpoint property if there exists a sequence  $\{x_n\} \subset X$  such that

$$\lim_{n \rightarrow \infty} \mathcal{H}_p(\{x_n\}, Tx_n) = 0 \text{ or equivalently if } \inf_{x \in X} \sup_{y \in Tx} p(x, y) = 0.$$

**Theorem 3.9.** Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow CB_p(X)$  be an  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ . Suppose the following conditions hold:

- (1)  $T$  is strictly  $(\alpha, \beta)$ -cyclic admissible mapping,
- (2) there exists  $x_0 \in X$  and  $x_1 \in Tx_0$  such that  $\alpha(x_0, x_1) > 1$  and  $\beta(x_0, x_1) > 1$ ,
- (3) If  $\{x_n\}$  is a sequence such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  and  $\alpha(x_n, x_{n+1}) > 1, \beta(x_n, x_{n+1}) > 1$ , then  $\alpha(x_n, x) > 1, \beta(x_n, x) > 1$  for all  $n \in \mathbb{N}$ .

Then  $T$  has an endpoint  $x$  if and only if  $T$  has the approximate endpoint property.

*Proof.* It is easy to see that if  $T$  has an endpoint and hence  $T$  has the approximate endpoint property.

Conversely, suppose that  $T$  has the approximate endpoint property, that is, we can find a sequence  $\{x_n\} \subset X$  such that  $\lim_{n \rightarrow \infty} \mathcal{H}_p(\{x_n\}, Tx_n) = 0$ . It has been established in Theorem 3.7 that  $x_n$  is a Cauchy sequence and that  $\{x_n\}$  converges to some  $x \in X$  with respect to the metric  $d_p$  and we also have

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_n, x_n) = 0. \tag{3.20}$$

Using condition (3) it is clear  $\alpha(x_n, x)\beta(x_n, x) > 1$ . Now using the fact that  $T$  is an  $(\alpha, \beta)$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$  and Remark 3.11, we observe that

$$\begin{aligned} \mathcal{H}_p(\{x_n\}, Tx) - \mathcal{H}_p(\{x_n\}, Tx_n) &\leq \mathcal{H}_p(Tx_n, Tx) - \inf_{y \in Tx_n} p(y, y) \\ &\leq \mathcal{H}_p(Tx_n, Tx) \\ &\leq \alpha(x_n, x)\beta(x_n, x)\mathcal{H}_p(Tx_n, Tx) \\ &\leq p(x_n, x). \end{aligned} \tag{3.21}$$

Using (3.20) and  $\lim_{n \rightarrow \infty} \mathcal{H}_p(\{x_n\}, Tx_n) = 0$  we obtain

$$\mathcal{H}_p(\{x\}, Tx) = 0, \tag{3.22}$$

and using Proposition 2.8 we have  $\{x\} = Tx$ . ■

**Definition 3.10.** Let  $(X, p)$  be a partial metric space,  $\alpha : X \times X \rightarrow (0, \infty)$  be a function. We say that  $T : X \rightarrow CB_p(X)$  is an  $\alpha$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ , if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\epsilon \leq p(x, y) < \epsilon + \delta \Rightarrow \alpha(x, y)\mathcal{H}_p(Tx, Ty) < \epsilon, \quad (3.23)$$

for all  $x, y \in X$ .

**Remark 3.11.** If  $T : X \rightarrow CB_p(X)$  is an  $\alpha$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ . Then we have

$$\alpha(x, y)\mathcal{H}_p(Tx, Ty) < p(x, y), \quad (3.24)$$

for all  $x, y \in X$  when  $p(x, y) > 0$ . On the other hand, observe that if  $p(x, y) = 0$ , we clearly have that  $H_p(Tx, Ty) = 0$  and using Proposition 2.8, we obtain that  $Tx = Ty$ . Thus for all  $x, y \in X$ , we get

$$\alpha(x, y)\mathcal{H}_p(Tx, Ty) \leq p(x, y). \quad (3.25)$$

**Corollary 3.12.** Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow CB_p(X)$  be an  $\alpha$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ . Suppose the following conditions hold:

- (1)  $T$  is strictly  $\alpha$ -admissible mapping,
- (2) there exists  $x_0 \in X$  and  $x_1 \in Tx_0$  such that  $\alpha(x_0, x_1) > 1$ ,
- (3) If  $\{x_n\}$  is a sequence such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  and  $\alpha(x_n, x_{n+1}) > 1$ , then  $\alpha(x_n, x) > 1$ , for all  $n \in \mathbb{N}$ .

Then  $T$  has a fixed point.

**Corollary 3.13.** Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow CB_p(X)$  be an  $\alpha$ -Meir-Keeler contraction with respect to the partial Hausdorff metric  $\mathcal{H}_p$ . Suppose the following conditions hold:

- (1)  $T$  is strictly  $\alpha$ -admissible mapping,
- (2) there exists  $x_0 \in X$  and  $x_1 \in Tx_0$  such that  $\alpha(x_0, x_1) > 1$ ,
- (3) If  $\{x_n\}$  is a sequence such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  and  $\alpha(x_n, x_{n+1}) > 1$ , then  $\alpha(x_n, x) > 1$ , for all  $n \in \mathbb{N}$ .

Then  $T$  has an endpoint  $x$  if and only if  $T$  has the approximate endpoint property.

**Corollary 3.14.** Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow CB_p(X)$  be a multivalued mapping such that for all  $x, y \in X$ , we have

$$\mathcal{H}_p(Tx, Ty) \leq kp(x, y),$$

where  $k \in (0, 1)$ . Then  $T$  has an endpoint  $x$  if and only if  $T$  has the approximate endpoint property.

**Corollary 3.15.** Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow CB_p(X)$  be a multivalued mapping such that for all  $x, y \in X$ , we have

$$\mathcal{H}_p(Tx, Ty) \leq kp(x, y),$$

where  $k \in (0, 1)$ . Then  $T$  has a fixed point.

**Remark 3.16.** It is worth mentioning that Corollary 3.15 is the main result of [15].

#### 4. CONCLUSION

Throughout this paper we have discussed the fixed point and endpoint theorems for an  $(\alpha, \beta)$ -Meir-Keeler contraction in the framework of complete partial metric spaces. We established that this class of an  $(\alpha, \beta)$ -Meir-Keeler contraction has an endpoint. But we have yet to establish that whether this endpoint is unique and this question remains open for interested mathematicians in this area of research.

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