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Efficient cut for a subset of prescribed area

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Abstract : We discuss an interesting isoperimetric problem on the plane. Given a set $S \subset \mathbb{R}^2$ of finite area A and a real number $0 < a < A$, we conjecture that there exists a set E such that $S \setminus E$ has connected components of area a and length(E) is less than or equal to the shortest length needed to enclose a component of area a inside a disk of area A. When S is convex with infinite area, such E may have length at most $\sqrt{2\pi a}$. We prove this for the case that $a = \frac{A}{2}$ and there is a point in which any line through equally bisects the area of S. Moreover, E can be chosen so that it is on a straight line. nearring.

1 Introduction

Around 900 BC., Princess Dido was challenged to enclose the biggest area by a given rope with fixed length. Intuitively, she came up with the right idea by making the rope a semicircle and put it against an almost straight sea shore, which is the only natural boundary found nearby. It is much later before mathematicians confirmed that this solution is the best possible. More precisely the shortest way to enclose a region of the given area on a half plane is by using a semicircle against the boundary of the half plane. Suppose A is the amount of area, then the length of the semicircle is $\sqrt{2\pi A}$.

During my study of soap bubble problem [5], I found an approach to partially help solving the planar triple bubble conjecture. The idea is for showing that a triple bubble with too many components is not minimizing. For this we need the following two steps. Step one, we must prove that a triple bubble with too many components has edges of too much total length between a pair of two regions R and S. Step two, we must prove that, after removing all the edges between the regions, we can find an efficient way to separate the union $R \cup S$ into two region of the original areas. The second step needs a lemma like

Conjecture 1.1. *Given a set $S \subset \mathbb{R}^2$ of finite area A and a real number $0 < a < A$, there exists a set E such that $S \setminus E$ has components of areas a and* length$(E) < \sqrt{2\pi a}$.

2 The conjecture

As we need perimeter $\sqrt{2\pi a}$ to enclose a subset of area a in a half plane, enclosing a region in a strictly convex set or a set with finite area should request less perimeter. This idea leads to 2 similar conjectures. Note that, as we use Lebesgue measure for area and length, we may ignore some parts of sets that are composed of isolated points or isolated curves.

1. When S has a strictly convex boundary, not necessary to have finite area, it is intuitively clear that we can find a single cut of length less than $\sqrt{2\pi a}$. Here we have the following conjecture.

Conjecture 2.1. *Suppose S is a convex subset of $\mathbb{R}^2$ and $a > 0$ is a real number. Then there exists a connected set E such that $S \setminus E$ has a component of area a and* $\text{length}(E) \leq \sqrt{2\pi a}$ *where the equality holds if and only if, almost everywhere, S is a half plane or a wide infinite strip.*

2. When S has finite area, not necessary to be connected, we intuitively believe that there is a cut of (total) length less than $\sqrt{2\pi a}$ as we can conjecture that

Conjecture 2.2. *Suppose S is a subset of $\mathbb{R}^2$ with finite area and $a > 0$ is a real number. Then there exists a set E such that $S \setminus E$ has components of total area a and* $\text{length}(E) < \sqrt{2\pi a}$.

Combining the previous 2 conjectures, we have

Conjecture 2.3. *Suppose $a > 0$ is a real number and $S \subset \mathbb{R}^2$ has a convex component of area at least a (including infinity), or has some components of total area at least a. Then there exists a set E such that $S \setminus E$ has components of total area a and* $\text{length}(E) \leq \sqrt{2\pi a}$.

Obviously these conjectures hold when S has smooth boundary and a is very small. In our personal communication, Frank Morgan provided interesting argument that might lead to a proof for the case that S is convex. When S is not convex, one may expect to find connected E. However, this is not possible due to an example by Ralph Alexander and I. David Berg after they attended my talk in the Geometric Potpourri seminar [6]. Due to early communication with Michael Hutchings, we have a stronger conjecture that $\text{length}(E)$ can be at most, $L(a, A)$, the length needed to enclose area a inside the circle of area A. Note that for all real numbers $0 < a < A$, we have $L(a, A) < \sqrt{2\pi a}$ and $L(a, A) = L(A - a, A)$. We may also say that $L(0, A) = 0$ and $L(A, A) = 0$.

Conjecture 2.4. *Suppose $S \subset \mathbb{R}^2$ has area A and $a > 0$ is a real number. Then there exists a set E such that $S \setminus E$ has components of area a and* $\text{length}(E) \leq L(a, A)$ *where the equality holds if and only if S is a disk almost everywhere.*

In addition, M. Hutchings also included that case $A = \infty$ into the previous conjecture as we can consider $\mathbb{R}^2$ as the circle of area ∞. This is to include the classical version that enclosing a subset of area a on a planar set needs perimeter at most $2\sqrt{\pi a}$. Here we may state that $L(a, \infty) = 2\sqrt{\pi a}$.

3 Bisecting a set into two equal area pieces

In this section we discuss the special case when $a = \frac{A}{2}$. There are number of papers that discuss this problem [3], [7], [4], [1]. For convenience, we assume $A = \pi R^2$ as we always compare S to the circle of area A, which then has radius R. Note that $L(\frac{A}{2}, A) = 2\sqrt{\frac{A}{\pi}} = 2R$. chosen to be collinear.

We say a line l ***bisects*** S or is a ***bisector*** of S if l divides S into two equal area pieces. We say S is ***centrally symmetric*** if S is point symmetric.

In 1958, Pólya proved the conjecture for the case that the boundary of S is a centrally symmetric closed curve.

Proposition 3.1. *[3] Suppose the boundary of S is a centrally symmetric closed curve. Then there is a segment E that bisect S and* $\text{length}(E) \leq 2R$ *where the equality holds if and only if S is a disk.*

Proof. We may assume that S is not a disk. Let C be the circle of area A centered at the center of S. Let x be a point of S inside C and let x' be the symmetric image of x. Hence the segment xx' is a bisector and is shorter than the diameter of C. □

We improve this result to more general sets after the following lemmas.

Lemma 3.2. *If $A \subset \mathbb{R}^+$, then $\frac{m(A)^2}{2} \leq \int_A xdx$ and the equality holds if and only if $A = [0, a]$ a.e. for some $a \geq 0$.*

Proof. Let A be a subset of $\mathbb{R}^+$. Therefore $\frac{m(A)^2}{2} = \int_0^{m(A)} xdx \leq \int_A xdx$. The condition for the equality to hold is clear. □

Lemma 3.3. *For $a, b \in \mathbb{R}^+$, we have $(a+b)^2 \leq 2a^2 + 2b^2$ and the equality holds if and only if $a = b$.*

Proof. Let a and b be positive real numbers. Since $2ab \leq a^2 + b^2$, hence $(a+b)^2 = a^2 + b^2 + 2ab \leq 2a^2 + 2b^2$. The condition for the quality to hold is clear. □

Lemma 3.4. *If $A \subset \mathbb{R}$, then $\frac{m(A)^2}{4} \leq \int_A |x|\, dx$ and the equality holds if and only if $A = [-a, a]$ a.e. for some $a \geq 0$.*

Proof. Let A be a subset of $\mathbb{R}$. By the previous 2 lemmas, we conclude that

$$\frac{m(A)^2}{4} = \frac{(m(A \cap \mathbb{R}^+) + m(A \cap \mathbb{R}^-))^2}{4} \leq \frac{m(A \cap \mathbb{R}^+)^2}{2} + \frac{m(A \cap \mathbb{R}^-)^2}{2}$$

$$= \int_0^{m(A \cap \mathbb{R}^+)} xdx + \int_0^{m(A \cap \mathbb{R}^-)} xdx \leq \int_{A \cap \mathbb{R}^+} xdx - \int_{A \cap \mathbb{R}^-} xdx = \int_A |x|\, dx.$$

The condition for the equality to hold is clear. □

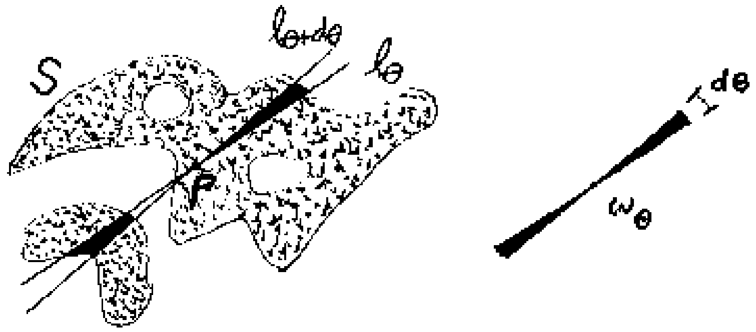

Figure 3.1: The area between l_θ and $l_{\theta+d\theta}$ is at most $(\frac{w_\theta}{2})^2 d\theta$.

Theorem 3.5. *Suppose S is a subset of $\mathbb{R}^2$ and p is a point in which every line through is a bisector of S. Then there exists a bisector l such that* $\text{length}(l \cap S) \leq 2R$ *and the equality holds if and only if S is a disk almost everywhere.*

Proof. Let S be a subset of $\mathbb{R}^2$, p a point in which every line through is a bisector of S, l_θ the line through p that makes angle θ to the horizontal line, and w_θ the ***width*** $\text{length}(l_\theta \cap S)$ of S along l_θ. By the previous lemmas with illustration by Figure 3.1, we have

$$A = \int_0^\pi \int_{-\infty}^{+\infty} \chi_S(p + r(\cos\theta, \sin\theta))\,|r|\,dr d\theta$$

$$= \int_0^\pi \int_{\{r|p+r(\cos\theta,\sin\theta)\in S\}} |r|\,dr d\theta \geq \int_0^\pi \frac{{w_\theta}^2}{4} d\theta.$$

As $A \geq \pi \frac{\min_\theta {w_\theta}^2}{4}$, we get $\min_\theta w_\theta \leq 2\sqrt{\frac{A}{\pi}} = 2R$. The condition for the equality to hold is clear. $\square$

To show that the condition of the theorem is weaker than being centrally symmetric, Figure 3.2 shows examples of S that satisfy the condition of Theorem 3.5 but are not centrally symmetric.

From the proof of Theorem 3.5, we can also conclude that $\max_\theta w_\theta \geq 2R$ and the equality holds if and only if S is a disk almost everywhere. In general, when bisectors do not meet at a point, M. Hutchings points out that such a cut E cannot always be chosen to be on a straight line. However, after the discussion with Chaiwat Maneesawang, it is likely to still have the same result on $\max_\theta w_\theta$.

4 Symmetricity measuring

Measuring of symmetricity is discussed in [2]. Here we introduce a natural way to measure how much a set is centrally symmetric. First, let S^p be the 180°-rotated

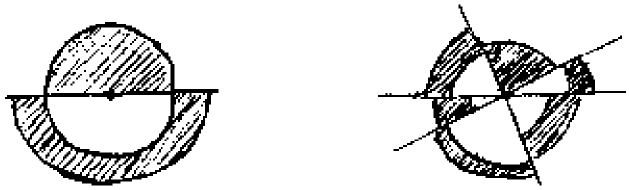

Figure 3.2: Two sets that satisfy the condition of Theorem 3.5 but are not centrally symmetric. The second set has no symmetric at all.

copy of S around p. Now we define a natural measure m of the symmetricity of S, when $\text{area}(S) > 0$, to be

$$m(S) = \frac{1}{\text{area}(S)} \sup_{p \in \mathbb{R}^2} \text{area}(S \cap S^p).$$

It is obvious that $0 \leq m(S) \leq 1$ and $m(S) = 1$ when S is centrally symmetric almost everywhere. We can also use this method for line symmetry, other symmetries, and all symmetries in higher dimensions.

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References

[1] H. Croft, K. Falconer, R. Guy, Unsolved Problems in Geometry, Problem Books in Mathematics, Unsolved Problems in Intuitive Mathematics Vol II, Springer-Verlag, 1991.

[2] B. Grünbaum, Measures of symmetry for convex sets, Convexity, Proceedings of Symposia in Pure Mathematics, v. VII, AMS, 1963, p. 233.

[3] G. Pólya, Aufgabe 283, *Elem. Math.* **13** (1958), 40-41.

[4] G. Pólya, Mathematics and Plausible Reasoning, v. I, 1967, 185-196.

[5] W. Wichiramala, The planar triple bubble problem, Ph.D. thesis, University of Illinois, Urbana-Champaign, 2002.

[6] W. Wichiramala, A length-area isoperimetric conjecture in the plane, Geometric Potpourri Seminar, University of Illinois, Urbana-Champaign, September 10th, 2002.

[7] K. Zarankiewicz, Bisection of plane convex sets by lines (Polish), *Wiadom. Mat.* (2) **2** (1959), 228-234.

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