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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

A Seqential Constraint Method for Solving Variational Inequality over the Intersection of Fixed Point Sets

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Abstract We consider the variational inequality problem over the intersection of fixed point sets of firmly nonexpansive operators. In order to solve the problem, we present an algorithm and subsequently show the strong convergence of the generated sequence to the solution of the considered problem.

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1. Introduction

It is well known that many problems arising in applications of mathematics can be formed as the finding a point that belongs to the nonempty intersection of finitely many closed convex sets, or in general, the fixed point sets of nonlinear operators in a Hilbert space $\mathcal{H}$, see for instance [1–4]. Namely, let a finite family of nonlinear operators $T_i : \mathcal{H} \to \mathcal{H}$ with, the set of all fixed points of the operator T_i, $\mathrm{Fix}\ T_i := \{x \in \mathcal{H} \mid T_i x = x\} \neq \emptyset$, $i = 1, 2, ..., m$, be given, the *common fixed point problem* is to find a point $x^* \in \mathcal{H}$ such that

$$x^* \in \bigcap_{i=1}^{m} \mathrm{Fix}\ T_i, \tag{1.1}$$

provided that the intersection is nonempty.

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According to its fruitful applications, there is a vast literature on solving the common fixed point problem (1.1). Notable methods and applications are proposed in [5–9] when dealing with the certain nonexpansivity of operators $T_i, i = 1, 2, ..., m$. For more approaches on wilder class of operators and many extrapolation variants, the reader can be found, for example, in [10–18] and many references therein.

Since the fixed point set of a nonexpansive operator is convex, it is clear that the intersection of such fixed point sets is also convex. This means that the problem (1.1) might have infinitely many solutions, otherwise it has a unique common fixed point. In this case it is customary to inquire that, under some prior criterion, which common fixed point is the best or at least a better common fixed point. A classical strategy is the *minimal norm solution problem* of finding a common fixed point in which it solves the minimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & x \in \bigcap_{i=1}^m \operatorname{Fix} T_i, \end{array}$$

provided that the problem has a solution. A number of iterative schemes for finding this minimal norm solution have been proposed, see for example, in [19–24] and references therein.

Along the line of selecting a specific solution among the common fixed points, and it is well known that the smooth convex optimization problem can be written as the so-called *variational inequality problem*. These observations motivated the solving a variational inequality problem over the common fixed point sets formulated as follows: given a monotone continuous operator $F : \mathcal{H} \to \mathcal{H}$, find $x^* \in \bigcap\limits_{i=1}^{m} \operatorname{Fix} T_i$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i. \tag{1.2}$$

Clearly, the minimal norm solution problem is an example of the problem (1.2) when $F(x)$ is the gradient of $\frac{1}{2}\|x\|^2$.

Among popular methods for dealing with this variational inequality problem (1.2), we underline, for instance, the classical work of Lions [25], where $T_i, i = 1, \ldots, m$ are supposed to be firmly nonexpansive and $F := Id - a$, for some $a \in \mathcal{H}$. After that, the case when $T_i, i = 1, \ldots, m$ are nonexpansive has been studied by Bauschke [26]. And, the most remarkable method is the so-called *hybrid steepest descent method* proposed by Yamada [27], where $T_i, i = 1, \ldots, m$ are supposed to be nonexpansive and the operator F is generally supposed to be strongly monotone and Lipschitz continuous. This starting point inspired many researchers to study in both generalizations of the problem setting and accelerations of this introduced iterative scheme, see [28–38] for more insight developments and applications.

In this paper, we deal with the variational inequality problem over the intersection of fixed point sets of firmly nonexpansive operators. We present an iterative scheme for solving the investigated problem. The proposed algorithm can be viewed as a generalization of the well-known hybrid steepest descent method in the allowance of adding appropriated information when computing of operators values. We subsequently give sufficient conditions for the convergence of the proposed method.

This paper is organized in the following way. We collect some technical definitions and useful facts needed in the paper in Sect. 2. In Sect. 3, we state the problem of consideration, namely the variational inequality over the intersection of fixed point sets, and discuss some remarkable examples. Whereas in Sect. 4, the proposed algorithm is introduced and analyzed. Actually, to get on with the proving our main theorem, in Subsect. 4.1 we prove several key tool lemmas, and subsequently establish the strong convergence of the sequence generated by proposed algorithm in Subsect. 4.2.

2. Preliminaries

Throughout the paper, $\mathcal{H}$ is always a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and with the norm $\| \cdot \|$. The strong convergence and weak convergence of a sequence $\{x^n\}_{n=1}^{\infty}$ to $x \in \mathcal{H}$ are indicated as $x^n \to x$ and $x^n \rightharpoonup x$, respectively. Id denotes the identity operator on $\mathcal{H}$.

An operator $F : \mathcal{H} \to \mathcal{H}$ is said to be *κ-Lipschitz continuous* if there is a real number $\kappa > 0$ such that

$$\|F(x) - F(y)\| \leq \kappa \|x - y\|,$$

for all $x, y \in \mathcal{H}$, and *η-strongly monotone* if there is a real number $\eta > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2,$$

for all $x, y \in \mathcal{H}$.

Firstly, in order to prove our convergence result, we need the following proposition. The proof of this result can be found in [27, Theorem 3.1].

Proposition 2.1. *Suppose that $F : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitz continuous and η-strongly monotone. If $\mu \in (0, 2\eta/\kappa^2)$, then for each $\beta \in (0, 1]$, the mapping $U^\beta := Id - \mu\beta F$ satisfies*

$$\|U^\beta x - U^\beta y\| \leq (1 - \beta\tau)\|x - y\|,$$

for all $x, y \in \mathcal{H}$, where $\tau := 1 - \sqrt{1 + \mu^2\kappa^2 - 2\mu\eta} \in (0, 1]$.

Next, we recall some noticeable operators. An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be *ρ-strongly quasi-nonexpansive* (SQNE), where $\rho \geq 0$, if $\operatorname{Fix} T \neq \emptyset$ and

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \rho\|Tx - x\|^2,$$

for all $x \in \mathcal{H}$ and $z \in \operatorname{Fix} T$. If $\rho > 0$, we say that T is strongly quasi-nonexpansive. If $\rho = 0$, then T is said to be *quasi-nonexpansive* (QNE), that is

$$\|Tx - z\| \leq \|x - z\|,$$

for all $x \in \mathcal{H}$ and $z \in \operatorname{Fix} T$. An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be *nonexpansive* (NE), if T is 1-Lipschitz continuous, that is

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in \mathcal{H}$. It is clear that a nonexpansive operator with nonempty fixed point set is quasi-nonexpansive. An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be a *cutter* if $\operatorname{Fix} T \neq \emptyset$ and

$$\langle x - Tx, z - Tx \rangle \leq 0,$$

for all $x \in \mathcal{H}$ and all $z \in \operatorname{Fix} T$. Furthermore, an operator $T : \mathcal{H} \to \mathcal{H}$ is said to be *firmly nonexpansive* (FNE), if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2,$$

for all $x, y \in \mathcal{H}$.

Some important properties applied in the further part of this paper are stated as the following facts which can be found in [3, Chapter 2].

Fact 2.2. Let $T : \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive operator. Then T is nonexpansive and it is a cutter, and hence quasi-nonexpansive.

Fact 2.3. If $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive, then Fix T is closed and convex.

Fact 2.4. Let $T : \mathcal{H} \to \mathcal{H}$ be an operator. The following properties are equivalent:

(i) T is a cutter.
(ii) $\langle Tx - x, z - x\rangle \geq \|Tx - x\|^2$ for every $x \in \mathcal{H}$ and $z \in \text{Fix } T$.
(iii) T is 1-strongly quasi-nonexpansive.

Below, we present further properties of a composition of strongly quasi-nonexpansive operators.

Fact 2.5. Let $T_i : \mathcal{H} \to \mathcal{H}, i = 1, 2, ..., m$, be strongly quasi-nonexpansive with $\bigcap_{i=1}^{m} \text{Fix } T_i \neq \emptyset$. Then a composition $T := T_m T_{m-1} \cdots T_1$ is also strongly quasi-nonexpansive and has the property:

$$\text{Fix } (T) = \text{Fix } (T_m T_{m-1} \cdots T_1) = \bigcap_{i=1}^{m} \text{Fix } T_i.$$

An operator $T : \mathcal{H} \to \mathcal{H}$ is said to satisfy the *demi-closedness* (DC) principle if $T - Id$ is demi-closed at 0, i.e., for any weakly converging sequence $\{x^n\}_{n=1}^{\infty}$ such that $x^n \rightharpoonup y \in \mathcal{H}$ as $n \to \infty$ with $\|Tx^n - x^n\| \to 0$ as $n \to \infty$, we have $y \in \text{Fix } T$.

The following fact is well known and can be found in [2, Corollary 4.28].

Fact 2.6. If $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator with Fix $T \neq \emptyset$, then the operator $T - Id$ is demi-closed at 0.

In order to prove the convergence result, we need the following proposition which can be found in [2, Corollary 2.15].

Proposition 2.7. *The following equality holds for all $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{R}$:*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

We close this section by presenting a special case of [39, Proposition 4.6] which plays an important role in proving our convergence result.

Proposition 2.8. *Let $T_i : \mathcal{H} \to \mathcal{H}, i = 1, 2, ..., m$, be cutter operators with $\bigcap_{i=1}^{m} \text{Fix } T_i \neq \emptyset$. Denote the compositions $T := T_m T_{m-1} \cdots T_1$, and $S_i := T_i T_{i-1} \cdots T_1$, where $S_0 := Id$. Then, for any $x \in \mathcal{H}$ and $z \in \bigcap_{i=1}^{m} \text{Fix } T_i$, it holds that*

$$\frac{1}{2L} \sum_{i=1}^{m} \|S_i x - S_{i-1} x\|^2 \leq \|Tx - x\|, \tag{2.1}$$

for any $L \geq \|x - z\|$.

3. Problem Statement

In this section, we state our main problem as follows:

Problem 3.1 (VIP). Assume that

(i) $T_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, ..., m$, are firmly nonexpansive with $\bigcap_{i=1}^{m} \operatorname{Fix} T_i \neq \emptyset$.

(ii) $F : \mathcal{H} \to \mathcal{H}$ is η-strongly monotone and κ-Lipschitz continuous with $\kappa \geq \eta > 0$.

The problem is to find a point $x^* \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \qquad \text{for all } x \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i.$$

Remark 3.2. By the assumptions (i) and (ii), we know from [40, Theorem 2.3.3] that Problem **(VIP)** has the unique solution.

Problem **(VIP)** also lies in the models of the suitably selected choice among common fixed point problems as the following few examples.

Now, let $B : \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued operator. The monotone inclusion problem is to find a point $x^* \in \mathcal{H}$ such that

$$0 \in B(x^*),$$

provided it exists. Actually, we denote by $\operatorname{Gr}(B) := \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Bx\}$ its graph, and $\operatorname{zer}(B) := \{z \in \mathcal{H} : 0 \in B(z)\}$ the set of all zero points of the operator B. The set-valued operator B is said to be *monotone* if $\langle x-y, u-v \rangle \geq 0$, for all $(x, u), (y, v) \in \operatorname{Gr}(B)$, and it is called *maximally monotone* if its graph is not properly contained in the graph of any other monotone operators. For a set-valued operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$, we define the *resolvent* of B, $J_B : \mathcal{H} \rightrightarrows \mathcal{H}$, by

$$J_B := (Id + B)^{-1}.$$

Note that if B is maximally monotone and $r > 0$, then the resolvent J_{rB} of rB is (single-valued) FNE with

$$\operatorname{Fix} J_{rB} = \operatorname{zer}(B),$$

see [2, Proposition 23.8, Proposition 23.38].

Thus, for a given $r > 0$ and a finitely many maximally monotone operators $B_i : \mathcal{H} \rightrightarrows \mathcal{H}, i = 1, 2, ..., m$, we put $T_i := J_{rB_i}, i = 1, 2, ..., m$, Problem **(VIP)** is nothing else than, in particular, the problem of finding a point $x^* \in \bigcap_{i=1}^{m} \operatorname{zer}(B_i)$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \bigcap_{i=1}^{m} \operatorname{zer}(B_i).$$

Some interesting iterative methods for solving this type of problem and its particular situations are investigated in [41–43].

Moreover, recalling that for a given $r > 0$ and a proper convex lower semicontinuous function $f : \mathcal{H} \to (-\infty, +\infty]$, we denote by $\operatorname{prox}_{rf}(x)$ the *proximal point* of parameter r of f at x, which is the unique optimal solution of the optimization problem

$$\min \left\{ f(u) + \frac{1}{2r} \|u - x\|^2 : u \in \mathcal{H} \right\}.$$

It is known that $\text{prox}_{rf} = J_{r\partial f}$ (see [2, Example 23.3]) which is FNE and Fix $\text{prox}_{r\varphi} = \arg\min f := \{x \in \mathcal{H} : f(x) \leq f(u), \forall u \in \mathcal{H}\}$. Thus, for a finitely many proper convex lower semicontinuous functions $f_i : \mathcal{H} \to (-\infty, +\infty], i = 1, 2, ..., m$ and putting $T_i := \text{prox}_{rf_i}$, Problem **(VIP)** is reduced to the problem of finding a point $x^* \in \bigcap_{i=1}^{m} \arg\min f_i$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \bigcap_{i=1}^{m} \arg\min f_i,$$

see [36, 44] for more details about this problem. In these cases, Algorithm 1 and Theorem 4.2 below are also applicable for these two problems.

4. Algorithm and Its Convergence Analysis

In this section, we will propose an algorithm for solving Problem **(VIP)** and subsequently analyze their convergence properties under some certain conditions.

Firstly, we now present an iterative method for solving Problem **(VIP)** as follows:

Algorithm 1 Sequential Constraint Method (in short, SCM)

Initialization: The positive real sequences $\{\lambda_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, and real number μ. Take an arbitrary $x^1 \in \mathcal{H}$.

Iterative Step: For a given current iterate $x^n \in \mathcal{H}$ $(n \geq 1)$, set

$$\varphi_0^n := x^n - \mu\beta_n F(x^n).$$

Define

$$\varphi_i^n := T_i \varphi_{i-1}^n + e_i^n, \qquad i = 1, \ldots, m,$$

where $e_i^n \in \mathcal{H}$ is added information when computing $T_i\varphi_{i-1}^n$'s value. Compute

$$x^{n+1} := (1 - \lambda_n)\varphi_0^n + \lambda_n \varphi_m^n.$$

Update $n = n + 1$.

Remark 4.1. Some useful comments are in order:

(i) It is important to point out that the term $e_i^n, i = 1 \ldots, m$, can be viewed as added information when computing the operator T's values, for instance, a feasible like direction. Actually, in constrained optimization problem, we call a vector d a feasible direction at the current iterate x_k if the estimate $x_k + d$ belongs to the constrained set. Notice that, in our situation, we can not ensure that each estimate $T_i\varphi_{i-1}^n$ belongs to the fixed point set Fix T_i. Thus, adding an appropriated term e_i^n possibly helps the estimate φ_i^n get closer to Fix T_i so that the convergence can be improved.

(ii) Apart from (i), the presence of added information $e_i^n, i = 1 \ldots, m$, can be viewed as the allowance of possible numerical errors on the computations of T_i's operator value. This situation may occur when the explicit form of T_i is not known, or even when T_i's operator value can be found approximately by solving a subproblem, for instance a metric projection onto a nonempty closed convex set, a proximity operator of a proper convex and lower semicontinuous function, or even the resolvent operator of a maximally monotone operator.

The main theorem of this section is as follows:

Theorem 4.2. *Suppose that* $\mu \in (0, 2\eta/\kappa^2)$, $\{\beta_n\}_{n=1}^{\infty} \subset (0,1]$ *satisfies* $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = +\infty$, *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [\varepsilon, 1-\varepsilon]$ *for some constant* $\varepsilon \in (0, 1/2]$. *If* $\sum_{n=1}^{\infty} \|e_i^n\| < +\infty$ *for each* $i = 1, 2, ..., m$, *then the sequence* $\{x^n\}_{n=1}^{\infty}$ *generated by Algorithm 1 converges strongly to the unique solution to Problem* **(VIP)**.

Remark 4.3. It is worth underlining that the assumptions on step sizes sequence $\{\beta_n\}_{n=1}^{\infty}$ hold true for several choices which include, for instance, $\beta_n := \beta/n, n \geq 1$, for any choice of $\beta \in (0, 1]$. Moreover, the parameter μ, which is used in Theorem 4.2, need to be chosen in the interval $(0, 2\eta/\kappa^2)$ so that the operator $Id - \mu\beta_n F$ is a contraction (see, Proposition 2.1) for any choice of the step sizes $\{\beta_n\}_{n=1}^{\infty}$.

In order to proceed the convergence analyses, we will consider the following into 2 parts. Actually, we start in the first part with a series of preliminary convergence results, and subsequently, present the main convergence proof of Theorem 4.2.

4.1. Preliminary Convergence Results

Before we present some useful lemmas used in proving Theorem 4.2, we make use of the following notations: the compositions

$$T := T_m T_{m-1} \cdots T_1,$$

$$S_0 := Id, \quad \text{and} \quad S_i := T_i T_{i-1} \cdots T_1, \quad i = 1, 2, ..., m.$$

Moreover, the iterate x^{n+1} is the combination

$$x^{n+1} = w^n + u^n, \tag{4.1}$$

where

$$\begin{aligned} w^n &:= \varphi_0^n + \lambda_n (T\varphi_0^n - \varphi_0^n), \\ u^n &:= \lambda_n (\varphi_m^n - T\varphi_0^n), \end{aligned}$$

for all $n \geq 1$.

Now, we start the convergence proof with the following technical result.

Lemma 4.4. *The series* $\sum_{n=1}^{\infty} \|u^n\|$ *converges.*

Proof. Let $z \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$ and $n \geq 1$ be fixed. By using the triangle inequality, we note that

$$\|x^{n+1} - z\| = \|w^n + u^n - z\| \leq \|w^n - z\| + \|u^n\|. \tag{4.2}$$

By using Proposition 2.7 and the quasi-nonexpansitivity of T, we obtain

$$\begin{aligned}\|w^n - z\|^2 &= \|\varphi_0^n + \lambda_n(T\varphi_0^n - \varphi_0^n) - z\|^2 \\ &= \|\lambda_n(T\varphi_0^n - z) + (1-\lambda_n)\varphi_0^n - (1-\lambda_n)z\|^2 \\ &= \|\lambda_n(T\varphi_0^n - z) + (1-\lambda_n)(\varphi_0^n - z)\|^2 \\ &= \lambda_n\|T\varphi_0^n - z\|^2 + (1-\lambda_n)\|\varphi_0^n - z\|^2 - \lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2 \\ &\leq \lambda_n\|\varphi_0^n - z\|^2 + (1-\lambda_n)\|\varphi_0^n - z\|^2 - \lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2 \\ &= \|\varphi_0^n - z\|^2 - \lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2. \end{aligned} \tag{4.3}$$

Since the relaxation parameter $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$, we obtain that

$$\|w^n - z\| \leq \|\varphi_0^n - z\|, \tag{4.4}$$

and, subsequently, the inequality (4.2) becomes

$$\|x^{n+1} - z\| \leq \|\varphi_0^n - z\| + \|u^n\|. \tag{4.5}$$

On the other hand, the nonexpansitivity of $T_i, i = 1, \ldots, m$, and the triangle inequality yield

$$\begin{aligned}\|u^n\| &= \|\lambda_n(\varphi_m^n - T\varphi_0^n)\| \\ &\leq \|\varphi_m^n - T\varphi_0^n\| \\ &= \|T_m(T_{m-1}(\cdots T_2(T_1\varphi_0^n + e_1^n) + e_2^n \cdots) + e_{m-1}^n) + e_m^n - T_m T_{m-1}\cdots T_1\varphi_0^n\| \\ &\leq \|e_m^n\| + \|T_m(T_{m-1}(\cdots T_2(T_1\varphi_0^n + e_1^n) + e_2^n \cdots) + e_{m-1}^n) - T_m T_{m-1}\cdots T_1\varphi_0^n\| \\ &\leq \|e_m^n\| + \|T_{m-1}(\cdots T_2(T_1\varphi_0^n + e_1^n) + e_2^n \cdots) + e_{m-1}^n - T_{m-1}\cdots T_1\varphi_0^n\| \\ &\leq \|e_m^n\| + \|e_{m-1}^n\| + \|T_{m-1}(\cdots T_2(T_1\varphi_0^n + e_1^n) + e_2^n \cdots) - T_{m-1}\cdots T_1\varphi_0^n\| \\ &\;\;\vdots \\ &\leq \sum_{i=1}^{m} \|e_i^n\|. \end{aligned}$$

Since, for each $i = 1, \ldots, m$, $\sum_{n=1}^{\infty} \|e_i^n\| < +\infty$, we get

$$\sum_{n=1}^{\infty} \|u^n\| < +\infty,$$

as required. ∎

Before we proceed further convergence properties, we will show that the generated sequences are bounded as the following lemma.

Lemma 4.5. *The sequences $\{x^n\}_{n=1}^{\infty}$, $\{F(x^n)\}_{n=1}^{\infty}$ and $\{\varphi_0^n\}_{n=1}^{\infty}$ are bounded.*

Proof. Let $z \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$ and $n \geq 1$ be fixed. By using Proposition 2.1, we note that

$$\begin{aligned}\|\varphi_0^n - z\| &= \|x^n - \mu\beta_n F(x^n) - z\| \\ &= \|(x^n - \mu\beta_n F(x^n)) - (z - \mu\beta_n F(z)) - \mu\beta_n F(z)\| \\ &\leq \|(x^n - \mu\beta_n F(x^n)) - (z - \mu\beta_n F(z))\| + \mu\beta_n\|F(z)\| \\ &= \|(Id - \mu\beta_n F)x^n - (Id - \mu\beta_n F)z\| + \mu\beta_n\|F(z)\| \\ &\leq (1 - \beta_n\tau)\|x^n - z\| + \mu\beta_n\|F(z)\|, \end{aligned} \tag{4.6}$$

where $\tau = 1 - \sqrt{1 + \mu^2\kappa^2 - 2\mu\eta} \in (0, 1]$.

Now, by using (4.5) together with the above inequality, we have

$$\begin{aligned}\|x^{n+1} - z\| &\leq \|\varphi_0^n - z\| + \|u^n\| \\ &\leq (1 - \beta_n\tau)\|x^n - z\| + \mu\beta_n\|F(z)\| + \|u^n\| \\ &\leq \max\left\{\|x^n - z\|, \frac{\mu}{\tau}\|F(z)\|\right\} + \|u^n\|.\end{aligned}$$

By the induction argument, we obtain that

$$\|x^{n+1} - z\| \leq \max\left\{\|x^1 - z\|, \frac{\mu}{\tau}\|F(z)\|\right\} + \sum_{i=1}^{n}\|u^i\|, \quad \forall n \geq 1.$$

By Lemma 4.4, we know that $\sum\limits_{n=1}^{\infty}\|u^n\| < +\infty$, we obtain that $\{x^n\}_{n=1}^{\infty}$ is bounded. Moreover, the use of Lipschitz continuity of the operator F implies that $\{F(x^n)\}_{n=1}^{\infty}$ is bounded, and consequently, $\{\varphi_0^n\}_{n=1}^{\infty}$ is also bounded. ∎

For an element $z \in \bigcap\limits_{i=1}^{m} \operatorname{Fix} T_i$ and all $n \geq 1$, we denote from this point onward that

$$v := 2\left(\sup_{n\geq 1}\|x^n - z\| + \mu\|F(z)\|\right) + \sup_{n\geq 1}\|u^n\| < +\infty,$$

$$\xi_n := \mu^2\beta_{n+1}^2\|F(x^n)\|^2 + 2\mu\beta_n\|x^n - z\|\|F(x^n)\| + v\|u^n\|,$$

$$\delta_n := \frac{\beta_n}{\tau}\left(\mu^2\|F(z)\|^2 + 2\mu^2\langle F(x^n) - F(z), F(z)\rangle\right) + \frac{2\mu}{\tau}\langle x^n - z, -F(z)\rangle,$$

and

$$\alpha_n := \beta_n\tau.$$

Lemma 4.6. *The limit* $\lim_{n\to\infty}\xi_n = 0$.

Proof. Invoking the boundedness of the sequences $\{x^n\}_{n=1}^{\infty}$ and $\{F(x^n)\}_{n=1}^{\infty}$, Lemma 4.4, and the assumption that $\lim_{n\to\infty}\beta_n = 0$, we obtain

$$0 \leq \xi_n = \mu^2\beta_n^2\|F(x^n)\|^2 + 2\mu\beta_n\|x^n - z\|\|F(x^n)\| + v\|u^n\| \to 0,$$

as desired. ∎

The following lemma states a key tool inequality on the generated sequence which will be formed the basis relation for our convergence results.

Lemma 4.7. *The following statement holds:*

$$\|x^{n+1} - z\|^2 \leq \|x^n - z\|^2 - \frac{\lambda_n(1-\lambda_n)}{4L^2}\left(\sum_{i=1}^{m}\|S_i\varphi_0^n - S_{i-1}\varphi_0^n\|^2\right)^2 + \xi_n,$$

for all $z \in \bigcap\limits_{i=1}^{m} \operatorname{Fix} T_i$ *and all* $n \geq 1$.

Proof. Let $z \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$ and $n \geq 1$ be fixed. From the inequality (4.6), we have

$$\|\varphi_0^n - z\| \leq \|x^n - z\| + \mu\beta_n \|F(z)\|.$$

By using (4.2) together with (4.4) and the above inequality, we obtain that

$$\begin{aligned}
\|x^{n+1} - z\|^2 &\leq (\|w^n - z\| + \|u^n\|)^2 \\
&= \|w^n - z\|^2 + 2\|w^n - z\|\|u^n\| + \|u^n\|^2 \\
&\leq \|w^n - z\|^2 + (2\|\varphi_0^n - z\| + \|u^n\|)\|u^n\| \\
&\leq \|w^n - z\|^2 + [2(\|x^n - z\| + \mu\beta_n\|F(z)\|) + \|u^n\|]\|u^n\| \\
&\leq \|w^n - z\|^2 + \left[2\left(\sup_{n\geq 1}\|x^n - z\| + \mu\|F(z)\|\right) + \sup_{n\geq 1}\|u^n\|\right]\|u^n\| \\
&= \|w^n - z\|^2 + v\|u^n\|, \qquad (4.7)
\end{aligned}$$

where the fifth inequality holds from the assumption that $\{\beta_n\}_{n=1}^{\infty} \subset (0,1]$ and the boundedness of the sequences $\{x^n\}_{n=1}^{\infty}$ and $\{u^n\}_{n=1}^{\infty}$.

Invoking the obtained inequality (4.7) in (4.3), we obtain

$$\begin{aligned}
\|x^{n+1} - z\|^2 &\leq \|\varphi_0^n - z\|^2 - \lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2 + v\|u^n\| \\
&= \|x^n - \mu\beta_n F(x^n) - z\|^2 - \lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2 + v\|u^n\| \\
&= \|x^n - z\|^2 + \mu^2\beta_n^2\|F(x^n)\|^2 - 2\mu\beta_n\langle x^n - z, F(x^n)\rangle \\
&\quad -\lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2 + v\|u^n\| \\
&\leq \|x^n - z\|^2 + \mu^2\beta_n^2\|F(x^n)\|^2 + 2\mu\beta_n\|x^n - z\|\|F(x^n)\| \\
&\quad -\lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2 + v\|u^n\| \\
&= \|x^n - z\|^2 - \lambda_n(1-\lambda_n)\|T\varphi_0^n - \varphi_0^n\|^2 + \xi_n,
\end{aligned}$$

Putting $L := \sup_{n\geq 1}\|x^n - z\|$, using the above inequality, and Proposition 2.8, we arrive that

$$\|x^{n+1} - z\|^2 \leq \|x^n - z\|^2 - \frac{\lambda_n(1-\lambda_n)}{4L^2}\left(\sum_{i=1}^{m}\|S_i\varphi_0^n - S_{i-1}\varphi_0^n\|^2\right)^2 + \xi_n,$$

which completes the proof. ∎

The following lemma shows that the weak cluster point of the generated sequences belongs to the intersection of fixed point sets.

Lemma 4.8. *If the sequence $\{\varphi_0^n\}_{n=1}^{\infty}$ satisfies $\|S_i\varphi_0^n - S_{i-1}\varphi_0^n\| \to 0$ for all $i = 1, \ldots, m$, then the weak cluster point $z \in \mathcal{H}$ of $\{\varphi_0^n\}_{n=1}^{\infty}$ belongs to $\bigcap_{i=1}^{m} \operatorname{Fix} T_i$.*

Proof. Since $\{\varphi_0^n\}_{n=1}^{\infty}$ is bounded, we let $z \in \mathcal{H}$ be a weak cluster point of $\{\varphi_0^n\}_{n=1}^{\infty}$, and let $\{\varphi_0^{n_k}\}_{k=1}^{\infty} \subset \{\varphi_0^n\}_{n=1}^{\infty}$ be a subsequence such that $\varphi_0^{n_k} \rightharpoonup z$. Now, we note that

$$\|(T_1 - Id)\varphi_0^{n_k}\| = \|T_1\varphi_0^{n_k} - \varphi_0^{n_k}\| = \|S_1\varphi_0^{n_k} - S_0\varphi_0^{n_k}\| \to 0.$$

Since T_1 satisfies the DC principle, we obtain that

$$z \in \operatorname{Fix} T_1.$$

Note that

$$\|(T_1\varphi_0^{n_k} - T_1 z) - (\varphi_0^{n_k} - z)\| = \|(T_1 - Id)\varphi_0^{n_k}\| \to 0,$$

and $\varphi_0^{n_k} \rightharpoonup z$ together imply that

$$T_1\varphi_0^{n_k} \rightharpoonup T_1 z = z.$$

But we know that

$$\|(T_2 - Id)T_1\varphi_0^{n_k}\| = \|T_2T_1\varphi_0^{n_k} - T_1\varphi_0^{n_k}\| = \|S_2\varphi_0^{n_k} - S_1\varphi_0^{n_k}\| \to 0,$$

and, consequently, the DC principle of T_2 yields that

$$z \in \text{Fix } T_2.$$

By proceeding the above proving lines, we obtain that

$$z \in \text{Fix } T_i \quad \forall i = 1, 2, ..., m,$$

which means that $z \in \bigcap_{i=1}^m \text{Fix } T_i$. ∎

The following lemma presents the key relation for obtaining the strong convergence of the generated sequence.

Lemma 4.9. *The following statement holds:*

$$\|x^{n+1} - z\|^2 \leq (1 - \alpha_n)\|x^n - z\|^2 + \alpha_n\delta_n + v\|u^n\|,$$

for all $z \in \bigcap_{i=1}^{m} \text{Fix } T_i$ *and all* $n \geq 1$.

Proof. Let $z \in \bigcap_{i=1}^{m} \text{Fix } T_i$ and $n \geq 1$ be fixed. By utilizing the inequalities (4.4), (4.7), and Proposition 2.1, we note that

$$\begin{aligned}
\|x^{n+1} - z\|^2 &\leq \|w^n - z\|^2 + v\|u^n\| \\
&\leq \|\varphi_0^n - z\|^2 + v\|u^n\| \\
&= \|x^n - \mu\beta_n F(x^n) - z + \mu\beta_n F(z) - \mu\beta_n F(z)\|^2 + v\|u^n\| \\
&= \|[(x^n - \mu\beta_n F(x^n)) - (z - \mu\beta_n F(z))] - \mu\beta_n F(z)\|^2 + v\|u^n\| \\
&= \|(x^n - \mu\beta_n F(x^n)) - (z - \mu\beta_n F(z))\|^2 + \|\mu\beta_n F(z)\|^2 \\
&\quad -2\langle x^n - \mu\beta_n F(x^n) - z + \mu\beta_n F(z), \mu\beta_n F(z)\rangle + v\|u^n\| \\
&= \|(Id - \mu\beta_n F)x^n - (Id - \mu\beta_n F)z\|^2 + \mu^2\beta_n^2\|F(z)\|^2 \\
&\quad -2\langle (x^n - z) - (\mu\beta_n F(x^n) - \mu\beta_n F(z)), \mu\beta_n F(z)\rangle + v\|u^n\| \\
&\leq (1 - \beta_n\tau)^2\|x^n - z\|^2 + \mu^2\beta_n^2\|F(z)\|^2 + v\|u^n\| \\
&\quad -2\langle (x^n - z) - \mu\beta_n(F(x^n) - F(z)), \mu\beta_n F(z)\rangle \\
&\leq (1 - \beta_n\tau)\|x^n - z\|^2 + \mu^2\beta_n^2\|F(z)\|^2 + v\|u^n\| \\
&\quad -2\mu\beta_n\langle x^n - z, F(z)\rangle + 2\mu^2\beta_n^2\langle F(x^n) - F(z), F(z)\rangle \\
&= (1 - \beta_n\tau)\|x^n - z\|^2 + v\|u^n\| \\
&\quad +\beta_n\left[\mu^2\beta_n\|F(z)\|^2 - 2\mu\langle x^n - z, F(z)\rangle + 2\mu^2\beta_n\langle F(x^n) - F(z), F(z)\rangle\right] \\
&= (1 - \beta_n\tau)\|x^n - z\|^2 + v\|u^n\| \\
&\quad +\beta_n\tau\left[\frac{\beta_n}{\tau}\left(\mu^2\|F(z)\|^2 + 2\mu^2\langle F(x^n) - F(z), F(z)\rangle\right) + \frac{2\mu}{\tau}\langle x^n - z, -F(z)\rangle\right] \\
&= (1 - \alpha_n)\|x^n - z\|^2 + \alpha_n\delta_n + v\|u^n\|,
\end{aligned}$$

which completes the proof. ∎

4.2. Convergence Proof

In order to prove our main theorem, we need the following proposition which was proven in [45].

Proposition 4.10. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n,$$

where $\{\alpha_n\}_{n=1}^{\infty} \subseteq [0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = +\infty$, $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\limsup_{n\to 0} \beta_n \leq 0$ and $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} \gamma_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

We are now in a position to prove Theorem 4.2.

Proof. Let $\bar{u}$ be the unique solution to Problem **(VIP)**. Then, $\bar{u} \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$ and all above results hold true with replacing $z = \bar{u}$. Now, for simplicity, we denote $a_n := \|x^n - \bar{u}\|^2$. Firstly, it should be remembered from Lemma 4.4 and Lemma 4.6 that $\lim_{n\to\infty} v\|u^n\| = 0$ and $\lim_{n\to\infty} \xi_n = 0$, respectively.

We will show that the generated sequence $\{x^n\}_{n=1}^{\infty}$ converges strongly to $\bar{u}$ by considering the two following cases.

Case 1. Suppose that $\{a_n\}_{n=1}^{\infty}$ is eventually decreasing, i.e., there exists $n_0 \geq 1$ such that $a_{n+1} < a_n$ for all $n \geq n_0$. In this case, $\{a_n\}_{n=1}^{\infty}$ must be convergent. Setting $\lim_{n\to\infty} a_n = r$. In view of Lemma 4.7 with $z = \bar{u}$ and using Lemma 4.6, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \frac{\lambda_n(1-\lambda_n)}{4L^2}\left(\sum_{i=1}^{m} \|S_i\varphi_0^n - S_{i-1}\varphi_0^n\|^2\right)^2 \\ &\leq \limsup_{n\to\infty} (a_n - a_{n+1} + \xi_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} a_{n+1} + \lim_{n\to\infty} \xi_n = 0, \end{aligned}$$

and hence

$$\lim_{n\to\infty} \frac{\lambda_n(1-\lambda_n)}{4L^2}\left(\sum_{i=1}^{m} \|S_i\varphi_0^n - S_{i-1}\varphi_0^n\|^2\right)^2 = 0.$$

Since $\{\lambda_n\}_{n=1}^{\infty} \subset [\varepsilon, 1-\varepsilon]$, we have $\lambda_n(1-\lambda_n) \geq \varepsilon^2$ for all $n \geq 1$, and, consequently,

$$\lim_{n\to\infty} \sum_{i=1}^{m} \|S_i\varphi_0^n - S_{i-1}\varphi_0^n\|^2 = 0,$$

which implies that, for all $i = 1, 2, ..., m$,

$$\lim_{n\to\infty} \|S_i\varphi_0^n - S_{i-1}\varphi_0^n\| = 0. \tag{4.8}$$

On the other hand, since the sequence $\{\varphi_0^n\}_{n=1}^{\infty}$ is bounded, we have $\{\langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle\}_{n=1}^{\infty}$ is also bounded. Now, let $\{\varphi_0^{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{\varphi_0^n\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty} \langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle = \lim_{k\to\infty} \langle \varphi_0^{n_k} - \bar{u}, -F(\bar{u})\rangle.$$

Since $\{\varphi_0^{n_k}\}_{k=1}^{\infty}$ is of course bounded, it indeed has a weakly cluster point $z \in \mathcal{H}$ and a subsequence $\{\varphi_0^{n_{k_j}}\}_{j=1}^{\infty}$ such that $\varphi_0^{n_{k_j}} \rightharpoonup z \in \mathcal{H}$. Thus, it follows from Lemma 4.8 and (4.8) that $z \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$. Since $\bar{u}$ is the unique solution to Problem **(VIP)**, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle &= \lim_{k\to\infty}\langle \varphi_0^{n_k} - \bar{u}, -F(\bar{u})\rangle \\ &= \lim_{j\to\infty}\langle \varphi_0^{n_{k_j}} - \bar{u}, -F(\bar{u})\rangle = \langle z - \bar{u}, -F(\bar{u})\rangle \le 0. \end{aligned} \tag{4.9}$$

Now, let us note that

$$\begin{aligned}\langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle &= \langle x^n - \mu\beta_n F(x^n) - \bar{u}, -F(\bar{u})\rangle \\ &= \langle x^n - \bar{u}, -F(\bar{u})\rangle - \mu\beta_n\langle F(x^n), -F(\bar{u})\rangle,\end{aligned}$$

and by setting $p := \sup_{n\ge1}\|F(x^n)\| < +\infty$, we have

$$\begin{aligned}\langle x^n - \bar{u}, -F(\bar{u})\rangle &= \langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle + \mu\beta_n\langle F(x^n), -F(\bar{u})\rangle \\ &\le \langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle + \mu\beta_n\|F(x^n)\|\| - F(\bar{u})\rangle\| \\ &\le \langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle + \mu\beta_n\|F(\bar{u})\|\sup_{n\ge1}\|F(x^n)\| \\ &= \langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle + \mu p\beta_n\|F(\bar{u})\|.\end{aligned}$$

Invoking the assumption $\lim_{n\to\infty}\beta_n = 0$ and (4.9), we obtain

$$\limsup_{n\to\infty}\langle x^n - \bar{u}, -F(\bar{u})\rangle \le \limsup_{n\to\infty}\langle \varphi_0^n - \bar{u}, -F(\bar{u})\rangle + \mu p\|F(\bar{u})\|\lim_{n\to\infty}\beta_n \le 0. \tag{4.10}$$

In view of δ_n with replacing $z = \bar{u}$, we get

$$\begin{aligned}\delta_n &= \frac{\beta_n}{\tau}\left(\mu^2\|F(\bar{u})\|^2 + 2\mu^2\langle F(x^n) - F(\bar{u}), F(\bar{u})\rangle\right) + \frac{2\mu}{\tau}\langle x^n - \bar{u}, -F(\bar{u})\rangle \\ &\le \frac{\beta_n}{\tau}\left(\mu^2\|F(\bar{u})\|^2 + 2\mu^2\sup_{n\ge1}\langle F(x^n) - F(\bar{u}), F(\bar{u})\rangle\right) + \frac{2\mu}{\tau}\langle x^n - \bar{u}, -F(\bar{u})\rangle \\ &= \frac{1}{\tau}\left(\mu^2\|F(\bar{u})\|^2 + 2\mu^2 q\right)\beta_n + \frac{2\mu}{\tau}\langle x^n - \bar{u}, -F(\bar{u})\rangle,\end{aligned}$$

where $q := \sup_{n\ge1}\langle F(x^n) - F(\bar{u}), F(\bar{u})\rangle < +\infty$. Again, the assumption $\lim_{n\to\infty}\beta_n = 0$ and (4.10) yield that

$$\begin{aligned}\limsup_{n\to\infty}\delta_n &= \frac{1}{\tau}\left(\mu^2\|F(\bar{u})\|^2 + 2\mu^2 q\right)\lim_{n\to\infty}\beta_n + \frac{2\mu}{\tau}\limsup_{n\to\infty}\langle x^n - \bar{u}, -F(\bar{u})\rangle \\ &= \frac{2\mu}{\tau}\limsup_{n\to\infty}\langle x^n - \bar{u}, -F(\bar{u})\rangle \le 0.\end{aligned} \tag{4.11}$$

Finally, in view of Lemma 4.9 with $z = \bar{u}$, we have

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\delta_n + v\|u^n\|.$$

Since $\alpha_n = \beta_n\tau$, and we know that $\tau \le 1$, we have $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1]$. Moreover, since $\sum_{n=1}^{\infty}\beta_n = +\infty$, we have $\sum_{n=1}^{\infty}\alpha_n = \tau\sum_{n=1}^{\infty}\beta_n = +\infty$. Hence, by using (4.11), Lemma 4.4, and applying Proposition 4.10, we conclude that $\lim_{n\to\infty}\|x^n - \bar{u}\| = 0$.

Case 2. Suppose that $\{a_n\}_{n=1}^{\infty}$ is not eventually decreasing. Thus, we can find an integer n_0 such that $a_{n_0} \leq a_{n_0+1}$. Now, for each $n \geq n_0$, we define

$$J_n := \{k \in [n_0, n] : a_k \leq a_{k+1}\}.$$

Observe that $n_0 \in J_n$, i.e., J_n is nonempty and satisfies $J_n \subseteq J_{n+1}$. For each $n \geq n_0$, we denote

$$\nu(n) := \max J_n.$$

Note that $\nu(n) \to \infty$ as $n \to \infty$ and $\{\nu(n)\}_{n \geq n_0}$ is nondecreasing. Furthermore, we have

$$a_{\nu(n)} \leq a_{\nu(n)+1} \quad \forall n \geq n_0. \tag{4.12}$$

Next, we will show that

$$a_n \leq a_{\nu(n)+1} \quad \forall n \geq n_0. \tag{4.13}$$

For all $n \geq n_0$, we have from the definition of J_n that it is either $\nu(n) = n$ or $\nu(n) < n$. Thus, in order to prove the above inequality, we consider these 2 cases:

For $\nu(n) = n$, we immediately get $a_n = a_{\nu(n)} \leq a_{\nu(n)+1}$.

For $\nu(n) < n$, we notice that if $\nu(n) = n - 1$, then the inequality (4.13) is trivial as $a_n = a_{\nu(n)+1}$. So, we suppose that $\nu(n) < n - 1$. Note that $a_{\nu(n)+1} > a_{\nu(n)+2} > \cdots > a_{n-1} > a_n$ (otherwise, if $a_{\nu(n)+1} \leq a_{\nu(n)+2}$, then it means that $\nu(n) + 1 \in J_n$, but $\nu(n) = \max J_n$ which brings a contradiction, and the other terms are likewise), which implies that the inequality (4.13) holds true.

On the other hand, invoking Lemma 4.7 and the inequality (4.12), we have for all $n \geq n_0$

$$0 \leq a_{\nu(n)+1} - a_{v(n)} \leq -\frac{\lambda_{\nu(n)}\left(1-\lambda_{\nu(n)}\right)}{4L^2}\left(\sum_{i=1}^{m} \|S_i\varphi_0^{\nu(n)} - S_{i-1}\varphi_0^{\nu(n)}\|^2\right)^2 + \xi_{\nu(n)},$$

and, consequently,

$$\frac{\lambda_{\nu(n)}\left(1-\lambda_{\nu(n)}\right)}{4L^2}\left(\sum_{i=1}^{m} \|S_i\varphi_0^{\nu(n)} - S_{i-1}\varphi_0^{\nu(n)}\|^2\right)^2 \leq \xi_{\nu(n)}.$$

Since $\lim_{n\to\infty} \xi_{\nu(n)} = \lim_{n\to\infty} \xi_n = 0$, we get

$$\lim_{n\to\infty} \frac{\lambda_{\nu(n)}\left(1-\lambda_{\nu(n)}\right)}{4L^2}\left(\sum_{i=1}^{m} \|S_i\varphi_0^{\nu(n)} - S_{i-1}\varphi_0^{\nu(n)}\|^2\right)^2 \leq 0.$$

Since we know that $\lambda_{\nu(n)}\left(1-\lambda_{\nu(n)}\right) \geq \varepsilon^2$, it follows

$$\lim_{n\to\infty} \|S_i\varphi_0^{\nu(n)} - S_{i-1}\varphi_0^{\nu(n)}\| = 0 \quad \forall i = 1, 2, ..., m. \tag{4.14}$$

Now, let $\{\varphi_0^{\nu(n_k)}\}_{k=1}^{\infty} \subseteq \{\varphi_0^{\nu(n)}\}_{n=1}^{\infty}$ be a subsequence such that

$$\limsup_{n\to\infty} \langle\varphi_0^{\nu(n)} - \bar{u}, -F(\bar{u})\rangle = \lim_{k\to\infty} \langle\varphi_0^{\nu(n_k)} - \bar{u}, -F(\bar{u})\rangle.$$

Following the same arguments as in **Case 1**, for a subsequence $\{\varphi_0^{\nu(n_{k_j})}\}_{j=1}^\infty$ of $\{\varphi_0^{\nu(n_k)}\}_{k=1}^\infty$ such that $\varphi_0^{\nu(n_{k_j})} \rightharpoonup z \in \bigcap_{i=1}^m \operatorname{Fix} T_i$ (by (4.14) and the DC principle of each T_i), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\varphi_0^{\nu(n)} - \bar{u}, -F(\bar{u})\rangle &= \lim_{k\to\infty}\langle\varphi_0^{\nu(n_k)} - \bar{u}, -F(\bar{u})\rangle \\ &= \lim_{j\to\infty}\langle\varphi_0^{\nu(n_{k_j})} - \bar{u}, -F(\bar{u})\rangle = \langle z-\bar{u}, -F(\bar{u})\rangle \leq 0,\end{aligned}$$

and also obtain that

$$\limsup_{n\to\infty}\delta_{\nu(n)} \leq 0. \tag{4.15}$$

Again, by using Lemma 4.9, we have

$$0 \leq a_{\nu(n)+1} \leq \left(1-\alpha_{\nu(n)}\right)a_{\nu(n)} + \alpha_{\nu(n)}\delta_{\nu(n)} + v\|u^{\nu(n)}\|,$$

and then

$$\begin{aligned}0 \leq a_{\nu(n)+1} - a_{\nu(n)} &\leq \alpha_{\nu(n)}\left(\delta_{\nu(n)} - a_{\nu(n)}\right) + v\|u^{\nu(n)}\| \\ &= \tau\beta_{\nu(n)}\left(\delta_{\nu(n)} - a_{\nu(n)}\right) + v\|u^{\nu(n)}\| \\ &\leq \tau\left(\delta_{\nu(n)} - a_{\nu(n)}\right) + v\|u^{\nu(n)}\|.\end{aligned}$$

The fact that the constant $\tau > 0$ yields

$$0 \leq a_{\nu(n)} \leq \delta_{\nu(n)} + \frac{v\|u^{\nu(n)}\|}{\tau}.$$

Note that $\lim_{n\to\infty} v\|u^{\nu(n)}\| = 0$ and by utilizing this together with (4.15), we obtain

$$0 \leq \limsup_{n\to\infty} a_{\nu(n)} \leq \limsup_{n\to\infty}\delta_{\nu(n)} + \lim_{n\to\infty}\frac{v\|u^{\nu(n)}\|}{\tau} \leq 0,$$

and, this implies that

$$\lim_{n\to\infty} a_{\nu(n)} = 0 \text{ and } \lim_{n\to\infty}\left(a_{\nu(n)+1} - a_{\nu(n)}\right) = 0.$$

As we have shown that $a_n \leq a_{\nu(n)+1}$, we note that

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_{\nu(n)+1} = \limsup_{n\to\infty}\left[\left(a_{\nu(n)+1} - a_{\nu(n)}\right) + a_{\nu(n)}\right] = 0,$$

and, consequently, $\lim_{n\to\infty} a_n = 0$. Therefore, we can conclude that $\lim_{n\to\infty}\|x^n - \bar{u}\| = 0$, which completes the proof. ■

Remark 4.11. Some useful remarks are in order:

(i) Let us take a look at Algorithm 1 when the operator F is identically zero. Notice that it is related to [6, Algorithm 1.2] and [7, Iterative scheme (3.17)] for solving the common fixed point problem (1.1). According to the absence of F, the operator $T_i, i = 1,\ldots,m$, considered in [7, Theorem 3.5] can be relaxed to be in the class of averaged nonexpansive operators, whereas in our work we need the use of Proposition 2.8 so that the firm nonexpansivity of T_i must be assumed here. To discuss Theorem 4.2 with these previous results, we derive in Theorem 4.2 the strong convergence of the generated sequence to the unique solution to the variational inequality over the common fixed point sets, however the results in [6] and [7] are weak convergences of the sequences provided that every weak cluster point of their generated sequences is in the intersection of fixed point sets. To obtain strong convergence, the nonemptiness of interior of the common fixed point set needs to be imposed in their works.

(ii) Algorithm 1 is related to the relaxed hybrid steepest descent method in [46] in the sense that the added information terms $e_i^n, i = 1, \ldots, m$, are absent. One can see that Algorithm 1 reduces to

$$x^{n+1} = (1 - \lambda_n)x^n + \lambda_n T\left(x^n - \mu\beta_n F(x^n)\right)$$

where the nonexpansive operator T is defined by $T := T_m T_{m-1} \cdots T_2 T_1$, and the convergence results can be followed the proving lines in [46, Theorem 3,1] with the additional assumption $\lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$.

5. Conclusion

This paper discussed the variational inequality problem over the intersection of fixed point sets of firmly nonexpansive operators. To solve the problem, we derived the so-called sequential constraint method based on iterative technique of the celebrated hybrid steepest descent method and presented its convergence analyses.

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