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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Inertial Relaxed CQ Algorithm with an Application to Signal Processing

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Abstract In this paper, we introduce inertial relaxed algorithm involving the self-adaptive technique for solving the split feasibility problem in Hilbert spaces which we approximate the original convex subset by a sequence of closed balls instead of half spaces. Then, the convergence results are established under mild conditions. Finally, we applied our algorithm in signal processing.

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1. Introduction

We focus on the split feasibility problem (SFP), introduced by Censor and Elfving [1], which is to find a point

$$u \in C \quad \text{such that } Au \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^M$ and $\mathbb{R}^N$, and A is an $M \times N$ matrix.

The CQ algorithm introduced by Byrne [2, 3] is a very successful method for solving the SFP which generates a sequence $\{u_n\}$ as follows:

$$u_{n+1} = P_C(u_n - \lambda_n A^*(I - P_Q)Au_n), \tag{1.2}$$

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where P_C is the metric projections onto C. If the stepsize λ_n is chosen so that $\lambda_n \equiv \lambda \in (0, 2\|A\|^{-2})$, then the CQ algorithm is weakly convergent to a solution of the SFP.

More specifically, C and Q are respectively given by

$$C = \{u \in H_1 : c(u) \leq 0\},$$

and

$$Q = \{v \in H_2 : q(v) \leq 0\},$$

where $c : H_1 \to (-\infty, +\infty]$ and $q : H_2 \to (-\infty, +\infty]$ are two proper convex functions.

However, to implement the CQ algorithm, one has to compute or estimate the norm $\|A\|$, which is in general not an easy task in practice.

Yang [4] introduced the relaxed CQ algorithm, by replacing P_C and P_Q by P_{C_n} and P_{Q_n}, respectively, presented the relaxed CQ algorithm:

$$u_{n+1} = P_{C_n}(u_n - \lambda_n \nabla h_n(u_n)), \tag{1.3}$$

where

$$C_n = \{u \in H_1 \,|\, c(u_n) + \langle \zeta_n, u - u_n \rangle \leq 0\}, \tag{1.4}$$

with $\zeta_n \in \partial c(u_n)$, and

$$Q_n = \{v \in H_2 \,|\, q(Au_n) + \langle \eta_n, v - Au_n \rangle \leq 0\}, \tag{1.5}$$

with $\eta_n \in \partial q(Au_n)$,

$$h_n(u_n) = \frac{1}{2}\|(I - P_{Q_n})Au_n\|^2, \text{ and } \nabla h_n(u_n) = A^*(I - P_{Q_n})Au_n. \tag{1.6}$$

Fenghui Wang and Hai Yu [5] approximated the original convex subset by a sequence of closed balls instead of half spaces and constructed $\widetilde{C}_n$ as

$$\widetilde{C}_n = \{u \in H_1 \,|\, c(w_n) + \langle \zeta_n, u - w_n \rangle + \frac{\alpha}{2}\|u - w_n\|^2 \leq 0\}, \tag{1.7}$$

where $\zeta_n \in \partial c(w_n)$. If $\alpha = 0$, then $\widetilde{C}$ above is reduced to the half space (1.4).

The set $\widetilde{Q}_n$ is defined as

$$\widetilde{Q}_n = \{v \in H_2 \,|\, q(Aw_n) + \langle \eta_n, v - Aw_n \rangle + \frac{\beta}{2}\|v - w_n\|^2 \leq 0\}, \tag{1.8}$$

where $\eta_n \in \partial q(Aw_n)$. If $\beta = 0$, then $\widetilde{Q}$ above is reduced to the half space (1.5).

Here, for each $n \geq 0$, we define

$$h_n(u) = \frac{1}{2}\|(I - P_{\widetilde{Q}_n})Au\|^2, \text{ and } \nabla h_n(u) = A^*(I - P_{\widetilde{Q}_n})Au. \tag{1.9}$$

López et al. [6] , to overcome this difficulty, introduced a new way to select the stepsize and also practiced this way of selecting stepsizes for variants of the CQ algorithm, including a relaxed CQ algorithm. They introduced the stepsize λ_n which is defined as follows:

$$\lambda_n = \frac{\tau_n h_n(u_n)}{\|\nabla h_n(u_n)\|^2}, \tag{1.10}$$

where $\{\tau_n\}$ is a sequence in $(0,4)$ such that $\inf_{n\in\mathbb{N}} \tau_n(4-\tau_n) > 0$. It was proved that the sequence $\{u_n\}$ generated by (1.3) with the stepsize defined by (1.10) converges weakly to a solution of SFP. Many researchers extended to study SFP, *i.e.*, [5–20].

Inspired by mentioned above, we propose a new relaxed CQ algorithm for SFP which is a new method for choosing stepsizes and for creating relaxation the convex subsection was developed.

2. Preliminaries

Let H be a Hilbert space and C is a nonempty closed convex subset in H. Recall that a mapping $S : C \to H$ is said to be nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall\, u, v \in C.$$

A mapping $S : C \to H$ is said to be firmly nonexpansive if

$$\|Su - Sv\|^2 \leq \langle u - v, Su - Sv\rangle, \quad \forall\, u, v \in C.$$

A mapping $S : C \to H$ is said to be ν-inverse strongly monotone (ν-ism) if there is $\nu > 0$ such that

$$\langle Su - Sv, u - v\rangle \geq \nu\|Su - Sv\|^2 \quad \forall\, u, v \in C.$$

The projection of a nonempty closed convex set C onto H is defined as

$$P_C u = \arg\min_{v\in C} \|u - v\|^2,\ u \in H.$$

Lemma 2.1. [21] *For all $u, v \in H$ and $w \in C$, we have*

(i) $\langle u - P_C u, w - P_C u\rangle \leq 0$;

(ii) *P_C and $I - P_C$ are both 1-ism;*

(iii) *P_C and $I - P_C$ are both firmly nonexpansive.*

Lemma 2.2. [22] *Let $t > 0$ and $x \in H$. Then u^* solves SFP (1.1) if and only if u^* solves the fixed point equation:*

$$u^* = P_C(u^* - tA^*(I - P_Q)Au^*).$$

A function $h : H \to (-\infty, +\infty]$ is is proper if

$$\{u \in H \,|\, h(u) < \infty\} \neq \emptyset.$$

A proper function h is convex if for each $t \in (0,1)$,

$$h(tu + (1-t)v) \leq th(u) + (1-t)h(v), \quad \forall\, u, v \in H.$$

A differentiable function h is convex if and only if there holds the inequality:

$$h(w) \geq h(u) + \langle \nabla h(u), w - u\rangle \quad \forall\, w \in H.$$

A function $h : H \to (-\infty, +\infty]$ is said to be weakly lower semi-continuous at u if $u_n \rightharpoonup u^*$ implies

$$h(u^*) \leq \liminf_{n\to\infty} h(u_n).$$

Lemma 2.3. [21] *Let $h : H \to (-\infty, +\infty]$ be a proper convex function. Then h is lower semi-continuous if and only if it is weakly lower semi-continuous.*

Lemma 2.4. [21] *Assume that* $\{u_n\}$ *is a sequence in* H *such that*

(i) *for each* $w \in C$*, the limit of sequence* $\{\|u_n - w\|\}$ *exists;*

(ii) *any weak cluster point of sequence* $\{u_n\}$ *belongs to* C.

Then the sequence $\{u_n\}$ *is weakly convergent to* $w \in C$.

Lemma 2.5. [21] *Let* $u, v \in H$*. It then follows that*

(i) $\|au + bv\|^2 = a(a+b)\|u\|^2 + b(a+b)\|v\|^2 - ab\|u - v\|^2, \quad a, b \in \mathbb{R};$

(ii) $\langle u, v\rangle = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{2}\|u - v\|^2;$

(iii) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle.$

Lemma 2.6. [23] *Let* $\{\varphi_n\}$ *and* $\{\vartheta_n\}$ *be two nonnegative real sequences such that*

$$\varphi_{n+1} - \varphi_n \leq \theta_n(\varphi_n - \varphi_{n-1}) + \vartheta_n, \quad \sum_{n=0}^{\infty} \vartheta_n < +\infty,$$

where $\{\theta_n\} \subset [0, \theta]$ *with* $0 < \theta < 1$*. Then the sequence* $\{\varphi_n\}$ *is convergent.*

3. Main Results

We next introduce a new CQ algorithm and derive the weakly convergence of the proposed method.

Algorithm 1 : The proposed algorithm for a weak convergence.

Initial: Given $\{\theta_n\} \subset [0, \theta)$ for some $\theta > 0$ and $u_0, u_1 \in \mathcal{H}$ arbitrarily and $n = 1$.
Step 1. Compute

$$\begin{cases} w_n = u_n + \theta_n(u_n - u_{n-1}), \\ s_n = w_n - \lambda_n \nabla h_n(w_n), \\ u_{n+1} = P_{\widetilde{C}_n}(s_n - \delta_n \nabla h_n(s_n)), \end{cases}$$

where $\widetilde{C}_n$ is given as (1.7),

$$\lambda_n = \frac{\tau_n h_n(w_n)}{\|\nabla h_n(w_n)\|^2}, \quad \text{and} \quad \delta_n = \frac{\tau_n h_n(s_n)}{\|\nabla h_n(s_n)\|^2}, \; 0 < \tau_n < 4.$$

Set $n =: n + 1$ and go back to **Step 1.**

In this paper, we denote $\mathcal{S}$ by the solution set of SFP and assume that $\mathcal{S}$ is nonempty.

Lemma 3.1. *Suppose that the sequence* $\{w_n\}$ *is generated by Algorithm 1. Then,* $h_n(w_n) = 0$ *if and only if* $\|\nabla h_n(w_n)\| = 0$.

Proof. If $h_n(w_n) = 0$, then we obtain

$$\begin{aligned} \|\nabla h_n(w_n)\|^2 &= \|A^*(I - P_{\widetilde{Q}_n})Aw_n\|^2 \\ &\leq \|A\|^2\|(I - P_{\widetilde{Q}_n})Aw_n\|^2 \\ &= 2\|A\|^2 h_n(w_n). \end{aligned} \tag{3.1}$$

This yields $\|\nabla h_n(w_n)\| = 0$. To see the converse, let $\|\nabla h_n(w_n)\| = 0$ and $u^* \in \mathcal{S}$. Using Lemma 2.1(ii), we have

$$\begin{aligned} h_n(w_n) &= \frac{1}{2}\|(I - P_{\widetilde{Q}_n})Aw_n\|^2 \\ &\le \langle (I - P_{\widetilde{Q}_n})Aw_n, Aw_n - Au^*\rangle \\ &= \frac{1}{2}\langle A^*(I - P_{\widetilde{Q}_n})Aw_n, w_n - u^*\rangle \\ &\le \frac{1}{2}\|A^*(I - P_{\widetilde{Q}_n})Aw_n\|\|w_n - u^*\| \\ &= \frac{1}{2}\|\nabla h_n(w_n)\|\|w_n - u^*\|. \end{aligned} \tag{3.2}$$

Hence, $h_n(w_n) = 0$. ■

Lemma 3.2. *Suppose that the sequence $\{s_n\}$ is generated by Algorithm 1. Then, $h_n(s_n) = 0$ if and only if $\|\nabla h_n(s_n)\| = 0$.*

Proof. If $h_n(s_n) = 0$, then we obtain

$$\begin{aligned} \|\nabla h_n(s_n)\|^2 &= \|A^*(I - P_{\widetilde{Q}_n})As_n\|^2 \\ &\le \|A\|^2\|(I - P_{\widetilde{Q}_n})As_n\|^2 \\ &= 2\|A\|^2 h_n(s_n). \end{aligned} \tag{3.3}$$

This yields $\|\nabla h_n(s_n)\| = 0$. To see the converse, let $\|\nabla h_n(s_n)\| = 0$ and $u^* \in \mathcal{S}$. Using Lemma 2.1(ii), we have

$$\begin{aligned} h_n(s_n) &= \frac{1}{2}\|(I - P_{\widetilde{Q}_n})As_n\|^2 \\ &\le \langle (I - P_{\widetilde{Q}_n})As_n, As_n - Au^*\rangle \\ &= \frac{1}{2}\langle A^*(I - P_{\widetilde{Q}_n})As_n, s_n - u^*\rangle \\ &\le \frac{1}{2}\|A^*(I - P_{\widetilde{Q}_n})As_n\|\|s_n - u^*\| \\ &= \frac{1}{2}\|\nabla h_n(s_n)\|\|s_n - u^*\|. \end{aligned} \tag{3.4}$$

Hence, $h_n(s_n) = 0$. ■

Lemma 3.3. *Let $\{u_n\}$ be a sequence generated by Algorithm 1. Then, for any $\delta \in (0, 1]$ and $u^* \in \mathcal{S}$, it follows that*

$$\begin{aligned} \|u_{n+1} - u^*\|^2 \le \|w_n - u^*\|^2 - (1-\delta)\|s_n - u_{n+1}\|^2 - \tau_n(4 - \tau_n)\frac{h_n^2(w_n)}{\|\nabla h_n(w_n)\|^2} \\ - \tau_n(4 - \frac{\tau_n}{\delta})\frac{h_n^2(s_n)}{\|\nabla h_n(s_n)\|^2}. \end{aligned} \tag{3.5}$$

Proof. Let $u^* \in \mathcal{S}$. From definition of s_n, we have

$$\begin{aligned} \|s_n - u^*\|^2 &= \|w_n - \lambda_n \nabla h_n(w_n) - u^*\|^2 \\ &= \|w_n - u^*\|^2 + \lambda_n^2 \|\nabla h_n(w_n)\|^2 - 2\lambda_n \langle \nabla h_n(w_n), w_n - u^* \rangle. \end{aligned} \tag{3.6}$$

Using Lemma 2.1(ii), and $\nabla h_n(u^*) = 0$, we get

$$\begin{aligned} \langle \nabla h_n(w_n), w_n - u^* \rangle &= \langle \nabla h_n(w_n) - \nabla h_n(z), s_n - u^* \rangle \\ &= \langle A^*(I - P_{Q_n})Aw_n - A^*(I - P_{Q_n})Au^*, w_n - u^* \rangle \\ &\geq \|(I - P_{Q_n})Aw_n\|^2 \\ &= 2h_n(w_n). \end{aligned} \tag{3.7}$$

It also follows that

$$\langle \nabla h_n(s_n), s_n - u^* \rangle \geq 2h_n(s_n). \tag{3.8}$$

From (3.8), we get

$$\|s_n - u^*\|^2 \leq \|w_n - u^*\|^2 + \lambda_n^2 \|\nabla h_n(w_n)\|^2 - 4\lambda_n h_n(w_n). \tag{3.9}$$

From (3.8), we obtain

$$\begin{aligned} \|u_{n+1} - u^*\|^2 &= \|P_{\widetilde{C}_n}(s_n - \delta_n \nabla h_n(s_n)) - u^*\|^2 \\ &\leq \|s_n - \delta_n \nabla h_n(s_n) - u^*\|^2 - \|s_n - u_{n+1} - \delta_n \nabla h_n(s_n)\|^2 \\ &= \|s_n - u^*\|^2 + \delta_n^2 \|\nabla h_n(s_n)\|^2 - 2\delta_n \langle \nabla h_n(s_n), s_n - u^* \rangle \\ &\quad - \|s_n - u_{n+1}\|^2 - \delta_n^2 \nabla h_n(s_n)\|^2 \\ &\quad + 2\delta_n \| \langle \nabla h_n(s_n), s_n - u_{n+1} \rangle \\ &= \|s_n - u^*\|^2 - \|s_n - u_{n+1}\|^2 - 2\delta_n \langle \nabla h_n(s_n), s_n - u^* \rangle \\ &\quad + 2\delta_n \langle \nabla h_n(s_n), s_n - u_{n+1} \rangle \\ &\leq \|s_n - u^*\|^2 - \|s_n - u_{n+1}\|^2 - 4\delta_n h_n(s_n) \\ &\quad + 2\delta_n \langle \nabla h_n(s_n), s_n - u_{n+1} \rangle. \end{aligned} \tag{3.10}$$

By Young's inequality, we get

$$\begin{aligned} 2\delta_n |\langle \nabla h_n(s_n), s_n - u_{n+1} \rangle| &\leq 2\delta_n \|\nabla h_n(s_n)\| \|s_n - u_{n+1}\| \\ &\leq \delta \|s_n - u_{n+1}\|^2 + \frac{\delta_n^2}{\delta} \|\nabla h_n(s_n)\|^2. \end{aligned} \tag{3.11}$$

From (3.6)-(3.10), we have

$$
\begin{aligned}
\|u_{n+1}-u^*\|^2 &\le \|w_n-u^*\|^2+\lambda_n^2\|\nabla h_n(w_n)\|^2-4\lambda_n h_n(w_n)\\
&\quad -\|s_n-u_{n+1}\|^2-4\delta_n h_n(s_n)\\
&\quad +\delta\|s_n-u_{n+1}\|^2+\frac{\delta_n^2}{\delta}\|\nabla h_n(s_n)\|^2\\
&=\|w_n-u^*\|^2-(1-\delta)\|s_n-u_{n+1}\|^2\\
&\quad +\frac{\tau_n^2 h_n^2(w_n)}{(\|\nabla h_n(w_n)\|^2)^2}\|\nabla h_n(w_n)\|^2-\frac{4\tau_n h_n^2(w_n)}{\|\nabla h_n(w_n)\|^2}\\
&\quad -\frac{4\tau_n h_n^2(s_n)}{\|\nabla h_n(s_n)\|^2}+\frac{\tau_n^2 h_n^2(s_n)}{\delta(\|\nabla h_n(s_n)\|^2)^2}\|\nabla h_n(s_n)\|^2\\
&=\|w_n-u^*\|^2-(1-\delta)\|s_n-u_{n+1}\|^2 \qquad (3.12)\\
&\quad +\frac{\tau_n^2 h_n^2(w_n)}{\|\nabla h_n(w_n)\|^2}-\frac{4\tau_n h_n^2(w_n)}{\|\nabla h_n(w_n)\|^2}\\
&\quad -\frac{4\tau_n h_n^2(s_n)}{\|\nabla h_n(s_n)\|^2}+\frac{\tau_n^2 h_n^2(s_n)}{\delta\|\nabla h_n(s_n)\|^2}\\
&=\|w_n-u^*\|^2-(1-\delta)\|s_n-u_{n+1}\|^2\\
&\quad -\tau_n(4-\tau_n)\frac{h_n^2(w_n)}{\|\nabla h_n(w_n)\|^2}\\
&\quad -\tau_n(4-\frac{\tau_n}{\delta})\frac{h_n^2(s_n)}{\|\nabla h_n(s_n)\|^2}.
\end{aligned}
$$

■

Lemma 3.4. *Let $\{u_n\}$ and $\{s_n\}$ be the sequences generated by Algorithm 1. Suppose that $\{u_n\}$ and $\{s_n\}$ are bounded such that*

$$\lim_{n\to\infty}\|u_{n+1}-s_n\|=\lim_{n\to\infty}\|s_n-u_n\|=\lim_{n\to\infty}h_n(s_n)=0. \qquad (3.13)$$

Then each weak cluster point of $\{u_n\}$ belongs to $\mathcal{S}$.

Proof. Let $u^*\in\mathcal{S}$. Because ∂c and ∂q are bounded on bounded sets, we could suppose that for all $n\ge 0$, there is $\mathcal{M}>0$ such that

$$\|\zeta_n\|+\|\eta_n\|^2+\|A\|^2\|u_n-u^*\|\le\mathcal{M},\ \zeta_n\in\partial c(s_n),\ \eta_n\in\partial q(As_n).$$

Let z be any weak cluster point of $\{u_n\}$. Thus, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $\{u_{n_i}\}\rightharpoonup z$. It follows from (3.13) that that $\{s_{n_i}\}\rightharpoonup z$. Because A is linear and bounded, this yields that $\{As_{n_i}\}\rightharpoonup Az$. By the definition of $\widetilde{C}_n$ and the fact that $u_{n+1}\in\widetilde{C}_n$, and follow prove of [5], we have $c(s_n)\le\mathcal{M}\|s_n-u_{n+1}\|\to 0$, and c is weakly semi-continuous, so $c(z)\le\liminf_{i\to\infty}c(s_{n_i})\le 0$, this is $z\in C$. In fact, $P_{\widetilde{Q}_n}(As_n)\in\widetilde{Q}_n$, it follows that $q(As_n)\le\mathcal{M}\|(I-P_{\widetilde{Q}_n})As_n\|\to 0$. Because q is clearly weakly lower semi-continuous, $q(Az)\le\liminf_{i\to\infty}q(As_{n_i})\le 0$. Hence, $Az\in Q$. ■

Theorem 3.5. *Suppose that $\inf_n\tau_n(4-\tau_n)>0$ and $\sum_{n=1}^{\infty}\theta_n\|u_n-u_{n-1}\|^2<\infty$. Then, the sequence $\{u_n\}$ generated by Algorithm 1 converges weakly to a solution in $\mathcal{S}$.*

Proof. Let $u^* \in \mathcal{S}$. It follows from definition of w_n, we obtain

$$\begin{aligned} \|w_n - u^*\|^2 &= \|u_n + \theta_n(u_n - u_{n-1}) - u^*\|^2 \\ &= \|u_n - u^*\|^2 + 2\langle u_n - u^*, \theta_n(u_n - u_{n-1}\rangle + \|\theta_n(u_n - u_{n-1})\|^2 \\ &= \|u_n - u^*\|^2 + 2\theta_n\langle u_n - u^*, u_n - u_{n-1}\rangle + \theta_n^2\|u_n - u_{n-1}\|^2. \end{aligned} \tag{3.14}$$

Using Lemma 2.5(ii), we obtain

$$\langle u_n - u^*, u_n - u_{n-1}\rangle = \frac{1}{2}\|u_n - u^*\|^2 + \frac{1}{2}\|u_n - u_{n-1}\|^2 - \frac{1}{2}\|u_{n-1} - u^*\|^2. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\begin{aligned} \|w_n - u^*\|^2 &= \|u_n - u^*\|^2 + \theta_n\|u_n - u^*\|^2 + \theta_n\|u_n - u_{n-1}\|^2 \\ &\quad - \theta_n\|u_{n-1} - u^*\|^2 + \theta_n^2\|u_n - u_{n-1}\|^2 \\ &\leq \|u_n - u^*\|^2 + \theta_n(\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2) \\ &\quad + 2\theta_n\|u_n - u_{n-1}\|^2. \end{aligned} \tag{3.16}$$

From Lemma 3.3 with $\delta = 1$, we have

$$\begin{aligned} \|u_{n+1} - u^*\|^2 \leq\ & \|u_n - u^*\|^2 + \theta_n(\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2) \\ & + 2\theta_n\|u_n - u_{n-1}\|^2 - \tau_n(4 - \tau_n)\frac{h_n^2(w_n)}{\|\nabla h_n(w_n)\|^2} \\ & - \tau_n(4 - \tau_n)\frac{h_n^2(s_n)}{\|\nabla h_n(s_n)\|^2}. \end{aligned} \tag{3.17}$$

Using condition $\sum_{n=1}^{\infty}\theta_n\|u_n - u_{n-1}\|^2 < \infty$ and Lemma 2.6 in (3.17), we obtain that $\lim_{n\to\infty}\|u_n - u^*\|$ exists and $\{u_n\}$ is bounded.

Using Lemma 3.4 and $\{u_n\}$ is bounded, and hence, the sequence $\{s_n\}$. Therefore, we could suppose that for all $n \geq 0$, there is $\mathcal{M} > 0$ such that

$$8\|u_n - u^*\| + \|A\|^2\|s_n - u^*\| \leq \mathcal{M}.$$

From (3.17) and our hypothesis on τ_n, we obtain

$$\lim_{n\to\infty}\frac{h_n^2(w_n)}{\|\nabla h_n(w_n)\|^2} = 0, \tag{3.18}$$

and

$$\lim_{n\to\infty}\frac{h_n^2(s_n)}{\|\nabla h_n(s_n)\|^2} = 0. \tag{3.19}$$

From definition of λ_n, we have

$$\begin{aligned} h_n(w_n) &= \sqrt{\lambda_n}\|\nabla h_n(w_n)\| \\ &= \sqrt{\lambda_n}\|\nabla h_n(w_n) - \nabla h_n(u^*)\| \\ &\leq \sqrt{\lambda_n}\|A\|^2\|w_n - u^*\| \\ &\leq \mathcal{M}\sqrt{\lambda_n} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.20}$$

By condition $\sum_{n=1}^{\infty}\theta_n\|u_n-u_{n-1}\|^2<\infty$, we have

$$\|w_n-u_n\|^2=\theta_n^2\|u_n-u_{n-1}\|^2\le\theta_n\|u_n-u_{n-1}\|^2\to 0 \quad \text{as} \quad n\to\infty. \tag{3.21}$$

which implies

$$\lim_{n\to\infty}\|w_n-u^*\|^2=\lim_{n\to\infty}\|u_n-u^*\|^2=0. \tag{3.22}$$

From definition of s_n, and (3.20), we have

$$\|s_n-w_n\|^2=\lambda_n^2\|\nabla h_n(w_n)\|^2\to 0 \quad \text{as} \quad n\to\infty. \tag{3.23}$$

From (3.9) and (3.22), we get

$$\|s_n-u^*\|^2\to 0 \quad \text{as} \quad n\to\infty. \tag{3.24}$$

Using (3.21) and (3.23), we get

$$\begin{aligned}\|u_n-s_n\|&=\|u_n-w_n+w_n-s_n\|\\&\le\|u_n-w_n\|+\|w_n-s_n\|\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{3.25}$$

From definition of δ_n, we have

$$\begin{aligned}h_n(s_n)&=\sqrt{\delta_n}\|\nabla h_n(s_n)\|\\&=\sqrt{\delta_n}\|\nabla h_n(s_n)-\nabla h_n(u^*)\|\\&\le\sqrt{\delta_n}\|A\|^2\|s_n-u^*\|\\&\le\mathcal{M}\sqrt{\delta_n}\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{3.26}$$

Indeed, using Lemma 2.1, we get

$$\langle s_n-\delta_n\nabla h_n(s_n)-u_{n+1},u_{n+1}-u^*\rangle\ge 0, \tag{3.27}$$

which

$$\langle u_{n+1}-s_n,u_{n+1}-u^*\rangle\le-\delta_n\langle\nabla h_n(s_n),u_{n+1}-u^*\rangle. \tag{3.28}$$

From (3.28), we obtain

$$\begin{aligned}\|u_{n+1}-s_n\|^2&=(\|s_n-u^*\|^2-\|u_{n+1}-u^*\|^2)+2\langle u_{n+1}s_n,u_{n+1}-u^*\rangle\\&\le(\|s_n-u^*\|^2-\|u_{n+1}-u^*\|^2)-2\delta_n\langle\nabla h_n(s_n),u_{n+1}-u^*\rangle\\&\le(\|s_n-u^*\|^2-\|u_{n+1}-u^*\|^2)+2\delta_n\|\langle\nabla h_n(s_n)\|\|u_{n+1}-u^*\|\\&\le(\|s_n-u^*\|^2-\|u_{n+1}-u^*\|^2)+\mathcal{M}\sqrt{\delta_n}.\end{aligned} \tag{3.29}$$

Using (3.24) and (3.22), we obtain $\lim_{n\to\infty}\|u_{n+1}-s_n\|=0$. By Lemma 3.4, we conclude that each weak cluster point of $\{u_n\}$ belongs to $\mathcal{S}$. Using Lemma 2.4 that the sequence $\{u_n\}$ converges weakly to a solution of SFP (1.1). ■

4. Application

The compressed sensing can be modeled as the linear equation:

$$\Psi = \Upsilon u + \varphi, \tag{4.1}$$

where $u \in \mathbb{R}^N$ is a recovered vector with m non-zero components, $\Psi \in \mathbb{R}^M$ is the observed data with noisy φ, and $\Upsilon : \mathbb{R}^N \to \mathbb{R}^M (M < N)$. It is noted that (4.1) could be seen as solving the LASSO problem:

$$\min_{u \in \mathbb{R}^N} \frac{1}{2}\|\Psi - \Upsilon u\|^2 \quad \text{subject to} \quad \|u\|_1 \leq t, \tag{4.2}$$

where $t > 0$. In particular, in case $C = \{u \in \mathbb{R}^N : \|u\|_1 \leq t\}$ and $Q = \{\Psi\}$, the LASSO problem can be considered as the SFP(1.1). From this point of view, we could apply the CQ algorithm to solve (4.2). In our experiment, let matrix $\Upsilon \in \mathbb{R}^{M \times N}$ is generated from a normal distribution with mean zero and invariance one. The observation Ψ is generated by Gaussian noise distributed normally with mean 0 and variance 10^{-4}. The sparse vector $u \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-1, 1]$ with m nonzero elements.

The stopping criterion is defined by the mean square error (MSE):

$$MSE = \frac{1}{N}\|u^* - u\|^2 < 10^{-5}, \tag{4.3}$$

where u^* is an approximated signal of u.

We provide numerical experiments of the compressed sensing in signal recovery compare our CQ algorithm with Wang and Yu [5] by $\theta_n = 0.9$ and $\tau_n = 2$.

In our experiment, we test two cases as follows:
Case 1 : $N = 4096$, $M = 1024$, and $m = 10$.
Case 2 : $N = 4096$, $M = 1024$, and $m = 100$. The numerical results show in Figure 1 and Figure 2.

5. Conclusions

In this paper, we propose a new CQ algorithm with inertial extrapolation term and the self-adaptive technique for solving the split feasibility problem in Hilbert space. Applying our results in the compressed sensing problem comparing the proposed methods with Wang and Yu's algorithm. Our proposed algorithm has a better performance (MSE) than algorithm above.

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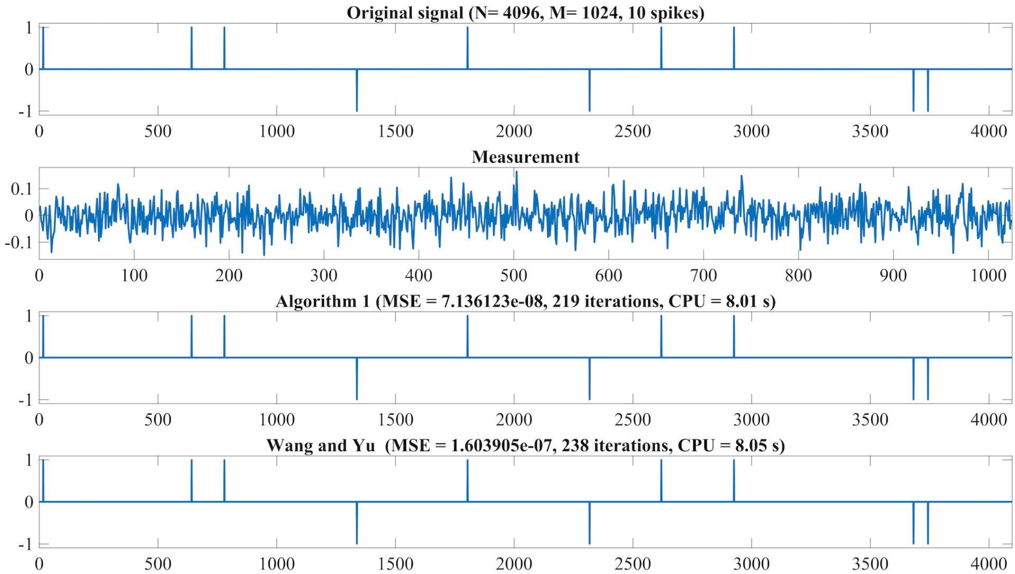

FIGURE 1. Numerical results of Case 1.

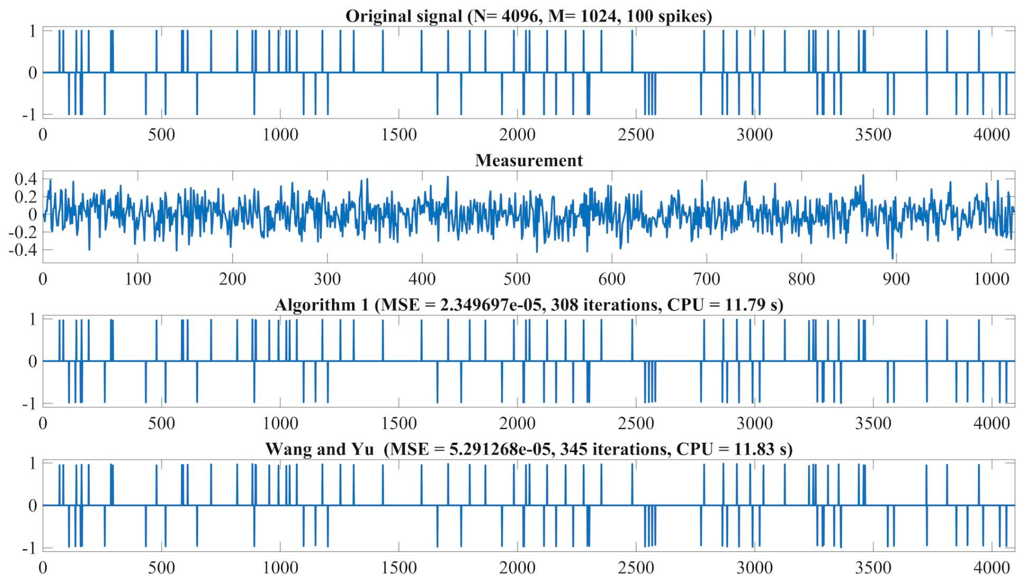

FIGURE 2. Numerical results of Case 2.

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