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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Statistical Approximation Properties of Lupaş q-Bernstein Shifted Operators

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Abstract In the present paper, statistical approximation properties of Lupaş q-Bernstein shifted operators are studied with the help of the Korovkin type approximation theorem. Rate of convergence by means of modulus of continuity has been investigated.

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1. Introduction

Approximation theory basically deals with approximation of functions by simpler functions or more easily calculated functions. Broadly it is divided into theoretical and constructive approximation. In 1912, S. N. Bernstein [1] was the first to construct sequence of positive linear operators to provide a constructive proof of well known Weierstrass approximation theorem [2] as follows:

$$B_m(f;u) = \sum_{k=0}^{m} \binom{m}{k} u^k (1-u)^{m-k} f\left(\frac{k}{m}\right), \tag{1.1}$$

using probabilistic approach. Here $C[0,1]$ denotes the set of all continuous functions on $[0,1]$ which is equipped with sup-norm $\|.\|_{C[0,1]}$. He showed that if $f \in C[0,1]$, then $B_m(f;u)$ converges to $f(u)$ uniformly.

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After development of q-calculus, Lupaş [3] in 1987 introduced the q-Lupaş operator (rational) as follows:

$$L_{m,q}(f;u)=\sum_{k=0}^{m}\frac{f\left(\frac{[k]_q}{[m]_q}\right)\left[\begin{array}{c}m\\k\end{array}\right]_q q^{\frac{k(k-1)}{2}}\ u^k\ (1-u)^{m-k}}{\prod\limits_{j=1}^{m}\{(1-u)+q^{j-1}u\}}, \tag{1.2}$$

and studied its approximation properties. Similarly, Phillips [4] in 1996 constructed another q-analogue of Bernstein operators (polynomials) as follows:

$$B_{m,q}(f;u)=\sum_{k=0}^{m}\left[\begin{array}{c}m\\k\end{array}\right]_q u^k\prod_{s=0}^{m-k-1}(1-q^s x)\ f\left(\frac{[k]_q}{[m]_q}\right),\ x\in[0,1] \tag{1.3}$$

where $B_{m,q}:C[0,1]\to C[0,1]$ defined for any $m\in\mathbb{N}$ and any function $f\in C[0,1]$. Basis of these operators have been used in computer aided geometric design(CAGD) to study curves and surfaces. Then onward it became an active area of research in approximation theory as well as CAGD. In Computer Aided Geometric Design, bases of Bernstein polynomials play a significant role in order to preserve the shape of the curves or surfaces. Popular programs, like Adobes illustrator and flash, and font imaging system such as Postscript, utilize polynomials to form what are known as Bezier curves. For more details one can see [5–7] In the recent past, q-analogues of various operators were investigated by several researchers (see [8, 9]).

In 1968 Stancu [10] showed that the Bernstein-Stancu polynomials

$$\left(P_m^{(\gamma,\delta)}f\right)(u)=\sum_{k=0}^{m}\binom{m}{k}u^k(1-u)^{m-k}f\left(\frac{k+\gamma}{m+\delta}\right), \tag{1.4}$$

converge to continuous function $f(u)$ uniformly in [0,1] for each real γ,δ such that $0\leq\gamma\leq\delta$.

In 2010, a new construction of Bernstein-Stancu type polynomials with shifted knots was introduced by Gadjiev and Gorhanalizadeh [11] as:

$$S_{m,\gamma,\delta}(f;u)=\left(\frac{m+\delta_2}{m}\right)^m\sum_{k=0}^{m}\binom{m}{k}\left(u-\frac{\gamma_2}{m+\delta_2}\right)^k\left(\frac{m+\gamma_2}{m+\delta_2}-u\right)^{m-k}f\left(\frac{k+\gamma_1}{m+\delta_1}\right), \tag{1.5}$$

where $\frac{\gamma_2}{m+\delta_2}\leq u\leq\frac{m+\gamma_2}{m+\delta_2}$ and γ_k,δ_k $(k=1,2)$ are positive real numbers provided $0\leq\gamma_1\leq\gamma_2\leq\delta_1\leq\delta_2$. It is clear that for $\gamma_2=\delta_2=0$, polynomials (1.5) turn into the Bernstein-Stancu polynomials (1.4) and if $\gamma_1=\gamma_2=\delta_1=\delta_2=0$ then these polynomials turn into the classical Bernstein polynomials.

Let us recall some basic definitions and notations of quantum calculus [12]. For any fixed real number $q>0$ satisfying the conditions $0<q<1$, the q-integer $[k]_q$, for $k\in\mathbb{N}$ are defined as

$$[k]_q:=\begin{cases}\frac{(1-q^k)}{(1-q)}, & q\neq 1\\ k, & q=1.\end{cases}$$

and the q-factorial by

$$[k]_q! := \begin{cases} [k]_q[k-1]_q...[1]_q, & k \geq 1 \\ 1, & k = 0. \end{cases}$$

The q-Binomial expansion is

$$(u+y)_q^m := (u+y)(u+qy)(u+q^2y)\cdots(u+q^{m-1}y),$$

and the q-binomial coefficients are as follows:

$$\left[\begin{array}{c} m \\ k \end{array}\right]_q := \frac{[m]_q!}{[k]_q![m-k]_q!}.$$

Gauss-formula is defined as:

$$(u+y)_q^m = \sum_{j=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_q q^{j(j-1)/2} y^j u^{m-j}.$$

Khalid et al. studied Bézier curves and surfaces using basis of shifted Bernstein polynomial in [7]. Recently Mursaleen et al. [13] introduced and studied Lupaş Bernstein shifted operators based on q-integers as follows:

$$S_{m,q}^{(\gamma,\delta)}(f;u) = \frac{1}{\left(\frac{[m]_q}{[m]_q+\delta}\right)_q^m} \sum_{j=0}^{m} \left[\begin{array}{c} m \\ j \end{array}\right]_q q^{\frac{j(j-1)}{2}} \left(\frac{[m]_q+\gamma}{[m]_q+\delta}-u\right)^{m-j} \left(u-\frac{a}{[m]_q+b}\right)^j f\left(\frac{[j]_q}{[m]_q}\right), \tag{1.6}$$

or

$$S_{m,q}^{(\gamma,\delta)}(f;u) = \frac{1}{\left(\frac{[m]_q}{[m]_q+\delta}\right)_q^m} \sum_{j=0}^{m} \left[\begin{array}{c} m \\ j \end{array}\right]_q \left(\frac{[m]_q+\gamma}{[m]_q+\delta}-u\right)^{m-j} \left(u-\frac{a}{[m]_q+\delta}\right)_q^j f\left(\frac{[j]_q}{[m]_q}\right),$$

where $\frac{\gamma}{[m]_q+\delta} \leq u \leq \frac{[m]_q+\gamma}{[m]_q+\delta}$ and $\gamma,\ \delta$ are positive real numbers provided $0 \leq \gamma \leq \delta$. In case $\gamma = \delta = 0$, the above operators (1.6) reduce to Lupaş q-Bernstein operators [3].

Gadjiev and Orhan [14] applied the concept of statistical convergence to positive linear operators in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence. The real life example of statistical convergence is computational intelligence.

This paper is mainly organized as follows: In the next section we will investigate the statistical approximation properties of Lupaş q-Bernstein shifted operators [13] with the help of the Korovkin type approximation theorem. We will also estimate the rate of convergence by means of modulus of continuity in Section 3.

In order to prove our main results, we now recall the following lemmas for the operator $S_{m,q}^{(\gamma,\delta)}(f;u)$.

Lemma 1.1. (See [13]) *Let* $S_{m,q}^{(\gamma,\delta)}(f;u)$ *be given by* (1.6)*. Then the following properties hold:*

$$\begin{aligned}
S_{m,q}^{(\gamma,\delta)}(1;u) &= 1,\\
S_{m,q}^{(\gamma,\delta)}(t;u) &= \frac{[m]_q+\delta}{[m]_q}\left(u-\frac{\gamma}{[m]_q+\delta}\right),\\
S_{m,q}^{(\gamma,\delta)}(t^2;u) &= \Big(\frac{q^2[m-1]_q}{[m]_q}\Big)\left(\frac{[m]_q+\delta}{[m]_q}\right)\frac{\left(u-\frac{\gamma}{[m]_q+\delta}\right)^2}{\left\{\frac{[m]_q+\gamma}{[m]_q+\delta}-u+q\left(u-\frac{\gamma}{[m]_q+\delta}\right)\right\}}\\
&\quad+\left(\frac{[m]_q+\delta}{[m]_q^2}\right)\Big(u-\frac{\gamma}{[m]_q+\delta}\Big).
\end{aligned}$$

Lemma 1.2. (See [13]) *For all* $u\in\left[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}\right]$*, we have*

$$\begin{aligned}
S_{m,q}^{(\gamma,\delta)}((t-u)^2;u) &= \Big(\frac{q^2[m-1]_q}{[m]_q}\Big)\left(\frac{[m]_q+\delta}{[m]_q}\right)\frac{\left(u-\frac{\gamma}{[m]_q+\delta}\right)^2}{\left\{\frac{[m]_q+\gamma}{[m]_q+\delta}-u+q\left(u-\frac{\gamma}{[m]_q+\delta}\right)\right\}}\\
&\quad+\left(\frac{1}{[m]_q}-2u\right)\left(\frac{[m]_q+\delta}{[m]_q}\right)\left(u-\frac{\gamma}{[m]_q+\delta}\right)+u^2. \qquad (1.7)
\end{aligned}$$

2. Statistical Approximation

In this section, we obtain the Korovkin type statistical approximation by the operators defined in (1.6). Let us recall the concept of statistical convergence which was given by Fast [15] and further studied by many authors.

Let $K\subseteq N$ and $K_m=\{i\leq m: i\in K\}$. Then the natural density or we can say asymptotic density of K is defined by $\delta(K)=\lim_m \frac{1}{m}|K_m|$ whenever the limit exists, where $|K_m|$ denotes the of the set K_m.

A sequence $u=(u_i)$ of real numbers is said be statistically convergent to L if for every $\epsilon>0$ the set $\{i\in N:|u_i-L|\geq\epsilon\}$ has natural density zero; that is, for each $\epsilon>0$,

$$\lim_m \frac{1}{m}|\{i\leq m:|u_i-L|\geq\epsilon\}|=0.$$

In this case, we write $st-\lim_m u_m=L$. Note that convergent sequences are statistically convergent since all finite subset of of natural no have density zero. However, its converse is not true. This is demonstrated by the following example.

Example 2.1. let us consider the sequences,

$$u=(u_m):=\begin{cases}\frac{1}{2m}+1, & \text{otherwise},\\ 0, & m=k^2 \text{ for some k},\end{cases}$$

and

$$v=(v_m):=\begin{cases}1, & n=k^2 \text{ for some k},\\ 0, & m=k^2+1 \text{ for some k},\\ 2, & \text{otherwise}.\end{cases}$$

Then, it is easy to see that the sequences $\{u\}$ and $\{v\}$ are not convergent in the ordinary sense, but $st - \lim_m u_m = 1$ and $st - \lim_m v_m = 2$. All properties of convergent sequences are not true for statistical convergence. For instance, it is known that a subsequence of a convergent sequence is convergent. However, for statistical convergence this is not true. Indeed, the sequence $l = \{i; i = 1, 2, 3, \}$ is a subsequence of the statistically convergent sequence u from Example 2.1. At the same time, l is statistically divergent.

Recently, some approximation theorems are proved by using the idea of statistical convergence, in particular, Korovkin type approximation theorems [16] by various authors, see [14, 17] and it has been found that the statistical version is stronger than the classical ones. Classical and test functions of various type were used by many authors to study Korovkin type approximation theorems. In other branch of mathematics, one can see that Korovkin type approximation theorem has many useful connections, other than classical approximation theory.

For other literatures related to statistical convergence and other related papers one can refer [18–22]

We recall the following theorem given by Gadjiev and Orhan [14].

Theorem 2.2. (See [14]) *If the sequence of positive linear operators $A_n : C_M[a,b] \to C[a,b]$ satisfies the conditions $st - \lim_{n\to\infty} ||A_n(e_v;\cdot) - e_v||_{C[a,b]} = 0$, with $e_v(t) = t^v$ for $v = 0, 1, 2$, then for any function $f \in C_M[a,b]$, we have*

$$st - \lim_{n\to\infty} ||A_n(f;\cdot) - f||_{C[a,b]} = 0,$$

where $C_M[a,b]$ denotes the space of all functions f which are continuous in $[a,b]$ and bounded on the all positive axis.

Now, consider sequence $q = (q_m)$, $q_m \in (0,1)$, such that

$$st - \lim_m q_m = 1. \tag{2.1}$$

The above condition (2.1) guarantees that $[m]_{q_m} \to \infty$ as $m \to \infty$.

Theorem 2.3. *Let $S_{m,q}^{(\gamma,\delta)}(f;u)$ be the sequence of operators (1.6), and the sequence $q = q_m$ satisfies (2.1) then for any function $f \in C[0,1]$*

$$st - \lim_m \| \, S_{m,q_m}^{(\gamma,\delta)}(f;\cdot) - f \| = 0 \tag{2.2}$$

Proof. According to Theorem 2.2, we only need to prove $st - \lim_m ||S_{m,q_m}^{(\gamma,\delta)}(e_v;\cdot) - e_v|| = 0$, with $e_v(t) = t^v$ for $v = 0, 1, 2$.

For $\nu = 0$, since $S_{m,q_m}^{(\gamma,\delta)}(1;u) = 1$, which implies

$$st - \lim_m ||S_{m,q_m}^{(\gamma,\delta)}(e_0;\cdot) \; - e_0 \; || \; = \; 0.$$

For $\nu \; = \; 1$,

$$\begin{aligned}\left|S_{m,q_m}^{(\gamma,\delta)}(t;u) \; - \; u\;\right| &\le \left|\frac{[m]_{q_m} + \delta}{[m]_{q_m}}\left(u - \frac{\gamma}{[m]_{q_m} + \delta}\right) \; - \; u\right| \\ &= \left|\left(\frac{[m]_{q_m} + \delta}{[m]_{q_m}} \; - \; 1\right)u \; - \; \frac{\gamma}{[m]_{q_m}}\right| \\ &\le \left|\frac{[m]_{q_m} + \delta}{[m]_{q_m}} \; - \; 1\right| \; + \; \left|\frac{\gamma}{[m]_{q_m}}\right|.\end{aligned}$$

For a given $\epsilon > 0$, let us define the following sets.

$$\begin{aligned} W &= \{m : \|S^{(\gamma,\delta)}_{m,q_m}(t;u) - u\| \geq \epsilon\}, \\ W' &= \{m : 1 - \frac{[m]_{q_m} + \delta}{[m]_{q_m}} \geq \epsilon\}, \\ W'' &= \{m : \frac{\gamma}{[m]_{q_m}} \geq \epsilon\}. \end{aligned}$$

It is obvious that $W \subseteq W'' \cup W'$. Then it can be written as:

$$\begin{aligned} & \delta\{K \leq m : \|S^{(\gamma,\delta)}_{m,q_m}(t;u) - u\| \geq \epsilon\} \\ \leq \; & \delta\{k \leq m : 1 - \frac{[m]_{q_m} + \delta}{[m]_{q_m}} \geq \epsilon\} + \delta\{K \leq m : \frac{\gamma}{[m]_{q_m}}\| \geq \epsilon\}. \end{aligned}$$

By using (2.1), we get

$$st - \lim_{m} \|S^{(\gamma,\delta)}_{m,q_m}(e_1;\cdot) - e_1\| = 0.$$

Lastly, for $\nu = 2$, we have

$$\begin{aligned} & \left|S^{(\gamma,\delta)}_{m,q_m}(t^2;u) - u^2\right| \\ \leq \; & \Bigg|(\frac{q^2[m-1]_{q_m}}{[m]_q})\left(\frac{[m]_{q_m}+\delta}{[m]_{q_m}}\right)\frac{\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)^2}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta} - u + q\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}} \\ & + \left(\frac{[m]_{q_m}+\delta}{[m]^2_{q_m}}\right)(u - \frac{\gamma}{[m]_{q_m}+\delta}) - u^2\Bigg| \\ \leq \; & \Bigg|\left((\frac{q^2[m-1]_{q_m}}{[m]_q})\left(\frac{[m]_{q_m}+\delta}{[m]_{q_m}}\right)\frac{1}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta} - u + q\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}} - 1\right)u^2 \\ & + \left(\frac{[m]_{q_m}+\delta}{[m]^2_{q_m}} - \frac{\frac{2\gamma}{[m]_{q_m}+\delta}}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta} - u + q\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}}\right)u \\ & + \frac{\left(\frac{\gamma}{[m]_{q_m}+\delta}\right)^2}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta} - u + q\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}}(\frac{\gamma}{[m]_{q_m}+\delta})\Bigg| \\ \leq \; & \left|(\frac{q^2[m-1]_{q_m}}{[m]_q})\left(\frac{[m]_{q_m}+\delta}{[m]_{q_m}}\right)\frac{1}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta} - u + q\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}} - 1\right| \\ & + \left|\frac{[m]_{q_m}+\delta}{[m]^2_{q_m}} - \frac{\frac{2\gamma}{[m]_{q_m}+\delta}}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta} - u + q\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}}\right| \\ & + \left|\frac{\left(\frac{\gamma}{[m]_{q_m}+\delta}\right)^2}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta} - u + q\left(u - \frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}}(\frac{\gamma}{[m]_{q_m}+\delta})\right|. \end{aligned}$$

If we choose

$$\alpha_m = \Big(\frac{q^2[m-1]_{q_m}}{[m]_q}\Big)\left(\frac{[m]_{q_m}+\delta}{[m]_{q_m}}\right)\frac{1}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta}-u+q\left(u-\frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}}-1,$$

$$\beta_m = \frac{[m]_{q_m}+\delta}{[m]_{q_m}^2}-\frac{\frac{2\gamma}{[m]_{q_m}+\delta}}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta}-u+q\left(u-\frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}},$$

$$\gamma_m = \frac{\left(\frac{\gamma}{[m]_{q_m}+\delta}\right)^2}{\left\{\frac{[m]_{q_m}+\gamma}{[m]_{q_m}+\delta}-u+q\left(u-\frac{\gamma}{[m]_{q_m}+\delta}\right)\right\}}\Big(\frac{\gamma}{[m]_{q_m}+\delta}\Big),$$

then, by (2.1), we can write

$$st-\lim_m \alpha_m = st-\lim_m \beta_m = st-\lim_m \gamma_m = 0. \tag{2.3}$$

Now given $\epsilon>0$, we define the following four sets:

$$W = \{m : \|S^{(\gamma,\delta)}_{m,q_m}(t^2;u)-u^2\|\geq\epsilon\},\quad W_1=\{m:\alpha_m\geq\frac{\epsilon}{3}\},$$
$$W_2 = \{m:\beta_m\geq\frac{\epsilon}{3}\},\quad W_3=\{m:\gamma_m\geq\frac{\epsilon}{3}\}.$$

It is obvious that $W\subseteq W_1\bigcup W_2\bigcup W_3$. Thus we obtain

$$\begin{aligned}&\delta\{K\leq m:\|S^{(\gamma,\delta)}_{m,q_m}(t^2;u)-u^2\|\geq\epsilon\}\\ \leq\ &\delta\{K\leq m:\alpha_m\geq\frac{\epsilon}{3}\}+\delta\{K\leq m:\beta_m\geq\frac{\epsilon}{3}\}+\delta\{K\leq m:\gamma_m\geq\frac{\epsilon}{3}\}.\end{aligned}$$

Using (2.3), we get

$$st-\lim_m\|S^{(\gamma,\delta)}_{m,q_m}(e_2;\cdot)-e_2\|=0$$

holds and thus the proof is completed. ■

3. Rates of Convergence

In this section, we compute the rates of convergence of the operators $S^{(\gamma,\delta)}_{m,q_m}(f;u)$ to the functions f by means of modulus continuity. we denote the space of real-valued continuous and bounded functions f defined on the interval $[\frac{\gamma}{[m]_q+\delta}\leq u\leq\frac{[m]_q+\gamma}{[m]_q+\delta}]$. The norm $\|\cdot\|$ on the space $C[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}]$ is given by

$$\| f \|=\sup_{[\frac{\gamma}{[m]_q+\delta}\leq u\leq\frac{[m]_q+\gamma}{[m]_q+\delta}]}\mid f(u)\mid.$$

Further let us consider the following K-functional:

$$K_2(f,\sigma)=\inf_{g\in W^2}\{\| f-g\| +\sigma\| g^{''}\|\},$$

where $\sigma>0$ and $W^2=\{g\in C[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}]:g^{'},g^{''}\in C[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}]\}$. By Devore and Lorentz ([24], p. 177, Theorem 2.4), there exists an absolute constant $C>0$ such that

$$K_2(f,\sigma)\leq C\omega_2(f,\sqrt{\sigma}). \tag{3.1}$$

Second order modulus of smoothness is as follows

$$\omega_2(f,\sqrt{\sigma}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{[\frac{\gamma}{[m]_q+\delta}\leq u\leq \frac{[m]_q+\gamma}{[m]_q+\delta}]} \mid f(u+2h)-2f(u+h)+f(u)\mid$$

where $f\in C[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}]$. The usual modulus of continuity of $f\in C[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}]$ is defined by

$$\omega(f,\sigma) = \sup_{0<h\leq\delta} \sup_{[\frac{\gamma}{[m]_q+\delta}\leq u\leq \frac{[m]_q+\gamma}{[m]_q+\delta}]} \mid f(u+h)-f(u)\mid .$$

Theorem 3.1. *If f be a continuous function on $[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}]$ and taking $0<q<1$, then*

$$\|S_{m,q}^{(\gamma,\delta)}(f;\cdot)-f\|\leq 2\omega_f(\sqrt{\sigma_m}),$$

where σ_m

$$\Big(\frac{q^2[m-1]_q}{[m]_q}\Big)\left(\frac{[m]_q+\delta}{[m]_q}\right)\frac{\left(u-\frac{\gamma}{[m]_q+\delta}\right)^2}{\left\{\frac{[m]_q+\gamma}{[m]_q+\delta}-u+q\left(u-\frac{\gamma}{[m]_q+\delta}\right)\right\}} + \left(\frac{1}{[m]_q}-2u\right)\left(\frac{[m]_q+\delta}{[m]_q}\right)\left(u-\frac{\gamma}{[m]_q+\delta}\right)+u^2.$$

Proof. For any $u,y\in[\frac{\gamma}{[m]_q+\delta},\frac{[m]_q+\gamma}{[m]_q+\delta}]$, it is known that

$$|f(y)-f(u)|\leq\omega_f(\sigma)\left(1+\frac{(y-u)^2}{\sigma^2}\right).$$

Therefore, we get

$$\begin{aligned}|S_{m,q}^{(\gamma,\delta)}(f;u)-f(u)| &\leq S_{m,q}^{(\gamma,\delta)}(|f(t)-f(u)|;u)\\ &\leq \omega_f(\sigma)\left(1+\frac{1}{\sigma^2}S_{m,q}^{(\gamma,\delta)}\left((t-u)^2;u\right)\right)\\ &\leq \omega_f(\sigma)\left(1+\frac{\sigma_m}{\sigma^2}\right).\end{aligned}$$

Choosing $\sigma=\sqrt{\sigma_m}$

$$= \Big(\frac{q^2[m-1]_q}{[m]_q}\Big)\left(\frac{[m]_q+\delta}{[m]_q}\right)\frac{\left(u-\frac{\gamma}{[m]_q+\delta}\right)^2}{\left\{\frac{[m]_q+\gamma}{[m]_q+\delta}-u+q\left(u-\frac{\gamma}{[m]_q+\delta}\right)\right\}} + \left(\frac{1}{[m]_q}-2u\right)\left(\frac{[m]_q+\delta}{[m]_q}\right)\left(u-\frac{\gamma}{[m]_q+\delta}\right)+u^2.$$

we have

$$\|S_{m,q}^{(\gamma,\delta)}(f;u)-f(u)\|\leq 2\omega_f(\sqrt{\sigma_m}),$$

Thus, we obtain the desired result. ∎

Conclusion

The bases of these operators can be used to draw curves and surfaces like as in Adobe reader and Corel draw, results derived for shifted intervals will be very helpful when it comes to implementation using computers for simulation purposes. Therefore, it has further scope of study in CAGD and Approximation Theory.

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