Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 1063–1079

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

On Some Bivariate Copula Transformations

Phuong Le[1,*], **Nguyen Kim Thai Ngoc**[2]

[1] *Department of Economic Mathematics, Banking University of Ho Chi Minh City, Ho Chi Minh City, Vietnam*
e-mail : phuongl@buh.edu.vn

[2] *Faculty of Finance, Banking University of Ho Chi Minh City, Ho Chi Minh City, Vietnam*
e-mail : nguyenkimthaingoc@gmail.com

Abstract Copulas are known as a fundamental tool to model dependence phenomena and have various applications in finance and risk management. In this paper, we propose a new class of copulas generated by means of function compositions of each two in three well-known bivariate copulas: the independence copula, the Fréchet-Hoeffding lower bound, and the Fréchet-Hoeffding upper bound. As consequences, we classify all such transformations in the class of quadratic polynomials.

MSC: 62E10; 39B62; 60E05

Keywords: bivariate copula; copula transformation; Fréchet-Hoeffding bounds; quadratic polynomial

Submission date: 16.05.2020 / Acceptance date: 18.09.2020

1. Introduction

Copulas play a central role in modeling multivariate dependence. This is due to the elegant Sklar's theorem (see [1, Theorem 2.3.3] or Theorem 2.3 below), which states that copulas link univariate marginal distributions together to form a multivariate distribution. Copulas are used in portfolio analysis and risk management to determine the nonparametric measure of the dependence between random variables. Therefore, they have vast applications in financial markets, banking, insurance, climate change, biometrics, aging population, and other fields (see a recent survey by Bhatti and Do [2]).

Fitting a suitable copula to given experimental data requires a large number of copulas. Therefore, any new construction method of copulas extends the possibility of their applications in empirical research.

Several different methods have been proposed to construct copulas. For example, Archimedian copulas via additive generators, elliptical copulas via inversion method, and Marshall-Olkin copulas via survival functions. One may consult the books by Nelsen [1] and Durante and Sempi [3] for these methods.

*Corresponding author.

Published by The Mathematical Association of Thailand.

A notable construction method is via transformations of known copulas to create the new ones. For example, the convex sums and the gluing method such as the ordinal sums and patchwork copulas. In 2015, Kolesárová et al. [4] searched for all quadratic polynomials P such that

$$(x, y) \mapsto P(x, y, C(x, y))$$

is a copula for all copulas C. Later, Wisadwongsa and Tasena [5] extended this result by characterizing all quadratic polynomials P such that

$$(x, y) \mapsto P(x, y, C_1(x, y), C_2(x, y))$$

is a copula whenever C_1, C_2 are copulas. As a consequence of the main result in [5], we have

Proposition 1.1 (Wisadwongsa and Tasena [5]). *Let P be a bivariate quadratic polynomial. The function*

$$(x, y) \mapsto P(C_1(x, y), C_2(x, y))$$

is a copula for any copulas C_1 and C_2 if and only if P assumes the form

$$P(x, y) = a(x - y)^2 + dx + (1 - d)y,$$

where $0 \leq d \leq 1$ *and* $\frac{|2d-1|-1}{2} \leq a \leq \frac{|2d-1|+1}{2}$.

Recently, Tasena [6] took a further step of characterizing all polynomial P such that

$$(x, y) \mapsto P(x, y, C_1(x, y), \ldots, C_n(x, y))$$

is a copula when $C_1, C_2, \ldots, C_n$ are arbitrary copulas ($n \geq 2$). He also described all such transformations in the class of quadratic polynomials.

In another direction, a class of bivariate copula mappings was found by Manstavičius and Bagdonas [7] and Girard [8]. More precisely, they gave necessary and sufficient conditions on a function $f : [0, 1] \to \mathbb{R}_+$ so that

$$(x, y) \mapsto C(x, y)f(1 - x - y + C(x, y))$$

is a copula for any bivariate copula C. Later, Saminger-Platz et al. [9] extended this result by searching for functions $f : [0, 1] \to \mathbb{R}_+$ and bivariate copulas D so that

$$(x, y) \mapsto D(C(x, y), f(C^*(x, y)))$$

is a copula for any bivariate copula C, where $C^*(x, y) = x + y - C(x, y)$.

Motivated by the above works, we are interested in copula transformations of the form

$$(x, y) \mapsto f(C_1(x, y), C_2(x, y))$$

where C_1, C_2 are limited to three well-known copulas: the independence copula, the Fréchet-Hoeffding lower bound, and the Fréchet-Hoeffding upper bound. Examples of such transformations are the convex sums and quadratic polynomials P described in Proposition 1.1. Obviously, by restricting C_1, C_2 in those only three copulas, we lower the criteria on functions f. This allows us to gain more controls on f and hence we can obtain more flexible copulas. We also classify all such transformations in the case that f belongs to the class of quadratic polynomials.

Throughout the paper, we will only consider bivariate copulas, which will be called copulas for brevity. The rest of the paper is organized as follows. In Section 2, we provide the basic definitions and properties of copulas that will be used later on. Section

3 provides the statement and proof of our main results (Theorems 3.1, 3.3 and 3.4). In Section 4, we classify the transformations in theclass of quadratic polynomials. Finally, we close the paper with a conclusion in Section 5.

2. Preliminaries

In this section, we recall some basic facts from copula theory which will be used later on. Although one can also consider n-variate copulas for any $n \geq 2$ (see [1, 3]), we only give the definition for bivariate copulas, which are the objects we are concerned with in this paper.

Definition 2.1. A (bivariate) copula is a function $C : [0,1]^2 \to \mathbb{R}$ such that

(i) $C(x,0) = C(0,x) = 0$ for any $x \in [0,1]$,
(ii) $C(x,1) = C(1,x) = x$ for any $x \in [0,1]$,
(iii) (2-increasing) for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$V_C([x_1,x_2] \times [y_1,y_2]) := C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) + C(x_1,y_1) \geq 0.$$

Remark 2.2. One may deduce from Definition 2.1 that $C : [0,1]^2 \to [0,1]$ and C is non-decreasing in each variables. Indeed, for any $x_1, x_2, y \in [0,1]$ with $x_1 \leq x_2$,

$$C(x_2,y) - C(x_1,y) = V_C([x_1,x_2] \times [0,y]) \geq 0.$$

Hence $C(x,y)$ is non-decreasing in x. Similarly, $C(x,y)$ is non-decreasing in y. Now for any $x, y \in [0,1]$, we have

$$C(x,y) \geq C(0,x) = 0$$

and

$$C(x,y) \leq C(1,x) = x \leq 1.$$

That means that $C : [0,1]^2 \to [0,1]$.

The importance of copulas is due to Sklar's theorem. Here we only recall a special version of this theorem for bivariate copulas, which are what we are concerned with.

Theorem 2.3 (Sklar's theorem [1, 3])**.** *For any joint distribution $H(x,y)$ with marginals $F_1(x)$, $F_2(y)$, there exists a copula C such that*

$$H(x,y) = C(F_1(x), F_2(y))$$

for all $x, y \in \mathbb{R} \cup \{\pm\infty\}$. If F_1, F_2 are continuous, then the copula C associated to H is unique and may be obtained by

$$C(x,y) = H(F_1^{-1}(x), F_2^{-1}(y)).$$

The most well-known copula should be the independence copula, which is given by

$$\Pi(x,y) = xy \quad \text{for all } x, y \in [0,1].$$

This copula is used to model the independence of random variables. Other examples of copula include the Fréchet-Hoeffding lower bound

$$W(x,y) := (x+y-1)^+$$

and the Fréchet-Hoeffding upper bound

$$M(x,y) := x \wedge y,$$

where we denote $x \wedge y = \min\{x,y\}$ and $x^+ = \max\{x,0\}$.

The Fréchet-Hoeffding bounds play an important role in the (partial) concordance order on the set of all copulas. This fact is stated in the following theorem.

Theorem 2.4 (see Theorem 2.2.3 in [1]). *For any copula C and any $x, y \in [0,1]$,*

$$W(x,y) \le C(x,y) \le M(x,y).$$

3. Copula Transformations by Function Compositions

In this section, we state and prove the main results of this paper. Our aim is to establish necessary and sufficient conditions on a bivariate function f such that its compositions with each two of three well-known copulas M, W, and Π are new copulas. These criteria will be used in Section 4 to classify f in form of a bivariate quadratic polynomial.

3.1. Composition of the Fréchet-Hoeffding Bounds

We begin with copula transformations of the Fréchet-Hoeffding upper bound M and lower bound W.

Theorem 3.1. *Let $f : [0,1]^2 \to \mathbb{R}$. The function*

$$C(x,y) := f(M(x,y), W(x,y))$$

is a copula if and only if the following conditions hold

(i) $f(x,x) = x$ *for any* $x \in [0,1]$,

(ii) $f(x,(2x-1)^+) + f(y,(2y-1)^+) \ge 2f(x,(x+y-1)^+)$ *for any* $x, y \in [0,1]$ *with* $x \le y$,

(iii) for any $x, y \in [0,1]$ *with* $x \le y$, *the function*

$$t \mapsto f(y,(y+t)^+) - f(x,(x+t)^+)$$

is non-decreasing in $[y-1, 0]$.

Remark 3.2. Since $M(x,y) \ge W(x,y)$ for any $x, y \in [0,1]$, the values of f can be arbitrarily defined on the set $\{(x,y) \in [0,1]^2 \mid x < y\}$. Therefore, assumptions $(i) - (iii)$ of Theorem 3.1 are only concerned with values of f in the set $\{(x,y) \in [0,1]^2 \mid x \ge y\}$.

Proof of Theorem 3.1. Notice that

$$\begin{aligned} V_C([x_1,x_2] \times [y_1,y_2]) = {} & f(x_2 \wedge y_2, (x_2+y_2-1)^+) - f(x_1 \wedge y_2, (x_1+y_2-1)^+) \\ & - f(x_2 \wedge y_1, (x_2+y_1-1)^+) + f(x_1 \wedge y_1, (x_1+y_1-1)^+). \end{aligned}$$

Step 1. Assume that C is a copula.

- For any $x \in [0,1]$ and $y = 1$, we have $f(x,x) = C(x,1) = x$.
- For any $x, y \in [0,1]$ with $x \le y$, we have

$$\begin{aligned} f(x,(2x-1)^+) + f(y,(2y-1)^+) - 2f(x,(x+y-1)^+) &= V_C([x,y] \times [x,y]) \\ &\ge 0. \end{aligned}$$

- For any $x, y \in [0,1]$ and $t_1, t_2 \in [y-1, 0]$ such that $x \le y$ and $t_1 \le t_2$, we have

$$\begin{aligned} & [f(y,(y+t_2)^+) - f(x,(x+t_2)^+)] - [f(y,(y+t_1)^+) - f(x,(x+t_1)^+)] \\ & = V_C([x,y] \times [t_1+1, t_2+1]) \\ & \ge 0. \end{aligned}$$

Therefore, f satisfies $(i) - (iii)$.

Step 2. Assume that f satisfies $(i)-(iii)$. We show that C is a copula.

Clearly, $C(x,0)=C(0,x)=f(0,0)=0$ for any $x\in[0,1]$.

Moreover, $C(x,1)=C(1,x)=f(x,x)=x$ for any $x\in[0,1]$.

It remains to prove that C is 2-increasing, i.e., $V_C([x_1,x_2]\times[y_1,y_2])\geq 0$ for any $x_1,x_2,y_1,y_2\in[0,1]$ with $x_1\leq x_2$ and $y_1\leq y_2$. Without loss of generality, we may assume $x_1\leq y_1$.

There are three cases to be considered.

Case 1: $x_1\leq x_2\leq y_1\leq y_2$. In this case,

$$\begin{aligned}
&V_C([x_1,x_2]\times[y_1,y_2])\\
&=[f(x_2,(x_2+y_2-1)^+)-f(x_1,(x_1+y_2-1)^+)]\\
&\quad-[f(x_2,(x_2+y_1-1)^+)-f(x_1,(x_1+y_1-1)^+)]\\
&\geq 0.
\end{aligned}$$

Case 2: $x_1\leq y_1\leq x_2\leq y_2$. In this case,

$$\begin{aligned}
&V_C([x_1,x_2]\times[y_1,y_2])\\
&=[f(x_2,(x_2+y_2-1)^+)-f(x_1,(x_1+y_2-1)^+)]\\
&\quad-[f(y_1,(x_2+y_1-1)^+)-f(x_1,(x_1+y_1-1)^+)]\\
&\geq[f(x_2,(2x_2-1)^+)-f(x_1,(x_1+x_2-1)^+)]\\
&\quad-[f(y_1,(x_2+y_1-1)^+)-f(x_1,(x_1+y_1-1)^+)]\\
&=[f(x_1,(x_1+y_1-1)^+)-f(x_1,(x_1+x_2-1)^+)]\\
&\quad-[f(y_1,(x_2+y_1-1)^+)-f(x_2,(2x_2-1)^+)]\\
&\geq[f(y_1,(2y_1-1)^+)-f(y_1,(y_1+x_2-1)^+)]\\
&\quad-[f(y_1,(x_2+y_1-1)^+)-f(x_2,(2x_2-1)^+)]\\
&=f(y_1,(2y_1-1)^+)+f(x_2,(2x_2-1)^+)-2f(y_1,(x_2+y_1-1)^+)\\
&\geq 0.
\end{aligned}$$

Case 3: $x_1\leq y_1\leq y_2\leq x_2$. In this case,

$$\begin{aligned}
&V_C([x_1,x_2]\times[y_1,y_2])\\
&=f(y_2,(x_2+y_2-1)^+)-f(x_1,(x_1+y_2-1)^+)\\
&\quad-f(y_1,(x_2+y_1-1)^+)+f(x_1,(x_1+y_1-1)^+)\\
&=[f(y_2,(x_2+y_2-1)^+)-f(y_1,(x_2+y_1-1)^+)]\\
&\quad+[f(x_1,(x_1+y_1-1)^+)-f(x_1,(x_1+y_2-1)^+)]\\
&\geq[f(y_2,(2y_2-1)^+)-f(y_1,(y_1+y_2-1)^+)]\\
&\quad+[f(y_1,(2y_1-1)^+)-f(y_1,(y_1+y_2-1)^+)]\\
&=f(y_2,(2y_2-1)^+)+f(y_1,(2y_1-1)^+)-2f(y_1,(y_1+y_2-1)^+)\\
&\geq 0.
\end{aligned}$$

Hence, C is 2-increasing. Consequently, C is a copula. ■

3.2. Composition of the Independence Copula and the Fréchet-Hoeffding Lower Bound

This subsection provides the criteria for copula transformations of the independence copula Π and the Fréchet-Hoeffding lower bound W.

Theorem 3.3. *Let $f : [0,1]^2 \to \mathbb{R}$. The function*

$$C(x,y) := f(\Pi(x,y), W(x,y))$$

is a copula if and only if the following conditions hold

(i) $f(x,x) = x$ *for any* $x \in [0,1]$,

(ii) for any $x, y \in [0,1]$ *with* $x \leq y$, *the function*

$$t \mapsto f(yt, (y+t-1)^+) - f(xt, (x+t-1)^+)$$

is non-decreasing in $[0,1]$.

Proof. Clearly,

$$\begin{aligned} V_C([x_1,x_2] \times [y_1,y_2]) = &f(x_2y_2, (x_2+y_2-1)^+) - f(x_1y_2, (x_1+y_2-1)^+) \\ &- f(x_2y_1, (x_2+y_1-1)^+) + f(x_1y_1, (x_1+y_1-1)^+). \end{aligned}$$

Step 1. Assume that C is a copula.
- For any $x \in [0,1]$ and $y = 1$, we have $f(x,x) = C(x,1) = x$.
- For any $x, y, t_1, t_2 \in [0,1]$ with $x \leq y$ and $t_1 \leq t_2$, we have

$$\begin{aligned} &[f(yt_2, (y+t_2-1)^+) - f(xt_2, (x+t_2-1)^+)] \\ &\quad - [f(yt_1, (y+t_1-1)^+) - f(xt_1, (x+t_1-1)^+)] \\ &= V_C([x,y] \times [t_1,t_2]) \\ &\geq 0. \end{aligned}$$

Therefore, f satisfies (i) and (ii).
Step 2. Assume that f satisfies (i) and (ii). We show that C is a copula.
Clearly, $C(x,0) = C(0,x) = f(0,0) = 0$ for any $x \in [0,1]$.
Moreover, $C(x,1) = C(1,x) = f(x,x) = x$ for any $x \in [0,1]$.
Now let any $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$. By assumption (ii), we have

$$\begin{aligned} &V_C([x_1,x_2] \times [y_1,y_2]) \\ &= [f(x_2y_2, (x_2+y_2-1)^+) - f(x_1y_2, (x_1+y_2-1)^+)] \\ &\quad - [f(x_2y_1, (x_2+y_1-1)^+) - f(x_1y_1, (x_1+y_1-1)^+)] \\ &\geq 0. \end{aligned}$$

Hence C is 2-increasing. Therefore, C is a copula. ∎

3.3. Composition of the Fréchet-Hoeffding Upper Bound and the Independence Copula

In this last subsection, we introduce a simple criteria for copula transformations of the Fréchet-Hoeffding upper bound M and the independence copula Π.

Theorem 3.4. *Let $f : [0,1]^2 \to \mathbb{R}$. The function*

$$C(x,y) := f(M(x,y), \Pi(x,y))$$

is a copula if and only if the following conditions hold

(i) $f(x,x) = x$ *for any* $x \in [0,1]$,
(ii) $f(x,x^2) + f(y,y^2) \geq 2f(x,xy)$ *for any* $x, y \in [0,1]$ *with* $x \leq y$,
(iii) *for any* $x, y \in [0,1]$ *with* $x \leq y$, *the function*

$$t \mapsto f(y,ty) - f(x,tx)$$

is non-decreasing in $[y,1]$.

Proof. We have

$$\begin{aligned} V_C([x_1,x_2] \times [y_1,y_2]) &= f(x_2 \wedge y_2, x_2y_2) - f(x_1 \wedge y_2, x_1y_2) \\ &\quad - f(x_2 \wedge y_1, x_2y_1) + f(x_1 \wedge y_1, x_1y_1). \end{aligned}$$

Step 1. Assume that C is a copula.
- For any $x \in [0,1]$ and $y = 1$, we have $f(x,x) = C(x,1) = x$.
- For any $x, y \in [0,1]$ with $x \leq y$, we have

$$\begin{aligned} &f(x,x^2) + f(y,y^2) - 2f(x,xy) \\ &= f(y \wedge y, yy) - f(x \wedge y, xy) - f(y \wedge x, yx) + f(x \wedge x, xx) \\ &= V_C([x,y] \times [x,y]) \\ &\geq 0. \end{aligned}$$

- For any $x, y, t_1, t_2 \in [0,1]$ with $x \leq y \leq t_1 \leq t_2$, we have

$$\begin{aligned} [f(y,t_2y) - f(x,t_2x)] - [f(y,t_1y) - f(x,t_1x)] &= V_C([x,y] \times [t_1,t_2]) \\ &\geq 0. \end{aligned}$$

Therefore, f satisfies $(i) - (iii)$.
Step 2. Assume that f satisfies $(i) - (iii)$. We show that C is a copula.
Clearly, $C(x,0) = C(0,x) = f(0,0) = 0$ for any $x \in [0,1]$.
Moreover, $C(x,1) = C(1,x) = f(x,x) = x$ for any $x \in [0,1]$.
It remains to prove that C is 2-increasing, i.e., $V_C([x_1,x_2] \times [y_1,y_2]) \geq 0$ for any $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$. Without loss of generality, we may assume $x_1 \leq y_1$.
There are three possible cases.
Case 1: $x_1 \leq x_2 \leq y_1 \leq y_2$. In this case,

$$\begin{aligned} &V_C([x_1,x_2] \times [y_1,y_2]) \\ &= [f(x_2,x_2y_2) - f(x_1,x_1y_2)] - [f(x_2,x_2y_1) - f(x_1,x_1y_1)] \\ &\geq 0. \end{aligned}$$

Case 2: $x_1 \leq y_1 \leq x_2 \leq y_2$. In this case,

$$\begin{aligned}
&V_C([x_1, x_2] \times [y_1, y_2]) \\
&= [f(x_2, x_2y_2) - f(x_1, x_1y_2)] - [f(y_1, x_2y_1) - f(x_1, x_1y_1)] \\
&\geq [f(x_2, x_2^2) - f(x_1, x_1x_2)] - [f(y_1, x_2y_1) - f(x_1, x_1y_1)] \\
&= [f(x_1, x_1y_1) - f(x_1, x_1x_2)] - [f(y_1, x_2y_1) - f(x_2, x_2^2)] \\
&\geq [f(y_1, y_1^2) - f(y_1, y_1x_2)] - [f(y_1, x_2y_1) - f(x_2, x_2^2)] \\
&= f(y_1, y_1^2) + f(x_2, x_2^2) - 2f(y_1, x_2y_1) \\
&\geq 0.
\end{aligned}$$

Case 3: $x_1 \leq y_1 \leq y_2 \leq x_2$. In this case,

$$\begin{aligned}
&V_C([x_1, x_2] \times [y_1, y_2]) \\
&= f(y_2, x_2y_2) - f(x_1, x_1y_2) - f(y_1, x_2y_1) + f(x_1, x_1y_1) \\
&= [f(y_2, x_2y_2) - f(y_1, x_2y_1)] + [f(x_1, x_1y_1) - f(x_1, x_1y_2)] \\
&\geq [f(y_2, y_2^2) - f(y_1, y_1y_2)] + [f(y_1, y_1^2) - f(y_1, y_1y_2)] \\
&= f(y_2, y_2^2) + f(y_1, y_1^2) - 2f(y_1, y_1y_2) \\
&\geq 0.
\end{aligned}$$

Hence, C is 2-increasing. Consequently, C is a copula. ■

4. Quadratic Transformations of Copulas

Clearly, the linear mappings

$$f(x, y) = \alpha x + (1 - \alpha)y \quad \text{for all } x, y \in [0, 1],$$

where $\alpha \in [0, 1]$, satisfy all conditions in Theorems 3.1, 3.3 and 3.4. This kind of f is associated with the convex sum transformations mentioned in the introduction section. It is natural to ask what type of quadratic polynomials satisfies those conditions.

In this section, we will give an answer to that question. More precisely, we classify all quadratic polynomials satisfying conditions in Theorems 3.1 and 3.3. We also search for a large class of quadratic polynomials which satisfies conditions in Theorem 3.4. These quadratic transformations provide us new classes of copulas which are easy to compute in practice.

We start with quadratic transformations on the Fréchet-Hoeffding bounds.

Theorem 4.1. *Let P be a bivariate quadratic polynomial. The function*

$$C(x, y) := P(M(x, y), W(x, y))$$

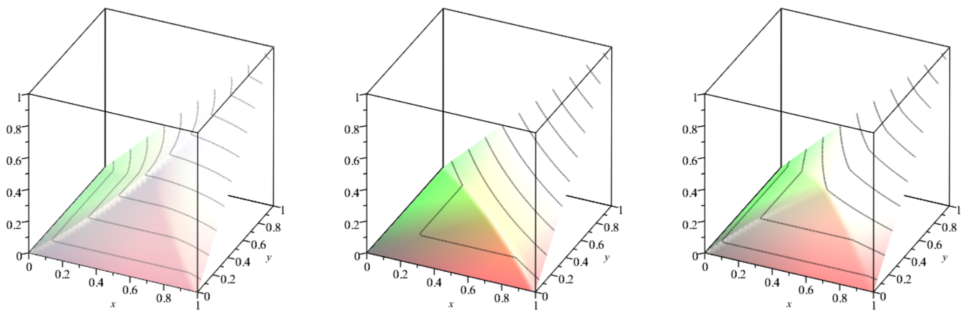
is a copula if and only if P assumes the form

$$P(x, y) = a(x^2 - xy) + b(y^2 - xy) + dx + (1 - d)y,$$

where

$$\begin{cases} 0 \le d \le 1, \\ a \ge -d, \\ b \le 1-d, \\ -d \le a-b \le 1-d, \\ a+b \le 2(1-d). \end{cases}$$

Example 4.2. Copula C under some special choices of a, b, d has the graphs presented in Figure 1. One should notice that if $a = b = 0$, then the graph of C contains four flat pieces. By choosing $(a, b) \neq (0, 0)$, we produce some curvature on its surface.



(A) $(a, b, d) = (0.5, 0.5, 0.5)$ (B) $(a, b, d) = (-0.3, 0, 0.3)$ (C) $(a, b, d) = (-0.6, 0.2, 0.8)$

FIGURE 1. Quadratic transformations of M and W.

Proof of Theorem 4.1. Since P is a quadratic polynomial, we have

$$P(x, y) = ax^2 + by^2 + cxy + dx + ey + f,$$

where $a, b, c, d, e, f \in \mathbb{R}$. Clearly, P satisfies Theorem 3.1 (i) if and only if

$$(a+b+c)x^2 + (d+e)x + f = x \quad \text{for all } x \in [0, 1].$$

This only happens when $a + b + c = f = 0$ and $d + e = 1$. Hence P has the form

$$P(x, y) = a(x^2 - xy) + b(y^2 - xy) + dx + (1-d)y.$$

We will exploit (ii) and (iii) in Theorem 3.1 to characterize all coefficients a, b, d. First, we search for conditions on the coefficients a, b, d such that

$$g(t) := P(y, (y+t)^+) - P(x, (x+t)^+)$$

is non-decreasing in $[y-1, 0]$ for any $0 \le x \le y \le 1$. That is, $g'(t) \ge 0$, where

$$g'(t) = \frac{\partial P}{\partial y}(y, (y+t)^+)\chi_{(-y,+\infty)}(t) - \frac{\partial P}{\partial y}(x, (x+t)^+)\chi_{(-x,+\infty)}(t).$$

Here χ_A denotes the characteristic function of a subset $A \subset \mathbb{R}$.

- If $t < -y$, then $g'(t) = 0$.
- If $-y < t < -x$, then

$$g'(t) = \frac{\partial P}{\partial y}(y, y+t) = (b-a)y + 2bt + 1 - d.$$

To ensure $g'(t) \geq 0$ for any $\max\{-y, y-1\} < t < -x$, where $0 \leq x \leq y \leq 1$, the necessary and sufficient condition is

$$\begin{cases} \min\limits_{0\leq y\leq 1}[(b-a)y + 1 - d] \geq 0, \\ \min\limits_{0\leq y\leq 1}[(b-a)y + 2b\max\{-y, y-1\} + 1 - d] \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \min\{1-d, b-a+1-d\} \geq 0, \\ \min\{1-d, b-a+1-d, \frac{b-a}{2} - b + 1 - d\} \geq 0. \end{cases}$$

That is,

$$1 - d \geq \max\{0, a-b, \frac{a+b}{2}\}. \tag{4.1}$$

- If $-x < t < 0$, then

$$g'(t) = \frac{\partial P}{\partial y}(y, y+t) - \frac{\partial P}{\partial y}(x, x+t) = (b-a)(y-x) + 2bt + 1 - d.$$

To ensure $g'(t) \geq 0$ for any $\max\{-x, y-1\} < t < 0$, where $0 \leq x \leq y \leq 1$, the necessary and sufficient condition is

$$\begin{cases} \min\limits_{0\leq x\leq y\leq 1}[(b-a)(y-x) + 1 - d] \geq 0, \\ \min\limits_{0\leq x\leq y\leq 1}[(b-a)(y-x) + 2b\max\{-x, y-1\} + 1 - d] \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \min\{1-d, b-a+1-d\} \geq 0, \\ \min\{1-d, b-a+1-d, -b+1-d\} \geq 0. \end{cases}$$

That is,

$$1 - d \geq \max\{0, a-b, b\}. \tag{4.2}$$

Next, we derive conditions on the coefficients a, b, d such that

$$h(x, y) := P(x, (2x-1)^+) + P(y, (2y-1)^+) - 2P(x, (x+y-1)^+)$$

is nonnegative for any $x, y \in [0, 1]$ with $x \leq y$.

- If $x \leq y \leq \frac{1}{2}$, then

$$\begin{aligned} h(x, y) &= P(x, 0) + P(y, 0) - 2P(x, 0) \\ &= [a(x+y) + d](y-x). \end{aligned}$$

To ensure that $h(x, y) \geq 0$ for any $0 \leq x \leq y \leq \frac{1}{2}$, the necessary and sufficient condition is

$$\min\{d, a+d\} \geq 0. \tag{4.3}$$

- If $x \leq \frac{1}{2} \leq y$ and $x + y \leq 1$, then

$$\begin{aligned} h(x, y) &= P(x, 0) + P(y, 2y-1) - 2P(x, 0) \\ &= a(y - y^2 - x^2) + b(2y^2 - 3y + 1) + d(1 - x - y) + (2y - 1). \end{aligned}$$

Taking $x = 1 - y$. In order to have

$$h(1-y, y) = [(b-a)(y-1) + 1](2y-1) \geq 0$$

for all $\frac{1}{2} \leq y \leq 1$, the necessary and sufficient condition is

$$\frac{a-b}{2} + 1 \geq 0. \tag{4.4}$$

Conversely, under condition (4.3), for any $x \leq \frac{1}{2} \leq y$ such that $x + y \leq 1$, we have

$$a(1-y+x) + d \geq 0.$$

Combining this with (4.4), we obtain

$$h(x,y) = h(1-y,y) + [a(1-y+x)+d](1-x-y) \geq 0$$

- If $x \leq \frac{1}{2} \leq y$ and $x + y \geq 1$, then

$$\begin{aligned} h(x,y) &= P(x,0) + P(y,2y-1) - 2P(x,x+y-1) \\ &= a(x^2+2xy-2x-y^2+y) + b(-2xy+2x+y-1) + d(x+y-1) + 1 - 2x \\ &= h(\frac{1}{2},y) + \frac{(2x-1)(2ax+4ay-4by-3a+4b+2d-4)}{4}. \end{aligned}$$

Notice that

$$h(\frac{1}{2},y) = \left[a(\frac{3}{2}-y)+d\right]\left(y-\frac{1}{2}\right) \geq 0 \quad \text{for all } \frac{1}{2} \leq y \leq 1$$

due to the fact $\min\{a+d, a/2+d\} \geq 0$ deduced from (4.3).

Moreover, from (4.2) and (4.3), we have

$$0 \leq d \leq 1, \quad a-b \leq 1-d, \quad b \leq 1-d,$$

which implies

$$\max\{a+2d-4, 2b+2d-4, 2a+2d-4\} \leq 0.$$

From this, one can easily verify that

$$2ax + 4ay - 4by - 3a + 4b + 2d - 4 \leq 0$$

and hence

$$h(x,y) \geq 0$$

for any $x \leq \frac{1}{2} \leq y$ with $x + y \geq 1$.

- If $\frac{1}{2} \leq x \leq y$, then

$$\begin{aligned} h(x,y) &= P(x,2x-1) + P(y,2y-1) - 2P(x,x+y-1) \\ &= [(x-y+1)a + (1-2x)b + d](y-x). \end{aligned}$$

To ensure $h(x,y) \geq 0$ for any $\frac{1}{2} \leq x \leq y \leq 1$, the necessary and sufficient condition is

$$\min\{a+d, a/2+d, a-b+d\} \geq 0. \tag{4.5}$$

Combining conditions (4.1)–(4.5), we finish the proof. ∎

Next, we find necessary and sufficient criteria for quadratic transformations on the independence copula and the Fréchet-Hoeffding lower bound.

Theorem 4.3. *Let P be a bivariate quadratic polynomial. The function*

$$C(x,y) := P(\Pi(x,y), W(x,y))$$

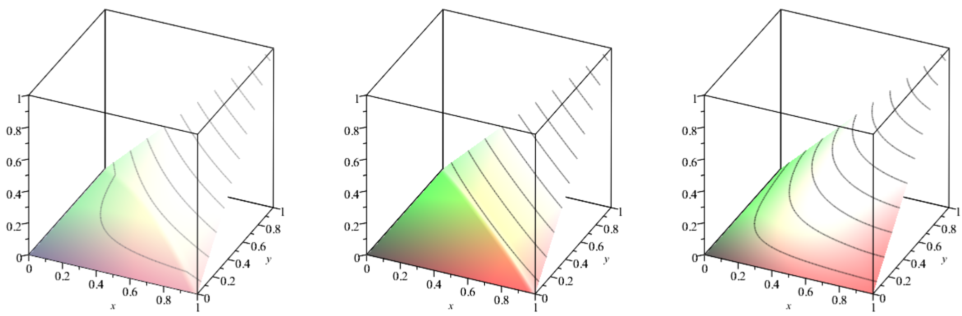
is a copula if and only if P assumes the form

$$P(x,y) = a(x^2 - xy) + b(y^2 - xy) + dx + (1-d)y,$$

where

$$\begin{cases} 0 \le d \le 1, \\ a + d \ge 0, \\ a + b \le 4(1-d), \\ -d \le b - a \le d, \\ \frac{(a+b)^2}{8a} + d \ge 0 & \text{if } a < 0 \text{ and } 0 \le \frac{a+b}{4a} \le 1, \\ b + d \ge 0 & \text{if } a < 0. \end{cases}$$

Example 4.4. Figure 2 contains some examples of copula $P(\Pi(x,y), W(x,y))$ under suitable choices of a, b, d.



(A) $(a,b,d) = (0, 0.5, 0.5)$ (B) $(a,b,d) = (-0.2, 0, 0.2)$ (C) $(a,b,d) = (0.5, -0.5, 1)$

FIGURE 2. Quadratic transformations of M and W.

Proof of Theorem 4.3. As in the proof of Theorem 4.1, P must have the form

$$P(x,y) = a(x^2 - xy) + b(y^2 - xy) + dx + (1-d)y.$$

By exploiting Theorem 3.3, we find conditions on the coefficients a, b, d such that

$$g(t) := P(yt, (y+t-1)^+) - P(xt, (x+t-1)^+)$$

is non-decreasing in $[0,1]$ for any $0 \le x \le y \le 1$. That is, $g'(t) \ge 0$.

Case 1. For $t \in [0, 1-y]$, we have

$$g(t) = P(yt, 0) - P(xt, 0) = a(y^2 - x^2)t^2 + d(y-x)t$$

and

$$g'(t) = [2a(x+y)t + d](y-x).$$

Hence $g'(t) \geq 0$ for all $t \in [0, 1-y]$ and $x \leq y$ if and only if

$$\begin{cases} d \geq 0, \\ \min\limits_{0 \leq x \leq y \leq 1} \{2a(x+y)(1-y) + d\} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} d \geq 0, \\ \min\limits_{0 \leq y \leq 1} \{2ay(1-y) + d, 4ay(1-y) + d\} \geq 0. \end{cases}$$

If $a \geq 0$, then

$$\min_{0 \leq y \leq 1} \{2ay(1-y) + d, 4ay(1-y) + d\} = d.$$

If $a < 0$, then

$$\min_{0 \leq y \leq 1} \{2ay(1-y) + d, 4ay(1-y) + d\} = a + d < d.$$

Combining these facts, we obtain the condition

$$\begin{cases} d \geq 0, & \text{if } a \geq 0, \\ a + d \geq 0, & \text{if } a < 0. \end{cases} \tag{4.6}$$

Case 2. For $1 - x \leq t \leq 1$, we have

$$\begin{aligned} g(t) &= P(yt, y+t-1) - P(xt, x+t-1) \\ &= [at^2x + at^2y - at^2 - atx - aty - bt^2 - btx - bty + at + 3bt \\ &\quad + bx + by + dt - 2b - d + 1](y - x) \end{aligned}$$

and

$$g'(t) = (2atx + 2aty - 2at - ax - ay - 2bt - bx - by + a + 3b + d)(y - x).$$

To ensure $g'(t) \geq 0$ for any $1 - x \leq t \leq 1$, where $0 \leq x \leq y \leq 1$, the necessary and sufficient condition is

$$\begin{cases} \min\limits_{0 \leq x \leq y \leq 1} \{ax + ay - bx - by - a + b + d\} \geq 0, \\ \min\limits_{0 \leq x \leq y \leq 1} \{-2ax^2 - 2axy + 3ax + ay + bx - by - a + b + d\} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \min\{-a + b + d, d, a - b + d\} \geq 0, \\ \min\limits_{0 \leq x \leq 1} \{-2ax^2 + (a+b)x + d\} \geq 0, \\ \min\limits_{0 \leq x \leq 1} \{-4ax^2 + 4ax - a + b + d\} \geq 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \min\{-a + b + d, d, a - b + d\} \geq 0, \\ \frac{(a+b)^2}{8a} + d \geq 0 & \text{if } a < 0 \text{ and } 0 \leq \frac{a+b}{4a} \leq 1, \\ b + d \geq 0 & \text{if } a < 0. \end{cases} \tag{4.7}$$

Case 3. For $t \in [1-y, 1-x]$, we have

$$\begin{aligned} g(t) &= P(yt, y+t-1) - P(xt, 0) \\ &= [(ayt - by - bt + b + d)(y-1)(t-1) + y + t - 1] - (ax^2t^2 + dxt) \end{aligned}$$

and

$$g'(t) = -2atx^2+2aty^2-2aty-ay^2-2bty-by^2+ay+2bt+3by-dx+dy-2b-d+1.$$

We have

$$g'(1-y) = [(2x^2-2y^2+y)a+by](y-1)+d(y-x-1)+1.$$

When $x = y$,

$$g'(1-y) = (a+b)(y^2-y)+1-d.$$

To ensure $g'(1-y) \geq 0$ for any $0 \leq x = y \leq 1$, the necessary and sufficient condition is

$$\min\{-\frac{a+b}{4}+1-d, 1-d\} \geq 0. \tag{4.8}$$

Conversely, under condition (4.8) and $x \leq y$, we have

$$\begin{aligned} g'(1-y) &= [(2x^2-2y^2+y)a+by](y-1)+d(y-x-1)+1 \\ &\geq (a+b)(y^2-y)+1-d \\ &\geq 0. \end{aligned} \tag{4.9}$$

Moreover,

$$\begin{aligned} g'(1-x) &= 2ax^3-2axy^2-2ax^2+2axy+ay^2+2bxy-by^2 \\ &\quad -ay-2bx+by-dx+dy-d+1 \\ &:= h(y). \end{aligned}$$

We have

$$h'(y) = (-4ax+2a-2b)y+2ax+2bx-a+b+d.$$

Using (4.7), we derive

$$\begin{aligned} h'(1) &= -2ax+2bx+a-b+d \\ &\geq \min_{0\leq x\leq 1}\{-2ax+2bx+a-b+d\} \\ &\geq \min\{a-b+d, -a+b+d\} \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} h'(x) &= -4ax^2+4ax-a+b+d \\ &\geq \min_{0\leq x\leq 1}\{-4ax^2+4ax-a+b+d\} \\ &\geq -a+b+d \\ &\geq 0. \end{aligned}$$

From this and the fact that $h'(y)$ is linear in y, we deduce $h'(y) \geq 0$ for all $x \leq y \leq 1$. That means that h is non-decreasing in $[x, 1]$.

Therefore,

$$g'(1-x) \geq h(x) = (a+b)(x^2-x)+1-d \geq 0, \tag{4.10}$$

where we have used (4.8).

Since $g'(t)$ is linear in t, from (4.9) and (4.10) we deduce that $g'(t) \geq 0$ for all $t \in [1-y, 1-x]$.

By collecting (4.6)–(4.8), we obtain the necessary and sufficient condition for C to be a copula. This completes the proof of the theorem. ∎

Finally, we introduce a sufficient condition for quadratic transformations on the Fréchet-Hoeffding upper bound and the independence copula.

Theorem 4.5. *Let P be a bivariate quadratic polynomial. The function*

$$C(x,y) := P(M(x,y), \Pi(x,y))$$

is a copula if P assumes the form

$$P(x,y) = a(x^2 - xy) + b(y^2 - xy) + dx + (1-d)y, \tag{4.11}$$

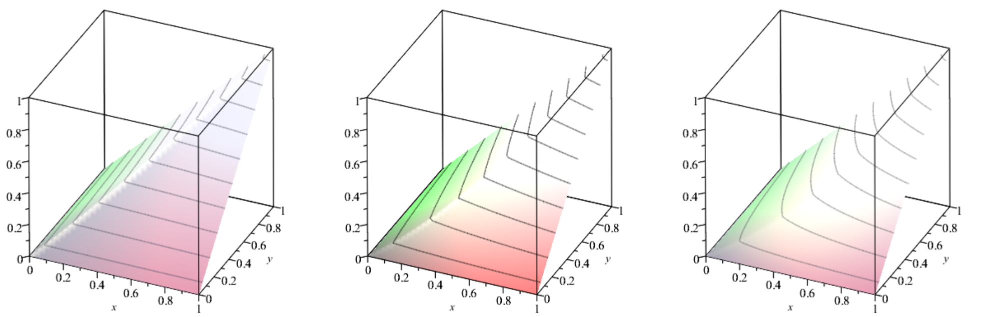
where

$$\begin{cases} 0 \le d \le 1, \\ 0 \le b \le a, \\ d-1 \le 2(b-a), \\ d-1 \le -\frac{(a+b)^2}{4b} & \text{if } 0 < a \le 3b \end{cases} \tag{4.12}$$

or

$$\begin{cases} 0 \le d \le 1, \\ \frac{|2d-1|-1}{2} \le a = b \le \frac{|2d-1|+1}{2}. \end{cases} \tag{4.13}$$

Example 4.6. We give some examples of copulas generated by quadratic transformations of M and W in Figure 3 below.



(A) $(a,b,d) = (0.2, 0.2, 0.8)$ (B) $(a,b,d) = (0.2, 0, 0.6)$ (C) $(a,b,d) = (-0.3, -0.3, 0.3)$

FIGURE 3. Quadratic transformations of M and W.

Proof of Theorem 4.5. As in the proof of Theorem 4.1, P must have the form

$$P(x,y) = a(x^2 - xy) + b(y^2 - xy) + dx + (1-d)y.$$

If a, b, d satisfy (4.13), then C is a copula due to Proposition 1.1. Hence, it suffices to consider assumption (4.12) in the rest of the proof.

First, we search for conditions on the coefficients a, b, d such that condition (iii) of Theorem 3.4 is satisfied. That is, q is non-decreasing in $[y, 1]$ for any $0 \leq x \leq y \leq 1$, where

$$\begin{aligned} q(t) &:= P(y, ty) - P(x, tx) \\ &= [a(y^2 - ty^2) + b(t^2y^2 - ty^2) + dy + (1-d)ty] \\ &\quad - [a(x^2 - tx^2) + b(t^2x^2 - tx^2) + dx + (1-d)tx]. \end{aligned}$$

Direct calculation yields

$$q'(t) = [(2bt - a - b)(x + y) - d + 1](y - x).$$

To ensure $q'(t) \geq 0$ for all $x \leq y \leq t \leq 1$, the necessary and sufficient condition is

$$\begin{cases} \min\limits_{0 \leq x \leq y \leq 1} \{(b-a)(x+y) - d + 1\} \geq 0, \\ \min\limits_{0 \leq x \leq y \leq 1} \{(2by - a - b)(x+y) - d + 1\} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \min\{-d+1, 2(b-a) - d + 1\} \geq 0, \\ \min\limits_{0 \leq y \leq 1} \{(2by - a - b)y - d + 1, 2(2by - a - b)y - d + 1\} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} d - 1 \leq \min\{2(b-a), 0\}, \\ d - 1 \leq 2 \min\limits_{0 \leq y \leq 1} \{(2by - a - b)y\}, \end{cases}$$

- If $b > 0$ and $\frac{a+b}{4b} \in [0, 1]$, then

$$\min_{0 \leq y \leq 1} \{(2by - a - b)y\} = -\frac{(a+b)^2}{8b} \leq \min\{0, b - a\}.$$

- Otherwise,

$$\min_{0 \leq y \leq 1} \{(2by - a - b)y\} = \min\{0, b - a\}.$$

Combining the above facts, we deduce that P satisfies Theorem 3.4 (iii) if and only if

$$d - 1 \leq \begin{cases} \min\{2(b-a), 0\}, \\ -\frac{(a+b)^2}{4b} & \text{if } b > 0 \text{ and } \frac{a+b}{4b} \in [0, 1]. \end{cases} \tag{4.14}$$

Clearly, (4.14) holds under assumption (4.12).

Next, we derive a sufficient condition on the coefficients a, b, d such that condition (ii) of Theorem 3.4 is satisfied. That is,

$$P(x, x^2) + P(y, y^2) \geq 2P(x, xy)$$

for any $x, y \in [0, 1]$ with $x \leq y$. By a direct calculation, we have

$$\begin{aligned} &P(x, x^2) + P(y, y^2) - 2P(x, xy) \\ &= [(a - by)(x^2 - xy - y^2 + x + y) + bx^2(1 - x) + d(1 + x - y) + (y - x)](y - x). \end{aligned}$$

Setting $g(y) := x^2 - xy - y^2 + x + y$ for $x \leq y \leq 1$. Since g does not admit local minimums, we have

$$g(y) \geq \min\{g(x), g(1)\} = \min\{2x - x^2, x^2\} \geq 0.$$

Combining the above facts, we deduce that P satisfies Theorem 3.4 (ii) if $a \geq b \geq 0$ and $0 \leq d \leq 1$.

This completes the proof of the theorem. ■

Remark 4.7. Unlike Theorems 4.1 and 4.3, we are not able to classify all quadratic transformations in Theorem 4.5 due to some technical difficulties. Nevertheless, it turns out from the proof of Theorem 4.5 that, if P is a bivariate quadratic polynomial such that $(x, y) \mapsto P(M(x, y), \Pi(x, y))$ is a copula, then P must have the form (4.11) with coefficients a, b, d satisfy

$$\begin{cases} 0 \le d \le 1, \\ d - 1 \le 2(b - a). \end{cases}$$

5. Conclusion

In this paper, we proposed three new classes of copulas generated by function compositions on three well-known copulas: the independence copula, the Fréchet-Hoeffding lower bound, and the Fréchet-Hoeffding upper bound (see Theorems 3.1, 3.3 and 3.4 in Section 3). In particular, some quadratic transformations of this type were completely classified (see Theorems 4.1, 4.3 and 4.5 in Section 4). For future research, we will investigate more general compositions of this type for multivariate copulas and find out the probabilistic properties of copulas generated by such transformations. This would open the door to applying new copulas to empirical research on dependence modeling.

References

[1] R.B. Nelsen, An Introduction to Copulas (second edition), Springer Series in Statistics, Springer, New York, 2006.

[2] M.I. Bhatti, H.Q. Do, Recent development in copula and its applications to the energy, forestry and environmental sciences, Int. J. Hydrog. Energy 44 (2019) 19453–19473.

[3] F. Durante, C. Sempi, Principles of Copula Theory, CRC Press; Boca Raton, FL, 2016.

[4] A. Kolesárová, G. Mayor, R. Mesiar, Quadratic constructions of copulas, Inform. Sci. 310 (2015) 69–76.

[5] S. Wisadwongsa, S. Tasena, Bivariate quadratic copula constructions, Internat. J. Approx. Reason. 92 (2018) 1–19.

[6] S. Tasena, Polynomial copula transformations, Internat. J. Approx. Reason. 107 (2019) 65–78.

[7] M. Manstavičius, G. Bagdonas, A class of bivariate copula mappings, Fuzzy Sets and Systems 354 (2019) 48–62.

[8] S. Girard, Transformation of a copula using the associated co-copula, Depend. Model. 6 (2018) 298–308.

[9] S. Saminger-Platz, A. Kolesárová, R. Mesiar, E.P. Klement, The key role of convexity in some copula constructions, Eur. J. Math. 6 (2020) 533–560.