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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

New Generalization of Hermite-Hadamard Type of Inequalities for Convex Functions Using Fourier Integral Transform

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Abstract The aim of this paper is to prove some new type of Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal inequalities for Fourier integral transform. These results pave the way to the researcher in obtaining complete unique functional inequalities for a well established inequalities in use and that involves convex functions.

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1. Introduction

In recent years several generalization and extensions of the classical thought of convex function have been presented, and the hypothesis of inequalities has produced significant commitments in that regard. The inequalities of Hermite-Hadamard and Fejer have been the object of extraordinary examination and have given numerous applications. The history of this inequalities can be found in [1–6]. Hermite- Hadamard derived the following inequalities: If a function $q : \mathbb{J} \subseteq R \to R$ is a convex function, $c, d \in \mathbb{J}$ with $c < d$, then

$$q\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d q(u)du \le \frac{q(c)+q(d)}{2}. \tag{1.1}$$

In 1906, L. Fejer [7] proved the following integral inequalities which are known in the literature as Fejer inequality:

$$q\left(\frac{c+d}{2}\right)\int_c^d w(u)du \le \frac{1}{d-c}\int_c^d q(u)w(u)du \le \frac{q(c)+q(d)}{2}\int_c^d w(u)du, \tag{1.2}$$

where $w : \mathbb{J} \subseteq R \to R^+$ is integrable and symmetric for $\frac{c+d}{2}$. If we consider in inequality (1.2) that $w \equiv 1$, then we reclaim the known Hermite-Hadamard inequality. In [8], Dragomir and Agarwal proved the following results related to the first part of inequality (1.1).

Let $q : \mathbb{J} \subseteq R \to R$ be a differentiable function and $|q'|$ is convex on $[c, d] \in \mathbb{J}$, then the following inequality holds:

$$\left|\frac{q(c)+q(d)}{2} - \frac{1}{d-c}\int_c^d q(u)du\right| \le \frac{d-c}{8}\left(|q'(c)| + |q'(d)|\right). \tag{1.3}$$

A several studies have been written on the generalized, extensions, providing new proofs and application of inequalities (1.1)-(1.3), see [9–19] and the references therein.

A convex function $q(u)$ which has great application in various field and closely related to inequalities. Since $q(u)$ is convex, it is absolutely continuous [20, 21]. If a function $q(u)$ is convex ($q''(u) > 0$), which implies that $q'(u) < 0$ and $q(u) > 0$. That is, convex functions possessing Fourier transform are also decreasing and positive [22, 23]. The most applied integral transform is a Fourier transform. The idea of Fourier transform was first suggested by the French mathematician Jean Baptiste Joseph Fourier in 1807. It transforms a function in the time domain into the frequency domain. The most number of researchers had researched on the transform of it in various streams. Fourier transform stands unique among the other transform because it is easier to comprehend and also the involvement of numerous works in it.

In [24], the author Chen to study the extensions of the Hermite-Hadamard inequality for harmonically convex functions via fractional integrals. In [25], Ahmad et al. generalized the Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir–Agarwal and Pachpatte type inequalities via fractional integrals. However, to the best of our knowledge, the Hermite-Hadamard and some new kind of inequalities for convex functions has not been fully reported based on Fourier integral transform approach.

In this paper, based on above observation, we develop a new inequalities of convex functions that are related to generalizing of the classical Hermite-Hadamard via Fourier integral transform. Furthermore, using the convolution concept and properties of Fourier transform, some new generalizing of Hermite-Hadamard-Fejer and Dragomir-Agarwal inequalities of convex function are established.

The contribution of the paper is outlined as follows: In Section 2, some definitions are introduced, which are closely connected to our main results. In Section 3, we established integral inequalities analogous to generalizing of Hermite-Hadamard, Hermite-Hadamard-Fejer and Dragomir-Agarwal inequalities of convex function. The conclusion are given in Section 4.

2. Preliminaries

Here, we give some definition and properties to further work. Throughout this section, we let $\mathbb{J} = [c, d]$ be an interval in R.

Definition 2.1. A function $q : \mathbb{J} \subseteq R \to R$ is said to be convex if

$$q\left(\eta u + (1-\eta)v\right) \leq \eta q(u) + (1-\eta)q(v), \tag{2.1}$$

holds for all $u, v \in \mathbb{J}$ and $\eta \in [0, 1]$.

The above inequality (2.1) holds in opposite direction for concave function. Now we recall some basic concept of Fourier transform of a function.

Definition 2.2. If a function $q : R \to F$ is piecewise continuous in each finite interval and is absolutely integrable in R, then the Fourier transform of $q \in L'(R)$ denoted by $\widehat{Q}(\xi)$ is given by the integral

$$\widehat{Q}(\xi) = \int_{-\infty}^{\infty} q(u)e^{i\xi u}du \quad (or) \quad \widehat{Q}(\xi) = \int_{-\infty}^{\infty} q(u)e^{-i\xi u}du. \tag{2.2}$$

The inverse Fourier transform is given by

$$q(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{Q}(\xi)e^{-i\xi u}d\xi \quad (or) \quad q(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{Q}(\xi)e^{i\xi u}d\xi. \tag{2.3}$$

The function $\widehat{Q}(\xi)$ consists of the frequency components of the time domain function $q(u)$, $\forall u \in R$.

Definition 2.3. (Convolution) The convolution of two function $q(u)$ and $r(u)$ is defined as

$$q(u) * r(u) = \int_{-\infty}^{\infty} q(u)r(t-u)du, \quad \forall u \in R.$$

Property of Fourier transform is given as,

(1) The Fourier transform of the convolution of $q(u)$ and $r(u)$ is the product of the Fourier transform of $q(u)$ and $r(u)$. That is,

$$F(q(u) * r(u)) = F(q(u))F(r(u)) = \widehat{Q}(\xi)\widehat{R}(\xi).$$

(2) If $\widehat{Q}(\xi)$ is Fourier transform of $q(u)$, then

$$F(q(u \pm a)) = e^{\pm ia}\widehat{Q}(\xi).$$

3. Main Results

In this section, we derive our main results related to Fourier integral transform of convex function.

Theorem 3.1. (Generalization of Hermite-Hadamard inequality) Let $q : \mathbb{J} \subseteq R \to R$ be a convex function with $c < d$ and $c, d \in \mathbb{J}$. Then, the following inequalities for Fourier integral transform Equations (2.2) and (2.3) hold:

$$q\left(\frac{c+d}{2}\right) \leq \frac{i\xi}{2(1-e^{-i\xi(d-c)})}\left(\widehat{Q}(\xi+c)+\widehat{Q}(\xi-d)\right) \leq \frac{q(c)+q(d)}{2}, \tag{3.1}$$

where $\widehat{Q}(\xi)$ is Fourier transform of function $q(u)$, $\forall u \in [c,d]$.

Proof. Since q is a convex function on $[c,d] \subseteq R$, for all $x \in [0,1]$ and $u, v \in [c,d]$. We have,

$$q\left(\frac{u+v}{2}\right) \leq \frac{q(u)+q(v)}{2}. \tag{3.2}$$

By making use of the substitution $u = xc + (1-x)d$ and $v = (1-x)c + xd$, we have

$$2q\left(\frac{c+d}{2}\right) \leq q(xc+(1-x)d) + q((1-x)c+xd). \tag{3.3}$$

Set $\rho = \xi(d-c)$, multiplying both sides of (3.3) by $e^{-i\rho x}$ and then integrating with respect to x over $[0,1]$. We have,

$$\begin{aligned}
2\int_0^1 q\left(\frac{c+d}{2}\right)e^{-i\rho x}dx \leq & e^{-i\rho x}\left(\int_0^1 q(xc+(1-x)d)dx \right.\\
& \left. + \int_0^1 q((1-x)c+xd)dx\right)\\
2\int_0^1 q\left(\frac{c+d}{2}\right)e^{-i\rho x}dx \leq & \int_0^1 q(xc+(1-x)d)e^{-i\rho x}dx\\
& + \int_0^1 q((1-x)c+xd)e^{-i\rho x}dx\\
2q\left(\frac{c+d}{2}\right)\left(\frac{1-e^{-i\xi(d-c)}}{i\xi(d-c)}\right) \leq & \frac{1}{d-c}\left(\int_c^d q(s)e^{-i\xi(d-s)}ds \right.\\
& \left. + \int_c^d q(s)e^{-i\xi(s-c)}ds\right)\\
= & \frac{1}{d-c}\left(e^{-i\xi d}\widehat{Q}(\xi) + e^{i\xi c}\widehat{Q}(\xi)\right)\\
= & \frac{1}{d-c}\left(\widehat{Q}(\xi-d) + \widehat{Q}(\xi+c)\right)\\
q\left(\frac{c+d}{2}\right) \leq & \frac{i\xi}{2\left(1-e^{-i\xi(d-c)}\right)}\left(\widehat{Q}(\xi+c)+\widehat{Q}(\xi-d)\right).
\end{aligned} \tag{3.4}$$

Thus the first inequality of (3.1) is established. The proof of second inequality is given as, since $q(u)$ is convex function, then it gives

$$q\left(xc+(1-x)d\right) \leq xq(c) + (1-x)q(d), \tag{3.5}$$

and

$$q((1-x)c + xd) \leq (1-x)q(c) + xq(d), \quad \forall x \in [0,1]. \tag{3.6}$$

Adding inequalities (3.5) and (3.6), we get

$$q\left(xc + (1-x)d\right) + q((1-x)c + xd) \leq q(c) + q(d). \tag{3.7}$$

Multiplying both sides of (3.7) by $e^{-i\rho x}$ and integrating the resulting inequality with respect to x over $[0,1]$,

$$e^{-i\rho x}\left(\int_0^1 q(xc+(1-x)d)dx + \int_0^1 q((1-x)c+xd)dx\right) \leq e^{-i\rho x}\int_0^1 (q(c)+q(d))dx$$

$$\int_0^1 q(xc+(1-x)d)e^{-i\rho x}dx + \int_0^1 q((1-x)c+xd)e^{-i\rho x}dx \leq \int_0^1 (q(c)+q(d))e^{-i\rho x}dx$$

$$\frac{1}{d-c}\int_c^d q(s)e^{-i\xi(s-c)}ds + \frac{1}{d-c}\int_c^d q(s)e^{-i\xi(d-s)}ds \leq 2\left(\frac{1-e^{-i\xi(d-c)}}{i\xi(d-c)}\right)(q(c)+q(d))$$

$$\frac{1}{d-c}\left(\widehat{Q}(\xi+c) + \widehat{Q}(\xi-d)\right) \leq 2\left(\frac{1-e^{-i\xi(d-c)}}{i\xi(d-c)}\right)(q(c)+q(d)$$

$$\frac{i\xi}{2\left(1-e^{-i\xi(d-c)}\right)}\left(\widehat{Q}(\xi+c) + \widehat{Q}(\xi-d)\right) \leq q(c)+q(d). \tag{3.8}$$

This completes the proof. ■

Corollary 3.2. Assume that $q : [c,d] \subseteq R \to R$ is a concave function on $\mathbb{J}$ with $c < d$ and $q \in L'(c,d)$, then we have

$$q\left(\frac{c+d}{2}\right) \geq \frac{i\xi}{2(1-e^{-i\xi(d-c)})}\left(\widehat{Q}(\xi+c) + \widehat{Q}(\xi-d)\right) \geq \frac{q(c)+q(d)}{2}. \tag{3.9}$$

Now we give proof of Hermite-Hadamard Fejer inequality utilizing Fourier transform of convex function.

Theorem 3.3. (Generalization of Hermite-Hadamard-Fejer Inequality) Let $q : [c,d] \subseteq R \to R^+$ be a convex function with $c < d$. If $w : [c,d] \subseteq R \to R$ is non-negative, integrable and $w(c+d-u) = w(u)$, then the following inequality hold:

$$q\left(\frac{c+d}{2}\right)\left(\widehat{W}(\xi-d) + \widehat{W}(\xi+c)\right) \leq \widehat{Q}(\xi+c)\widehat{W}(\xi+c) + \widehat{Q}(\xi-d)\widehat{W}(\xi-d) \leq \frac{q(c)+q(d)}{2}\left(\widehat{W}(\xi-d) + \widehat{W}(\xi+c)\right), \tag{3.10}$$

where $\widehat{W}(\xi)$ is Fourier transform of function $q(u)$, $\forall u \in [c,d]$.

Proof. Using the convexity of $q(u)$ on $[c,d]$, we get from the inequality (3.3) that

$$2q\left(\frac{c+d}{2}\right) \leq q(xc+(1-x)d) + q((1-x)c+xd), \quad \forall x \in [0,1]. \tag{3.11}$$

Multiply both-sides of above equation (3.11) by $w((1-x)c+xd)$, we get

$$\begin{aligned} 2q\left(\frac{c+d}{2}\right) * w((1-x)c+xd) \leq q(xc+(1-x)d) * w((1-x)c+xd) \\ + q((1-x)c+xd) * w((1-x)c+xd). \end{aligned} \tag{3.12}$$

Multiplying both sides of inequality (3.7) by $e^{-i\rho x}$ and integrate resulting inequality with respect to x over $[0,1]$, we obtain

$$\begin{aligned} 2q\left(\frac{c+d}{2}\right)\int_0^1 w((1-x)c+xd)e^{-i\rho x}dx \leq & \int_0^1 q(xc+(1-x)d) * w((1-x)c+xd) \\ & e^{-i\rho x}dx + \int_0^1 q((1-x)c+xd) \\ & * w((1-x)c+xd)e^{-i\rho x}dx \\ 2\frac{1}{d-c}q\left(\frac{c+d}{2}\right)\widehat{W}(\xi-d) \leq & \frac{1}{d-c}\int_c^d q(c+d-s) * w(s)e^{-i\xi(s-c)}ds \\ & + \frac{1}{d-c}\int_c^d q(s) * w(s)e^{-i\xi(s-c)}ds \\ = & \frac{1}{d-c}\int_c^d e^{-i\xi(d-s)}q(s) * w(c+d-s)ds \\ & + \frac{1}{d-c}e^{i\xi c}F(q(s) * w(s)) \\ = & \frac{1}{d-c}e^{-i\xi d}F(q(s) * w(s)) \\ & + \frac{1}{d-c}e^{i\xi c}F(q(s) * w(s)) \\ = & \frac{1}{d-c}e^{-i\xi d}\widehat{Q}(\xi)\widehat{W}(\xi) \\ & + \frac{1}{d-c}e^{i\xi c}\widehat{Q}(\xi)\widehat{W}(\xi), \end{aligned}$$

we get,

$$2q\left(\frac{c+d}{2}\right)\widehat{W}(\xi-d) \leq \widehat{Q}(\xi-d)\widehat{W}(\xi-d) + \widehat{Q}(\xi+c)\widehat{W}(\xi+d). \tag{3.13}$$

Multiplying both-sides of inequality (3.3) by $e^{-i\rho x}w(xc+(1-x)d)$ and then integrating above inequality with respect to x over $[0,1]$. In consequence, we obtain

$$2q\left(\frac{c+d}{2}\right)\widehat{W}(\xi+c) \leq \widehat{Q}(\xi-d)\widehat{W}(\xi-d) + \widehat{Q}(\xi+c)\widehat{W}(\xi+c). \tag{3.14}$$

In view of inequalities (3.13) and (3.14), we get

$$2q\left(\frac{c+d}{2}\right)\left(\widehat{W}(\xi+c)+\widehat{W}(\xi-d)\right)\leq \widehat{Q}(\xi-d)\widehat{W}(\xi-d)+\widehat{Q}(\xi+c)\widehat{W}(\xi+d). \tag{3.15}$$

To prove the second inequality, we make use of inequality (3.7)

$$q(xc+(1-x)d)+q((1-x)c+xd)\leq q(c)+q(d), \quad \forall x\in[0,1]. \tag{3.16}$$

Multiplying both-sides of inequality (3.16) by $e^{-i\rho x}w((1-x)c+xd)$ and then integrating resulting inequality with respect to x over $[0,1]$:

$$\int_0^1 e^{-i\rho x}q(xc+(1-x)d)*w((1-x)c+xd)dx+\int_0^1 e^{-i\rho t}q((1-x)c+xd)$$
$$*w((1-x)c+xd)dx\leq (q(c)+q(d))\int_0^1 e^{-i\rho x}w((1-x)c+xd)dx$$
$$\widehat{Q}(\xi+c)\widehat{W}(\xi+c)+\widehat{Q}(\xi-d)\widehat{W}(\xi-d)\leq \frac{q(c)+q(d)}{2}\left(\widehat{W}(\xi-d)+\widehat{W}(\xi+c)\right).$$

Thus the proof of theorem is complete. ■

The following Lemma will be used to established the generalization of Dragomir-Agarwal inequality.

Lemma 3.4. Let $q:\mathbb{J}\subseteq R\to R$ be a differentiable convex function on $[c,d]$ such that $c<d$. If q' is Lebesgue integral for $c,d\in\mathbb{J}$, then the following equality for Fourier integral transform holds:

$$\frac{q(c)+q(d)}{2}-\frac{i\xi}{2(1-e^{-i\rho})}(\widehat{Q}(\xi+c)+\widehat{Q}(\xi-d))=\left(\frac{d-c}{2(1-e^{-i\rho})}\right)$$
$$\int_0^1 e^{-i\rho x}q'(xc+(1-x)d)dx-\int_0^1 e^{-i\rho(1-x)}q'(xc+(1-x)d)dx,$$

where $\rho=\xi(d-c)$.

Proof. It suffices to note that,

$$\begin{aligned}
I =& \int_0^1 \left(e^{-i\rho x} - e^{-i\rho(1-x)}\right) q'(xc+(1-x)d)dx \\
=& \int_0^1 e^{-i\rho x} q'(cx+(1-x)d)dx - \int_0^1 e^{-i\rho(1-x)} q'(xc+(1-x)d)dx. \\
=& \left(\frac{e^{-i\rho x} q(xc+(1-x)d}{c-d}\right)_0^1 - \int_0^1 \frac{q(xc+(1-x)d)}{c-d} e^{-i\rho x}(-i\rho)dx \\
&- \left(\frac{e^{-i\rho(1-x)} q(xc+(1-x)d}{c-d}\right)_0^1 + \int_0^1 \frac{q(xc+(1-x)d)}{c-d} e^{-i\rho(1-x)}(i\rho)dx \\
=& \frac{e^{-i\rho} q(c)}{c-d} - \frac{q(d)}{c-d} + \frac{i\rho}{c-d}\int_0^1 e^{-i\rho x} q(xc+(1-x)d)dx - \frac{q(c)}{c-d} \\
&+ \frac{e^{-i\rho} q(d)}{c-d} + \frac{i\rho}{c-d}\int_0^1 e^{-i\rho(1-x)} q(xc+(1-x)d)dx \\
=& \frac{-e^{-i\rho} q(c)}{d-c} + \frac{q(d)}{d-c} - \frac{i\rho}{(d-c)^2}\int_0^1 e^{-i\xi(d-s))} q(s)ds + \frac{q(c)}{d-c} - \frac{e^{-i\rho} q(d)}{d-c} \\
&- \frac{i\rho}{(d-c)^2}\int_0^1 e^{-i\xi(s-c)} q(s)ds \\
=& \frac{q(c)+q(d)}{d-c}\left(1-e^{-i\rho}\right) - \frac{i\xi}{d-c}\left(\widehat{Q}(\xi-d) + \widehat{Q}(\xi+c)\right). \\
\frac{d-c}{2\left(1-e^{-i\rho}\right)} I =& \frac{q(c)+q(d)}{2} - \frac{i\xi}{2\left(1-e^{-i\rho}\right)}\left(\widehat{Q}(\xi-d) + \widehat{W}(\xi+c)\right).
\end{aligned}$$

■

Theorem 3.5. (Dragomir-Agarwal Inequality) Suppose that $q : \mathbb{J} \subseteq R \to R$ is a differentiable convex mapping on $\mathbb{J}$. If $|q'|$ is convex on $[c,d]$ and $c,d \in \mathbb{J}$, then the following inequality involving Fourier integral holds:

$$\left|\frac{q(c)+q(d)}{2} - \frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{Q}(\xi+c) + \widehat{Q}(\xi-d)\right)\right| \leq \frac{(1-2e^{-i\rho}+e^{-i\rho})}{i\xi(1-e^{-i\rho})}\left|q'(c)+q'(d)\right|,$$

where $\rho = \xi(d-c)$.

Proof. By Lemma (3.4), we have

$$\begin{aligned}
&\frac{q(c)+q(d)}{2} - \frac{i\xi}{2(1-e^{-i\rho})}(\widehat{Q}(\xi+c) + \widehat{Q}(\xi-d)) \\
&= \frac{d-c}{2(1-e^{-i\rho})}\int_0^1 e^{-i\rho x} q'(xc+(1-x)d)dx - \int_0^1 e^{-i\rho(1-x)} q'(xc+(1-x)d)dx.
\end{aligned}$$

Since the function $|q'|$ is convex on $[c,d]$, we have

$$
\begin{aligned}
&\left|\frac{q(c)+q(d)}{2} - \frac{i\xi}{2(1-e^{i\rho})}(\widehat{Q}(\xi+c)+\widehat{Q}(\xi-d))\right| \\
&\leq \frac{d-c}{2}\int_0^1 \frac{\left|e^{-i\rho x}-e^{-i\rho(1-x)}\right|}{1-e^{-i\rho}}\left|q'(xc+(1-x)d)\right| dx \\
&\leq \frac{d-c}{2}\int_0^1 \frac{\left|e^{-i\rho x}-e^{-i\rho(1-x)}\right|}{1-e^{-i\rho}} x\left|q'(c)\right| dx \\
&\quad + \frac{d-c}{2}\int_0^1 \frac{\left|e^{-i\rho x}-e^{-i\rho(1-x)}\right|}{1-e^{-i\rho}}(1-x)\left|q'(d)\right| dx \\
&= \frac{d-c}{2}\int_0^{1/2} \frac{\left(e^{-i\rho x}-e^{-i\rho(1-x)}\right)}{1-e^{-i\rho}} xq'(c)dx \\
&\quad + \frac{d-c}{2}\int_{1/2}^1 \frac{\left(e^{-i\rho(1-x)}-e^{-i\rho x}\right)}{1-e^{-i\rho}} xq'(d)dx \\
&\quad + \frac{d-c}{2}\int_0^{1/2} \frac{\left(e^{-i\rho x}-e^{-i\rho(1-x)}\right)}{1-e^{-i\rho}} (1-x)q'(c)dx \\
&\quad + \frac{d-c}{2}\int_{1/2}^1 \frac{\left(e^{-i\rho(1-x)}-e^{-i\rho x}\right)}{1-e^{-i\rho}} (1-x)q'(d)dx \\
&= \frac{d-c}{2(1-e^{-i\rho})}\left|q'(c)\right|\left(\frac{-e^{-\frac{i\rho}{2}}}{i\rho}+\frac{1}{(i\rho)^2}(1-e^{-i\rho})\right. \\
&\quad \left. + \frac{1}{i\rho}\left(1-e^{-\frac{i\rho}{2}}+e^{-i\rho}\right)-\frac{1}{(i\rho)^2}(1-e^{-i\rho})\right) \\
&\quad + \frac{d-c}{2(1-e^{-i\rho})}\left|q'(d)\right|\left(-\frac{e^{-\frac{i\rho}{2}}}{i\rho}+\frac{1}{i\rho}(1+e^{-i\rho})\right. \\
&\quad \left. -\frac{1}{(i\rho)^2}(1-e^{-i\rho})-e^{\frac{-i\rho}{2}}i\rho+\frac{1}{(i\rho)^2}(1-e^{-i\rho})\right) \\
&= \frac{(1-2e^{-i\rho}+e^{-i\rho})}{i\xi(1-e^{-i\rho})}\left|q'(c)+q'(d)\right|.
\end{aligned}
$$

Thus the proof of theorem is complete. ■

Remark 3.6. For $\xi \to 0$, notice that

$$
\lim_{\xi\to 0}\frac{i\xi}{2(1-e^{-i\rho})} = \frac{1}{2(d-a)}\lim_{\xi\to 0}\frac{d-c}{2(1-e^{-i\rho})}\left(\frac{1-2e^{-i\rho}+e^{-i\rho}}{i\rho}\right) = \frac{d-c}{8}. \quad (3.17)
$$

So, we can reduce to Hermite-Hadamard inequality from Theorem 3.5 with limit $\xi \to 0$.

Corollary 3.7. Let $q : \mathbb{J} \subseteq R \to R$ be a differentiable mapping on $\mathbb{J}$. If $|q'|$ is concave on $[c,d]$ and $c,d \in \mathbb{J}$, then the following inequality involving Fourier integral holds:

$$
\left|\frac{q(c)+q(d)}{2} - \frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{Q}(\xi+c)+\widehat{Q}(\xi-d)\right)\right| \geq \frac{(1-2e^{-i\rho}+e^{-i\rho})}{i\xi(1-e^{-i\rho})}\left|q'(c)+q'(d)\right|.
$$

4. Conclusion

The analyses of the results that the researcher had arrived at has re-affirmed the new generalization of Hermite-Hadamard and other type of integral inequalities for Fourier transform of convex functions. Moreover, the researcher believe that the existing the findings on the theory act as a source for the motivation. This had facilitated the investigator to identify and explore the similar literature that exists in relation with the research domain.

Compliance with Ethical Standards
Conflict of Interest The authors declare that they have no conflict of interest.

References

[1] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of the University of Craiova-Mathematics and Computer Science Series 34 (2007) 82–87.
[2] A. Olbryś, On a problem of T. Szostok concerning the Hermite-Hadamard inequalities, Journal of Mathematical Analysis and Applications 475 (1) (2019) 41–50.
[3] H. Chen, U. N. Katugampola, Hermite-Hadamard and Hermite-Hadamard–Fejér type inequalities for generalized fractional integrals, Journal of Mathematical Analysis and Applications 446 (2) (2017) 1274–1291.
[4] P. R. Mercer, Hadamard's inequality and Trapezoid Rules for the Riemann–Stieltjes integral, Journal of Mathematical Analysis and Applications 344 (2) (2008) 921–926.
[5] C. Li, The convolution of analytic functionals, The Journal of Analysis 27 (4) (2019) 1081–1094.
[6] M. Kian, On superquadratic functions of Hilbert space operators, The Journal of Analysis 27 (3) (2019) 663–671.
[7] L. Fejér, Uber die fourierreihen, ii, math, Naturwiss. Anz Ungar. Akad. Wiss 24 (1906) 369–390.
[8] S. Dragomir, R. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to Trapezoidal formula, Applied Mathematics Letters 11 (5) (1998) 91–95.
[9] M. Jleli, B. Samet, On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function, Journal of Nonlinear Sciences and Applications 9 (3) (2016) 1252–1260.
[10] K. Mehrez, P. Agarwal, New Hermite-Hadamard type integral inequalities for convex functions and their applications, Journal of Computational and Applied Mathematics 350 (2019) 274–285.
[11] B. G. Osgood, Lectures on the Fourier Transform and Its Applications, Vol. 33, American Mathematical Society, 2019.
[12] M. Dehghani, A. Zamani, Characterization of real inner product spaces by Hermite-Hadamard type orthogonalities, Journal of Mathematical Analysis and Applications 479 (1) (2019) 1364–1382.
[13] J. de la Cal, J. Cárcamo, L. Escauriaza, A general multidimensional Hermite-Hadamard type inequality, Journal of Mathematical Analysis and Applications 356 (2) (2009) 659–663.

[14] E. Set, İ. İşcan, M. Z. Sarikaya, M. E. Özdemir, On new inequalities of Hermite-Hadamard–Fejér type for convex functions via fractional integrals, Applied Mathematics and Computation 259 (2015) 875–881.
[15] G. Abbas, G. Farid, Hadamard and Fejér–Hadamard type inequalities for harmonically convex functions via generalized fractional integrals, The Journal of Analysis 25 (1) (2017) 107–119.
[16] E. Set, J. Choi, E. A. Alan, Hermite–Hadamard–Fejér type inequalities involving generalized fractional integral operators, The Journal of Analysis 27 (4) (2019) 1007–1027.
[17] M. M. Islam, A. R. Karim, An extension of the Hadamard-Type Inequality for a convex function defined on Modulus of complex integral functions, Journal of Mathematics Research 5 (3) (2013) 92.
[18] J. E. Nápoles Valdés, J. M. Rodríguez, J. M. Sigarreta, New Hermite–Hadamard Type Inequalities Involving Non-Conformable Integral Operators, Symmetry 11 (9) (2019) 1108.
[19] K. Mehrez, P. Agarwal, New Hermite–Hadamard type integral inequalities for convex functions and their applications, Journal of Computational and Applied Mathematics 350 (2019) 274–285.
[20] R. T. Rockafellar, Convex analysis, Vol. 28, Princeton university press, 1970.
[21] S. Wasowicz, Support-type properties of generalized convex functions, Journal of Mathematical Analysis and Applications 365 (1) (2010) 415–427.
[22] E. O. Tuck, On positivity of Fourier transforms, Bulletin of the Australian Mathematical Society 74 (1) (2006) 133–138.
[23] A. Iosevich, E. Liflyand, The Fourier transform of convex and oscillating functions, in: Decay of the Fourier Transform, Springer, 2014, pp. 47–91.
[24] F. Chen, Extensions of the Hermite-Hadamard inequality for harmonically convex functions via fractional integrals, Applied Mathematics and Computation 268 (2015) 121–128.
[25] B. Ahmad, A. Alsaedi, M. Kirane, B. T. Torebek, Hermite-Hadamard, Hermite-Hadamard–Fejér, Dragomir–Agarwal and pachpatte type inequalities for convex functions via new fractional integrals, Journal of Computational and Applied Mathematics 353 (2019) 120–129.