



Dedicated to Prof. Suthep Suantai on the occasion of his 60<sup>th</sup> anniversary

# On Magnifying Elements in Transformation Semigroups with a Fixed Point Set

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**Abstract** An element  $a$  of a semigroup  $S$  is called a left (right) magnifying element if there is a proper subset  $M$  of  $S$  such that  $aM = S$  ( $Ma = S$ ). For a fixed point set, a nonempty subset of a set  $X$ , we determine the conditions for functions in the transformation semigroups on  $X$  fixing all elements in the given fixed point set to be a left or right magnifying element as well as the existence of such elements.

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## 1. INTRODUCTION AND PRELIMINARIES

The notion of magnifying elements of a semigroup was originally purposed by Ljapin [1] in 1963. Recall that an element  $a$  of a semigroup  $S$  is called a left (right) magnifying element if there is a proper subset  $M$  of  $S$  such that  $aM = S$  ( $Ma = S$ ). Many authors studied the magnifying elements properties and the existence of such elements in various semigroups. For instance, Catino and Migliorini [2] gave the necessary and sufficient conditions of the existence of magnifying elements in simple, bisimple and regular semigroups in 1992. In 1994, Magill [3] investigated magnifying elements in some general transformation semigroups and applied the results to the semigroup of linear maps and the semigroup of all continuous selfmaps over the topological space. Moreover, the conditions for elements in any subsemigroup containing the identity of the full transformation semigroup to be left or right magnifying elements are provided. Let  $X$  be a nonempty set and  $T(X)$  denote the full transformation semigroups on  $X$ , which is the set of all functions from  $X$  into itself, i.e.,  $T(X) = \{\sigma : X \rightarrow X \mid \sigma \text{ is a function}\}$  and let  $P(X)$

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denote the partial transformation semigroups on  $X$  which is the set of all functions from all subsets of  $X$  to  $X$ , i.e.,  $P(X) = \{\sigma : A \rightarrow X \mid A \subseteq X \text{ and } \sigma \text{ is a function}\}$ . Recently, Luangchaisri, Changphas and Phanlert [4] characterized magnifying elements in  $P(X)$  by extending the Magill's results [3]. The magnifying elements in the generalizations of the full and partial transformation semigroups have been studied extensively over the last three years not only for their interesting properties, but also the necessary and sufficient conditions for elements to be left or right magnifying elements and the existence of such elements in these semigroups, e.g., [5], [6], [7] and [8].

For any functions  $\sigma, \tau$  and  $x \in X$ , the notations  $x\sigma$  and  $x\sigma\tau$  are used instead of  $\sigma(x)$  and  $(\tau \circ \sigma)(x)$ , respectively. The range of  $\sigma$  is denoted by  $\text{ran } \sigma$ . Let  $\mathcal{S}$  be a nonempty subset of  $X$ . A function  $\sigma$  on  $X$  is called a function fixing  $\mathcal{S}$  pointwise if  $x\sigma = x$  for every  $x \in \mathcal{S}$ . Then, the set  $\mathcal{S}$  is called a fixed point set (or a set of fixed points) of  $\sigma$  in  $T(X)$ . Fixed point theory is interesting and useful in a wide range of aspects, such as Nash equilibrium in theoretical economics, minimax theorem in game theory and fixed point properties in digital topology. As we can see from the late 19th century until now, there are many publications of the studies of fixed points of various functions such as commuting functions, continuous functions, non-continuous functions, compatible functions, Lipschitz function, and nonexpansive functions. Ljapin showed that one can study fixed point properties of transformations from an abstract point of view in [9], [10] and ([1], Theorem 5.3, p.66). In 2013, the Green's relations and ideals of  $T(X)$  with the fixed point set of  $X$  are determined by Honyam and Sanwong [11].

In this paper, for a nonempty subset  $\mathcal{S}$  of  $X$ , we let  $T_{\mathcal{F}(\mathcal{S})}(X)$  and  $P_{\mathcal{F}(\mathcal{S})}(X)$  be the set of the full and partial transformation semigroup on  $X$  fixing  $\mathcal{S}$  pointwise, respectively, i.e.,

$$T_{\mathcal{F}(\mathcal{S})}(X) = \{\sigma \in T(X) \mid x\sigma = x \text{ for all } x \in \mathcal{S}\}$$

and

$$P_{\mathcal{F}(\mathcal{S})}(X) = \{\sigma \in P(X) \mid x\sigma = x \text{ for all } x \in \mathcal{S} \cap \text{dom } \sigma\}.$$

The sets  $T_{\mathcal{F}(\mathcal{S})}(X)$  and  $P_{\mathcal{F}(\mathcal{S})}(X)$  are semigroups under the composition of functions, which are called the full and partial transformation semigroups with a fixed point set, respectively. The aims of this paper are to study the properties of magnifying elements in  $T_{\mathcal{F}(\mathcal{S})}(X)$  and  $P_{\mathcal{F}(\mathcal{S})}(X)$  and to provide the necessary and sufficient conditions for elements in these semigroups to be a left or right magnifying element. Moreover, the conditions of the existence of magnifying elements in the semigroups  $T_{\mathcal{F}(\mathcal{S})}(X)$  and  $P_{\mathcal{F}(\mathcal{S})}(X)$  are established as well.

## 2. MAGNIFYING ELEMENTS IN $T_{\mathcal{F}(\mathcal{S})}(X)$

In this section, we consider left and right magnifying elements in  $T_{\mathcal{F}(\mathcal{S})}(X)$ . Recall that a function  $\sigma$  of the semigroup  $T_{\mathcal{F}(\mathcal{S})}(X)$  is called a left (right) magnifying element if  $\sigma M = T_{\mathcal{F}(\mathcal{S})}(X)$  ( $M\sigma = T_{\mathcal{F}(\mathcal{S})}(X)$ ) for some proper subset  $M$  of  $T_{\mathcal{F}(\mathcal{S})}(X)$ .

### 2.1. LEFT MAGNIFYING ELEMENTS IN $T_{\mathcal{F}(\mathcal{S})}(X)$

**Lemma 2.1.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element, then  $\sigma$  is one-to-one.*

*Proof.* Assume that  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element. Then, there exists a proper subset  $M$  of  $T_{\mathcal{F}(\mathcal{S})}(X)$  such that  $\sigma M = T_{\mathcal{F}(\mathcal{S})}(X)$ . Note that the identity map  $id_X$

on  $X$  belongs to  $T_{\mathcal{F}(\mathcal{S})}(X)$ . Accordingly, there exists  $\tau \in M$  such that  $\sigma\tau = id_X$ . This implies that  $\sigma$  is one-to-one. ■

**Lemma 2.2.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is bijective, then  $\sigma$  is not a left magnifying element.*

*Proof.* Assume that  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is bijective. So,  $\sigma^{-1}$  is also bijective. Suppose to the contrary that  $\sigma$  is a left magnifying element. Then, there exists a proper subset  $M$  of  $T_{\mathcal{F}(\mathcal{S})}(X)$  such that  $\sigma M = T_{\mathcal{F}(\mathcal{S})}(X)$ . Thus, we have  $\sigma M \subseteq \sigma T_{\mathcal{F}(\mathcal{S})}(X)$  and  $\sigma T_{\mathcal{F}(\mathcal{S})}(X) \subseteq T_{\mathcal{F}(\mathcal{S})}(X) = \sigma M$ . This shows that  $\sigma M = \sigma T_{\mathcal{F}(\mathcal{S})}(X)$ . Hence,  $M = \sigma^{-1}\sigma M = \sigma^{-1}\sigma T_{\mathcal{F}(\mathcal{S})}(X) = T_{\mathcal{F}(\mathcal{S})}(X)$ , which is a contradiction. Therefore,  $\sigma$  is not a left magnifying element. ■

By Lemma 2.1 and Lemma 2.2, we obtain the following corollary.

**Corollary 2.3.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element, then  $\sigma$  is one-to-one but not onto.*

**Lemma 2.4.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is one-to-one but not onto, then  $\sigma$  is a left magnifying element.*

*Proof.* Assume that a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is one-to-one but not onto. Let  $M = \{\tau \in T_{\mathcal{F}(\mathcal{S})}(X) \mid \tau \text{ is not one-to-one}\}$ . Then,  $M$  is a proper subset of  $T_{\mathcal{F}(\mathcal{S})}(X)$  as  $id_X \in T_{\mathcal{F}(\mathcal{S})}(X)$  does not belong to  $M$ . Since  $\sigma$  fixes  $\mathcal{S}$  pointwise,  $S \subseteq \text{ran } \sigma$ . For each  $x \in \text{ran } \sigma$ , there is an element  $y_x \in X$  such that  $y_x\sigma = x$ . Note that if  $x \in \mathcal{S}$ , then we must choose  $y_x = x$ . Let  $s \in \mathcal{S}$  and  $\varsigma$  be any function in  $T_{\mathcal{F}(\mathcal{S})}(X)$ . Define a function  $\tau \in T_{\mathcal{F}(\mathcal{S})}(X)$  by

$$x\tau = \begin{cases} y_x\varsigma & \text{if } x \in \text{ran } \sigma, \\ s & \text{otherwise.} \end{cases}$$

Since the function  $\sigma$  is not onto and  $S \subseteq \text{ran } \sigma$ , then there is an element  $z \in X \setminus \text{ran } \sigma$  and hence  $z\tau = s = s\tau$ . This shows that  $\tau$  is not one-to-one, and hence  $\tau \in M$ . For each  $x \in X$ ,  $x\sigma\tau = y_x\sigma\varsigma = x\varsigma$  since  $y_x\sigma\sigma = x\sigma$  and  $\sigma$  is one-to-one. Therefore,  $\sigma$  is a left magnifying element. ■

**Theorem 2.5.** *A function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element if and only if  $\sigma$  is one-to-one but not onto.*

*Proof.* It follows by Corollary 2.3 and Lemma 2.4. ■

**Example 2.6.** Let  $X = \mathbb{N}$  and let  $\mathcal{S} = 2\mathbb{N}$ . Let  $\sigma$  be a function in  $T_{\mathcal{F}(\mathcal{S})}(X)$ , which is defined by

$$x\sigma = \begin{cases} x + 6 & \text{if } x \text{ is odd,} \\ x & \text{if } x \text{ is even.} \end{cases}$$

The function  $\sigma$  is one-to-one but not onto. By Theorem 2.5,  $\sigma$  is a left magnifying element. Let  $M = \{\tau \in T_{\mathcal{F}(\mathcal{S})}(X) \mid \tau \text{ is not one-to-one}\}$  and consider the function  $\varsigma \in T_{\mathcal{F}(\mathcal{S})}(X)$ , which is defined by  $x\varsigma = x$  for all  $x \in \mathcal{S}$  and for all positive integers in  $X \setminus \mathcal{S}$ ,  $x\varsigma = x + 4$  if  $x \leq 7$  and  $x\varsigma = x - 3$  if  $x > 7$ . Choose  $2 \in \mathcal{S}$  and define a function  $\tau \in T_{\mathcal{F}(\mathcal{S})}(X)$  by  $x\tau = x$  for all  $x \in \mathcal{S}$  and for all positive integers in  $X \setminus \mathcal{S}$ ,

$$x\tau = \begin{cases} x - 2 & \text{if } 7 \leq x \leq 13, \\ x - 9 & \text{if } x \geq 15, \\ 2 & \text{otherwise.} \end{cases}$$

We see that  $\tau$  is not one-to-one, and hence  $\tau \in M$ . For all  $x \in X$ , we obtain  $x\sigma\tau = x\varsigma$ , which implies  $\sigma\tau = \varsigma$ .

Next, the existence of left magnifying elements in  $T_{\mathcal{F}(\mathcal{S})}(X)$  is established.

**Lemma 2.7.** *If  $X \setminus \mathcal{S}$  is finite, then there exists no left magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$ .*

*Proof.* Suppose to the contrary that there is a left magnifying element  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$ . By Lemma 2.1,  $\sigma$  is one-to-one. Since  $\sigma$  fixes  $\mathcal{S}$  pointwise,  $\sigma|_{\mathcal{S}}$  is onto  $\mathcal{S}$  and hence  $\sigma|_{\mathcal{S}}$  is bijective on  $\mathcal{S}$ . Since  $X \setminus \mathcal{S}$  is finite and  $\sigma$  is one-to-one,  $\sigma|_{X \setminus \mathcal{S}}$  is onto  $X \setminus \mathcal{S}$  and hence  $\sigma|_{X \setminus \mathcal{S}}$  is bijective on  $X \setminus \mathcal{S}$ . Thus,  $\sigma$  is bijective, which is a contradiction, by Lemma 2.2. Therefore, there exists no left magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$ . ■

**Lemma 2.8.** *If  $X \setminus \mathcal{S}$  is infinite, then there exists a left magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$ .*

*Proof.* Assume that  $X \setminus \mathcal{S}$  is an infinite set. Then, there is a proper subset  $A$  of  $X \setminus \mathcal{S}$  such that  $|A| = |X \setminus \mathcal{S}|$ . So, there is a bijection  $\delta$  from  $X \setminus \mathcal{S}$  to  $A$ . Let  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  be defined by

$$x\sigma = \begin{cases} x\delta & \text{if } x \in X \setminus \mathcal{S}, \\ x & \text{otherwise.} \end{cases}$$

We see that  $\text{ran } \sigma = \mathcal{S} \cup A \neq X$ . Then,  $\sigma$  is one-to-one but not onto. By Theorem 2.5, the function  $\sigma$  is a left magnifying element. ■

**Theorem 2.9.** *There exists a left magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$  if and only if  $X \setminus \mathcal{S}$  is infinite.*

*Proof.* It follows by Lemma 2.7 and Lemma 2.8. ■

## 2.2. RIGHT MAGNIFYING ELEMENTS IN $T_{\mathcal{F}(\mathcal{S})}(X)$

**Lemma 2.10.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is a right magnifying element, then  $\sigma$  is onto.*

*Proof.* Assume that  $\sigma$  is a right magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$ . Then, there exists a proper subset  $M$  of  $T_{\mathcal{F}(\mathcal{S})}(X)$  such that  $M\sigma = T_{\mathcal{F}(\mathcal{S})}(X)$ . Note that the identity map  $id_X$  on  $X$  belongs to  $T_{\mathcal{F}(\mathcal{S})}(X)$ . Accordingly, there exists  $\tau \in M$  such that  $\tau\sigma = id_X$ . This implies that  $\sigma$  is onto. ■

**Lemma 2.11.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is bijective, then  $\sigma$  is not a right magnifying element.*

*Proof.* Assume that  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is bijective. So,  $\sigma^{-1}$  is also bijective. Suppose to the contrary that  $\sigma$  is a right magnifying element. Then, there exists a proper subset  $M$  of  $T_{\mathcal{F}(\mathcal{S})}(X)$  such that  $M\sigma = T_{\mathcal{F}(\mathcal{S})}(X)$ . Thus, we have  $M\sigma \subseteq T_{\mathcal{F}(\mathcal{S})}(X)\sigma$  and  $T_{\mathcal{F}(\mathcal{S})}(X)\sigma \subseteq T_{\mathcal{F}(\mathcal{S})}(X) = M\sigma$ . This shows that  $M\sigma = T_{\mathcal{F}(\mathcal{S})}(X)\sigma$ . Hence,  $M = M\sigma\sigma^{-1} = T_{\mathcal{F}(\mathcal{S})}(X)\sigma\sigma^{-1} = T_{\mathcal{F}(\mathcal{S})}(X)$ , which is a contradiction. Therefore,  $\sigma$  is not a right magnifying element. ■

By Lemma 2.10 and Lemma 2.11, we obtain the following corollary.

**Corollary 2.12.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is a right magnifying element, then  $\sigma$  is onto but not one-to-one.*

**Lemma 2.13.** *If a function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is onto but not one-to-one, then  $\sigma$  is a right magnifying element.*

*Proof.* Assume that  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is onto but not one-to-one. Let  $M = \{\tau \in T_{\mathcal{F}(\mathcal{S})}(X) \mid \tau \text{ is not onto}\}$ . Then,  $M$  is a proper subset of  $T_{\mathcal{F}(\mathcal{S})}(X)$  since the identity map  $id_X \in T_{\mathcal{F}(\mathcal{S})}(X)$  does not belong to  $M$ . Let  $\varsigma$  be any function in  $T_{\mathcal{F}(\mathcal{S})}(X)$ . Since  $\sigma$  is onto, for each  $x \in X$  there exists  $y_x \in X$  such that  $y_x\sigma = x\varsigma$  (if  $a\varsigma = b\varsigma$ , then choose  $y_a = y_b$ ). Define  $\tau \in T_{\mathcal{F}(\mathcal{S})}(X)$  by  $x\tau = y_x$  for all  $x \in X$ . Since  $\sigma$  is not one-to-one, there are distinct elements  $x, y \in X$  such that  $x\sigma = y\sigma$ . Then, at least one of  $x$  and  $y$  does not belong to  $\text{ran } \tau$ , and hence  $\tau$  is not onto. Thus,  $\tau \in M$ . For all  $x \in X$ , we see that  $x\tau\sigma = y_x\sigma = x\varsigma$ . We then get  $\tau\sigma = \varsigma$ , which implies  $M\sigma = T_{\mathcal{F}(\mathcal{S})}(X)$ . Therefore,  $\sigma$  is a right magnifying element. ■

**Theorem 2.14.** *A function  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  is a right magnifying element if and only if  $\sigma$  is onto but not one-to-one.*

*Proof.* It follows by Corollary 2.12 and Lemma 2.13. ■

**Example 2.15.** Let  $X = \mathbb{Z}$  and  $\mathcal{S} = \mathbb{Z}^-$ . Let  $\sigma$  be a function in  $T_{\mathcal{F}(\mathcal{S})}(X)$ , which is defined by

$$x\sigma = \begin{cases} x & \text{if } x < 0, \\ x - 1 & \text{if } 0 \leq x \leq 2, \\ x - 3 & \text{if } x \geq 2. \end{cases}$$

The function  $\sigma$  is onto but not one-to-one. By Theorem 2.14,  $\sigma$  is a right magnifying element. Let  $M = \{\tau \in T_{\mathcal{F}(\mathcal{S})}(X) \mid \tau \text{ is not onto}\}$  and consider the function  $\varsigma \in T_{\mathcal{F}(\mathcal{S})}(X)$ , which is defined by  $x\varsigma = x$  for all  $x \in \mathbb{Z}^- \cup \{0\}$  and  $x\varsigma = -x$  if  $x \in \mathbb{Z}^+$ . Define a function  $\tau \in F_{\mathcal{S}}(T(X))$  by

$$x\tau = \begin{cases} x & \text{if } x < 0, \\ -x & \text{if } x > 0, \\ 3 & \text{if } x = 0. \end{cases}$$

We see that  $\tau$  is not onto, and hence  $\tau \in M$ . For all  $x \in X$ , we obtain  $x\tau\sigma = x\varsigma$ , which implies  $\tau\sigma = \varsigma$ . Note that  $x\varsigma = (-x)\varsigma$  for all  $x \in X$ . Therefore, we choose  $y_{-x} = y_x$

Next, the existence of right magnifying elements in  $T_{\mathcal{F}(\mathcal{S})}(X)$  is established.

**Lemma 2.16.** *If  $X \setminus \mathcal{S}$  is finite, then there exists no right magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$ .*

*Proof.* Suppose to the contrary that there is a right magnifying element  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$ . By Lemma 2.10,  $\sigma$  is onto. Since  $\sigma$  is onto and  $\sigma$  fixes  $\mathcal{S}$  pointwise,  $\mathcal{S}\sigma = \mathcal{S}$  and  $(X \setminus \mathcal{S})\sigma^{-1} \subseteq X \setminus \mathcal{S}$ . Since  $X \setminus \mathcal{S}$  is finite,  $|(X \setminus \mathcal{S})\sigma^{-1}| \leq |X \setminus \mathcal{S}|$ . On the other hand, the fact that  $\sigma$  is onto implies that  $|(X \setminus \mathcal{S})\sigma^{-1}| \geq |X \setminus \mathcal{S}|$ . We obtain  $(X \setminus \mathcal{S})\sigma^{-1} = X \setminus \mathcal{S}$ , and hence  $(X \setminus \mathcal{S})\sigma = X \setminus \mathcal{S}$ . Thus,  $\sigma|_{X \setminus \mathcal{S}}$  is bijective on  $X \setminus \mathcal{S}$ . This implies that  $\sigma|_{\mathcal{S}}$  is bijective on  $\mathcal{S}$ . Thus,  $\sigma$  is bijective, which is a contradiction, by Lemma 2.11. Therefore, there exists no right magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$ . ■

**Lemma 2.17.** *If  $X \setminus \mathcal{S}$  is infinite, then there exists a right magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$ .*

*Proof.* Assume that  $X \setminus \mathcal{S}$  is an infinite set. Let  $s \in \mathcal{S}$ . Then, there is a bijection  $\delta$  from  $X \setminus \mathcal{S}$  to  $(X \setminus \mathcal{S}) \cup \{s\}$ . Let  $\sigma \in T_{\mathcal{F}(\mathcal{S})}(X)$  be defined by

$$x\sigma = \begin{cases} x\delta & \text{if } x \in X \setminus \mathcal{S}, \\ x & \text{otherwise.} \end{cases}$$

Then,  $\sigma$  is onto but not one-to-one because there is an element  $x \in X \setminus \mathcal{S}$  such that  $x\sigma = s = s\sigma$ . By Theorem 2.14, the function  $\sigma$  is a right magnifying element. ■

**Theorem 2.18.** *There exists a right magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$  if and only if  $X \setminus \mathcal{S}$  is infinite.*

*Proof.* It follows by Lemma 2.16 and Lemma 2.17. ■

### 3. MAGNIFYING ELEMENTS IN $P_{\mathcal{F}(\mathcal{S})}(X)$

In this section, we consider left and right magnifying elements in  $P_{\mathcal{F}(\mathcal{S})}(X)$ . Recall that  $P_{\mathcal{F}(\mathcal{S})}(X) = \{\sigma \in P(X) \mid x\sigma = x \text{ for all } x \in \mathcal{S} \cap \text{dom } \sigma\}$  where  $\mathcal{S}$  is a nonempty subset of  $X$ . A function  $\sigma$  of the semigroup  $P_{\mathcal{F}(\mathcal{S})}(X)$  is called a left (right) magnifying element if  $\sigma M = P_{\mathcal{F}(\mathcal{S})}(X)$  ( $M\sigma = P_{\mathcal{F}(\mathcal{S})}(X)$ ) for some proper subset  $M$  of  $P_{\mathcal{F}(\mathcal{S})}(X)$ .

#### 3.1. LEFT MAGNIFYING ELEMENTS IN $P_{\mathcal{F}(\mathcal{S})}(X)$

**Lemma 3.1.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element, then  $\sigma$  is one-to-one and  $\text{dom } \sigma = X$ .*

*Proof.* Assume that  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element. Then, there exists a proper subset  $M$  of  $P_{\mathcal{F}(\mathcal{S})}(X)$  such that  $\sigma M = P_{\mathcal{F}(\mathcal{S})}(X)$ . Note that the identity map  $id_X$  on  $X$  belongs to  $P_{\mathcal{F}(\mathcal{S})}(X)$ . Accordingly, there exists  $\tau \in M$  such that  $\sigma\tau = id_X$ . This implies that  $\sigma$  is one-to-one and  $\text{dom } \sigma = X$ . ■

**Lemma 3.2.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is bijective and  $\text{dom } \sigma = X$ , then  $\sigma$  is not a left magnifying element.*

*Proof.* This follows by the same method as in Lemma 2.2. ■

By Lemma 3.1 and Lemma 3.2, we obtain the following corollary.

**Corollary 3.3.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element, then  $\sigma$  is one-to-one but not onto and  $\text{dom } \sigma = X$ .*

**Lemma 3.4.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is one-to-one but not onto and  $\text{dom } \sigma = X$ , then  $\sigma$  is a left magnifying element.*

*Proof.* This follows by the same method as in Lemma 2.4. ■

**Theorem 3.5.** *A function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is a left magnifying element if and only if  $\sigma$  is one-to-one but not onto and  $\text{dom } \sigma = X$ .*

*Proof.* It follows from Corollary 3.3 and Lemma 3.4. ■

**Corollary 3.6.** *A function  $\sigma$  is a left magnifying element in  $T_{\mathcal{F}(\mathcal{S})}(X)$  if and only if  $\sigma$  is a left magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$ .*

*Proof.* It follows from Theorem 2.5 and Theorem 3.5. ■

Next, the existence of left magnifying elements in  $P_{\mathcal{F}(\mathcal{S})}(X)$  can be established instantaneously.

**Theorem 3.7.** *There exists a left magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$  if and only if  $X \setminus \mathcal{S}$  is infinite.*

### 3.2. RIGHT MAGNIFYING ELEMENTS IN $P_{\mathcal{F}(\mathcal{S})}(X)$

**Lemma 3.8.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is a right magnifying element, then  $\sigma$  is onto.*

*Proof.* Assume that  $\sigma$  is a right magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$ . Then, there exists a proper subset  $M$  of  $P_{\mathcal{F}(\mathcal{S})}(X)$  such that  $M\sigma = P_{\mathcal{F}(\mathcal{S})}(X)$ . Note that the identity map  $id_X$  on  $X$  belongs to  $P_{\mathcal{F}(\mathcal{S})}(X)$ . Accordingly, there exists  $\tau \in M$  such that  $\tau\sigma = id_X$ . This implies that  $\sigma$  is onto. ■

**Lemma 3.9.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is bijective and  $dom \sigma = X$ , then  $\sigma$  is not a right magnifying element.*

*Proof.* This follows by the same method as in Lemma 2.11. ■

By Lemma 3.8 and Lemma 3.9, we obtain the following corollary.

**Corollary 3.10.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is a right magnifying element and  $dom \sigma = X$ , then  $\sigma$  is onto but not one-to-one.*

**Lemma 3.11.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is onto but not one-to-one and  $dom \sigma = X$ , then  $\sigma$  is a right magnifying element.*

*Proof.* This follows by the same method as in Lemma 2.13. ■

**Lemma 3.12.** *If a function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is onto and  $dom \sigma \neq X$ , then  $\sigma$  is a right magnifying element.*

*Proof.* Assume that  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is onto and  $dom \sigma \neq X$ . Let  $M = \{\tau \in P_{\mathcal{F}(\mathcal{S})}(X) \mid \tau \text{ is not onto}\}$ . Then,  $M$  is a proper subset of  $P_{\mathcal{F}(\mathcal{S})}(X)$  since the identity map  $id_X$  on  $X$  does not belong to  $M$ . Let  $\varsigma$  be any function in  $P_{\mathcal{F}(\mathcal{S})}(X)$ . Since  $\sigma$  is onto, for each  $x \in dom \varsigma$ , there exists  $y_x \in dom \sigma$  such that  $y_x\sigma = x\varsigma$ . Define a function  $\tau \in P_{\mathcal{F}(\mathcal{S})}(X)$  by  $x\tau = y_x$  for all  $x \in dom \varsigma$ . Since  $ran \tau \subseteq dom \sigma \neq X$ ,  $\tau$  is not onto. Then,  $\tau \in M$ . For all  $x \in dom \varsigma$ ,  $x\tau\sigma = y_x\sigma = x\varsigma$ . This shows that  $\tau\sigma = \varsigma$ , which implies  $M\sigma = P_{\mathcal{F}(\mathcal{S})}(X)$ . Therefore,  $\sigma$  is a right magnifying element. ■

**Theorem 3.13.** *A function  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  is a right magnifying element if and only if  $\sigma$  is onto and either*

- (1)  $dom \sigma \neq X$  or
- (2)  $dom \sigma = X$  and  $\sigma$  is not one-to-one.

*Proof.* It follows by Corollary 3.10, Lemma 3.11 and Lemma 3.12. ■

**Example 3.14.** Let  $X = \mathbb{Z}^+$  and  $\mathcal{S} = 2\mathbb{Z}^+$ . Let  $\sigma$  be a function in  $P_{\mathcal{F}(\mathcal{S})}(X)$ , which is defined by

$$x\sigma = \begin{cases} x & \text{if } x \in 2\mathbb{Z}^+, \\ i & \text{if } x \in \{p_i \mid p_i \text{ is the } i^{th} \text{ odd prime number}\}. \end{cases}$$

The function  $\sigma$  is onto and  $dom \sigma \neq X$ . By Theorem 3.13,  $\sigma$  is a right magnifying element. Let  $M = \{\tau \in P_{\mathcal{F}(\mathcal{S})}(X) \mid \tau \text{ is not onto}\}$  and consider the function  $\varsigma \in P_{\mathcal{F}(\mathcal{S})}(X)$ , which

is defined by  $x_\zeta = x$  for all  $x \in 2\mathbb{Z}^+$  and  $x_\zeta = x - 2$  if  $x$  is odd and  $x \equiv 0 \pmod 3$ . Define a function  $\tau \in P_{\mathcal{F}(\mathcal{S})}(X)$  by  $x\tau = x$  for all  $x \in 2\mathbb{Z}^+$  and  $x\tau = p_{x_\zeta}$  if  $x$  is odd and  $x \equiv 0 \pmod 3$ . Then,  $\tau \in M$ . For all  $x \in \text{dom } \zeta$ ,  $x\tau\sigma = x_\zeta$ . Hence,  $\tau\sigma = \zeta$ . Note that  $\{3, 5, 7, 11, 13, 17, 19, \dots\}$  is a sequence of odd prime numbers. Hence,  $p_1 = 3$  and  $p_7 = 19$ . We then obtain  $3\sigma = 1$  and  $19\sigma = 7$ . Consider  $3\zeta = 1$  and  $9\zeta = 7$ , we must define  $3\tau = p_{3\zeta} = p_1 = 3$  and  $9\tau = p_{9\zeta} = p_7 = 19$ .

Next, the existence of right magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$  is established.

**Lemma 3.15.** *If  $X \setminus \mathcal{S}$  is finite, then there exists no right magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$ .*

*Proof.* Suppose to the contrary that there is a right magnifying element  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$ . By Lemma 3.8,  $\sigma$  is onto. Since  $\sigma$  is onto and fixes  $\mathcal{S}$  pointwise,  $(X \setminus \mathcal{S})\sigma^{-1} \subseteq X \setminus \mathcal{S}$ . Since  $X \setminus \mathcal{S}$  is finite,  $|(X \setminus \mathcal{S})\sigma^{-1}| \leq |X \setminus \mathcal{S}|$ . On the other hand, the fact that  $\sigma$  is onto implies that  $|(X \setminus \mathcal{S})\sigma^{-1}| \geq |X \setminus \mathcal{S}|$ . Then,  $(X \setminus \mathcal{S})\sigma^{-1} = X \setminus \mathcal{S}$ . We obtain  $X \setminus \mathcal{S} \subseteq \text{dom } \sigma$  and  $(X \setminus \mathcal{S})\sigma = X \setminus \mathcal{S}$ . Thus,  $\sigma|_{X \setminus \mathcal{S}}$  is bijective on  $X \setminus \mathcal{S}$ . So,  $\mathcal{S}\sigma^{-1} \subseteq \mathcal{S}$ . Since  $\sigma$  is onto and  $\sigma$  fixes  $\mathcal{S}$  pointwise,  $x\sigma^{-1} = \{x\}$  for all  $x \in \mathcal{S}$ . Then,  $\mathcal{S} \subseteq \text{dom } \sigma$  and hence  $\mathcal{S}\sigma = \mathcal{S}$ . So,  $\sigma|_{\mathcal{S}}$  is bijective on  $\mathcal{S}$ . Then,  $\text{dom } \sigma = X$  and  $\sigma$  is bijective, which is a contradiction, by Theorem 3.13. Therefore, there exists no right magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$ . ■

**Lemma 3.16.** *If  $X \setminus \mathcal{S}$  is infinite, then there exists a right magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$ .*

*Proof.* Assume that  $X \setminus \mathcal{S}$  is an infinite set. Let  $s \in \mathcal{S}$ . Then, there is a bijection  $\delta$  from  $X \setminus \mathcal{S}$  to  $(X \setminus \mathcal{S}) \cup \{s\}$ . Let  $\sigma \in P_{\mathcal{F}(\mathcal{S})}(X)$  be defined by

$$x\sigma = \begin{cases} x\delta & \text{if } x \in X \setminus \mathcal{S}, \\ x & \text{otherwise.} \end{cases}$$

We see that  $\text{dom } \sigma = X$  and  $\sigma$  is onto but not one-to-one since there is an element  $x \in X \setminus \mathcal{S}$  such that  $x\sigma = s = s\sigma$ . By Theorem 3.13, the function  $\sigma$  is a right magnifying element. ■

**Theorem 3.17.** *There exists a right magnifying element in  $P_{\mathcal{F}(\mathcal{S})}(X)$  if and only if  $X \setminus \mathcal{S}$  is infinite.*

*Proof.* It follows by Lemma 3.15 and Lemma 3.16. ■

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