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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Hyers-Ulam Stability of the Additive s-Functional Inequality and Hom-Derivations in Banach Algebras

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Abstract In this work, we solve the following additive s-functional inequality:

$$\|f(x+y)-f(x)-f(y)\| \leq \|s(f(x-y)-f(x)-f(-y))\|, \tag{0.1}$$

where s is a fixed nonzero complex number with $|s|<1$. We prove the HyersUlam stability of the additive s-functional inequality (0.1) in complex Banach spaces by using the fixed point method and the direct method. Moreover, we prove the HyersUlam stability of hom-derivations in complex Banach algebras.

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1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. These question form is the object of the stability theory. If the answers is affirmative, we say that the functional equation for homomorphism is stable. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. Park [6, 7] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and

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non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [8–15]).

Applications of stability theory of functional equations for the proof of new fixed point theorems with applications were the first to furnished by Isac and Rassias [16] in 1996. The stability problems of several functional equations by using fixed-point methods have been extensively investigated by a number of authors, see [17–21].

Recently, Park et al. [22] solved the additive s-functional inequality (0.1) in the case that f is an odd function. They proved the Hyers–Ulam stability of the additive s-functional inequality (0.1) in complex Banach spaces by using the fixed point method and the direct method. Also, they presented the Hyers–Ulam stability of hom-derivations in complex Banach algebras.

To obtain the desired results, the following definition is needed to be used later.

Definition 1.1. [22, Definition 1.1] Let A be a complex Banach algebra and $G : A \to A$ be a homomorphism. A $\mathbb{C}$-linear mapping $F : A \to A$ is called a *hom-derivation* on A if F satisfies

$$F(xy) = F(x)G(y) + F(x)G(y)$$

for all $x, y \in A$.

Now, to accomplish the purpose of this work, some important tools are needed as follows.

Theorem 1.2. [23, 24] *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

Lemma 1.3. [22, Lemma 2.1] *If a mapping $f : X \to Y$ satisfies (0.1) for all $x, y \in X$, then $f : X \to Y$ is additive.*

With preceding inspirational research, we solve the additive s-functional inequality (0.1) where the function f does not have to be an odd function. This paper is organized as follows: In Sections 2 and 3, we solve the additive s-functional inequality (0.1) and prove the HyersUlam stability of the additive s-functional inequality (0.1) in Banach spaces using the direct method and using the fixed point method. In Sections 4 and 5, using the direct method and using the fixed point method, we prove the HyersUlam stability of hom-derivations in Banach algebras, associated with the additive s-functional inequality (0.1).

Throughout this work, let X be a complex normed space and Y a complex Banach space. Assume that A is a complex Banach algebra and s is a fixed nonzero complex number with $|s| < 1$.

2. Additive s-Functional Inequality (0.1): a Direct Method

We now prove the HyersUlam stability of the additive s-functional inequality (0.1) in complex Banach spaces using the direct method.

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.1}$$

for all $x, y \in X$ and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \le \|s(f(x-y) - f(x) - f(-y))\| + \varphi(x,y) \tag{2.2}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \le \frac{1}{2(1-|s|^2)} [\Psi(x,x) + |s|\Psi(x,-x)] \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.2), we get

$$\|f(2x) - 2f(x)\| \le \|s(-f(x) - f(-x))\| + \varphi(x,x) \tag{2.4}$$

for all $x \in X$. Replacing y by $-x$ in (2.2), we also get

$$\| - f(x) - f(-x)\| \le \|s(f(2x) - 2f(x))\| + \varphi(x,-x) \tag{2.5}$$

for all $x \in X$. It follows from (2.4) and (2.5) that

$$\|f(2x) - 2f(x)\| \le \frac{1}{1-|s|^2} [\varphi(x,x) + |s|\varphi(x,-x)] \tag{2.6}$$

and so

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{1}{1-|s|^2} \left[\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + |s|\varphi\left(\frac{x}{2}, -\frac{x}{2}\right)\right]$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{1}{2(1-|s|^2)} \sum_{j=l+1}^{m} 2^j \left[\varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right) + |s|\varphi\left(\frac{x}{2^j}, -\frac{x}{2^j}\right)\right] \end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $P : X \to Y$ by

$$P(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (2.3). It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\|P(x+y) - P(x) - P(y)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - P\left(\frac{y}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \left\| s\left(f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - P\left(-\frac{y}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= \|s\left(P(x-y) - P(x) - P(-y)\right)\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 1.3, the mapping $P : X \to Y$ is additive. Now, let $T : X \to Y$ be another additive mapping satisfying (2.3). Then we have

$$\begin{aligned}
\|P(x) - T(x)\| &= \left\|2^q P\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \left\|2^q P\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\le \frac{2^q}{1-|s|^2}\left[\Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right) + |s|\Psi\left(\frac{x}{2^q}, -\frac{x}{2^q}\right)\right],
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x, z \in X$. So we can conclude that $P(x) = T(x)$ for all $x \in X$. This proves the uniqueness of P, as desired. ∎

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (2.1) and $\varphi(x, y) = (x, -y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then, there exists a unique additive mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\| \le \frac{1}{2(1-|s|)}\Psi(x, x) \tag{2.8}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.1. ∎

Theorem 2.3. *Let $\phi : X \to [0, \infty)$ be a function satisfying*

$$\Phi(x) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j}\right) < \infty \tag{2.9}$$

for all $x \in X$ and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \le \|s(f(x-y) - f(x) - f(-y))\| + \phi(x) + \phi(y) \tag{2.10}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \le \frac{1}{2(1-|s|^2)}\left[(2+|s|)\Phi(x) + |s|\Phi(-x)\right] \tag{2.11}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.1. ∎

Theorem 2.4. *Let $\phi : X \to [0, \infty)$ be an even function satisfying (2.9) and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10) Then, there exists a unique additive mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\| \le \frac{1}{1-|s|}\Phi(x) \tag{2.12}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.1. ∎

Corollary 2.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|f(x+y) - f(x) - f(y)\| \le \|s(f(x-y) - f(x) - f(-y))\| + \theta\left(\|x\|^r + \|y\|^r\right) \quad (2.13)$$

for all $x, y \in X$. Then, there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \le \frac{2\theta}{(1-|s|)(2^r-2)}\|x\|^r \quad (2.14)$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) = \theta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in X$ or Theorem 2.4 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in X$. ∎

Theorem 2.6. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty \quad (2.15)$$

for all $x, y \in X$ and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \le \frac{1}{2(1-|s|^2)}\left[\Psi(x,x) + |s|\Psi(x,-x)\right] \quad (2.16)$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\|\frac{1}{2}f(2x) - f(x)\right\| \le \frac{1}{2(1-|s|^2)}\left[\varphi(x,x) + |s|\varphi(x,-x)\right]$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \frac{1}{1-|s|^2}\sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\left[\varphi(2^j x, 2^j x) + |s|\varphi(2^j x, -2^j x)\right] \end{aligned} \quad (2.17)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.17) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $P : X \to Y$ by

$$P(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.17), we get (2.16). The rest of the proof is similar to the proof of Theorem 2.1. ∎

Theorem 2.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying (2.15) and $\varphi(x,y) = (x,-y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then, there exists a unique additive mapping $P : X \to Y$ satisfying (2.8).*

Proof. The proof is similar to the proof of Theorem 2.6. ∎

Theorem 2.8. *Let $\phi : X \to [0, \infty)$ be a function satisfying*

$$\Phi(x) := \sum_{j=0}^{\infty} \frac{1}{2^j} \phi\left(2^j x\right) < \infty \tag{2.18}$$

for all $x \in X$ and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10). Then, there exists a unique additive mapping $P : X \to Y$ satisfying (2.11)

Proof. The proof is similar to the proof of Theorem 2.6. ∎

Theorem 2.9. *Let $\phi : X \to [0, \infty)$ be an even function satisfying (2.18) and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10) Then, there exists a unique additive mapping $P : X \to Y$ satisfying (2.12).*

Proof. The proof is similar to the proof of Theorem 2.6. ∎

Corollary 2.10. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.13). Then, there exists a unique additive mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\| \leq \frac{2\theta}{(1 - |s|)(2 - 2^r)} \|x\|^r \tag{2.19}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x, y) = \theta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in X$ or Theorem 2.9 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in X$. ∎

3. Additive s-Functional Inequality (0.1): a Fixed Point Method

We present the Hyers-Ulam stability of the additive s-functional inequality (0.1) in complex Banach spaces by using the fixed point method.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2} \varphi(x, y) \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then, there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \leq \frac{L}{2(1 - L)} E(x) \tag{3.2}$$

where the function $E : X \to \mathbb{R}$ is defined as

$$E(x) = \frac{1}{1 - |s|^2} \left[\varphi(x, x) + |s|\varphi(x, -x)\right] \tag{3.3}$$

for all $x \in X$.

Proof. It follows from (2.4) and (2.5) that

$$\|f(2x) - 2f(x)\| \leq E(x) \tag{3.4}$$

for all $x \in X$. Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+ : \|g(x)-h(x)\|\le \mu E(x),\ \forall x\in X\right\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that (S,d) is complete (see [25]). Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=2g\left(\frac{x}{2}\right)$$

for all $x\in X$. Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$\|g(x)-h(x)\|\le\varepsilon E(x)$$

for all $x\in X$. Since

$$\|Jg(x)-Jh(x)\|=\left\|2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)\right\|\le 2\varepsilon E\left(\frac{x}{2}\right)\le L\varepsilon E\left(x\right)$$

for all $x\in X$, $d(Jg,Jh)\le L\varepsilon$. This means that

$$d(Jg,Jh)\le Ld(g,h)$$

for all $g,h\in S$. It follows from (3.4) that

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|\le E\left(\frac{x}{2}\right)\le\frac{L}{2}E\left(x\right)$$

for all $x,z\in X$. So $d(f,Jf)\le\frac{L}{2}$. By Theorem 1.2, there exists a mapping $P:X\to Y$ satisfying the following:

(1) P is a fixed point of J, i.e.,

$$P\left(x\right)=2P\left(\frac{x}{2}\right) \tag{3.5}$$

for all $x\in X$. The mapping P is a unique fixed point of J. This implies that P is a unique mapping satisfying (3.5) such that there exists a $\mu\in(0,\infty)$ satisfying

$$\|f(x)-P(x)\| \quad\le\quad \mu E(x)$$

for all $x\in X$;

(2) $d(J^lf,P)\to 0$ as $l\to\infty$. This implies the equality

$$\lim_{l\to\infty}2^lf\left(\frac{x}{2^l}\right)=P(x)$$

for all $x\in X$;

(3) $d(f,P)\le\frac{1}{1-L}d(f,Jf)$, which implies

$$\|f(x)-P(x)\|\le\frac{L}{2(1-L)}E(x)$$

for all $x\in X$. So we obtain (3.2). Since $2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\le L^n\varphi(x,y)$ tends to zero as $\to\infty$, from (2.2), we get

$$\begin{aligned}
&\|P(x+y)-P(x)-P(y)\|\\
&\quad=\lim_{n\to\infty}2^n\left\|f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-P\left(\frac{y}{2^n}\right)\right\|\\
&\quad\le\lim_{n\to\infty}2^n\left\|s\left(f\left(\frac{x-y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-P\left(-\frac{y}{2^n}\right)\right)\right\|+\lim_{n\to\infty}2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\\
&\quad=\|s\left(P(x-y)-P(x)-P(-y)\right)\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 1.3, the mapping $P : X \to Y$ is additive. ■

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (3.1) and $\varphi(x, y) = (x, -y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then, there exists a unique additive mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\| \le \frac{L}{2(1-L)(1-|s|)}\varphi(x, x)$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.1. ■

Theorem 3.3. *Let $\phi : X \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{2}\right) \le \frac{L}{2}\phi(x) \tag{3.6}$$

for all $x \in X$ and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10). Then, there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \le \frac{L}{2(1-L)}\mathcal{E}(x)$$

where the function $\mathcal{E} : X \to \mathbb{R}$ is defined as

$$\mathcal{E}(x) = \frac{1}{1-|s|^2}\left[(2+|s|)\phi(x) + |s|\phi(-x)\right] \tag{3.7}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.1. ■

Theorem 3.4. *Let $\phi : X \to [0, \infty)$ be an even function satisfying (3.6) and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10) Then, there exists a unique additive mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\| \le \frac{L}{(1-L)(1-|s|)}\phi(x)$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.1. ■

Corollary 3.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.13). Then, there exists a unique additive mapping $P : X \to Y$ satisfying (2.14).*

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in X$ or Theorem 3.4 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. ■

Theorem 3.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.8}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then, there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \le \frac{1}{2(1-L)}E(x)$$

for all $x \in X$, where the function $E : X \to \mathbb{R}$ is defined as in (3.3).

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (3.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2} E(x)$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 3.1. ■

Theorem 3.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (3.8) and $\varphi(x, y) = (x, -y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then, there exists a unique additive mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\| \leq \frac{1}{2(1 - L)(1 - |s|)} \varphi(x, x)$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.6. ■

Theorem 3.8. *Let $\phi : X \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x) \leq 2L\phi\left(\frac{x}{2}\right) \tag{3.9}$$

for all $x \in X$ and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10). Then, there exists a unique additive mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\| \leq \frac{1}{2(1 - L)} \mathcal{E}(x)$$

for all $x \in X$, where the function $\mathcal{E} : X \to \mathbb{R}$ is defined as in (3.7).

Proof. The proof is similar to the proof of Theorem 3.6. ■

Theorem 3.9. *Let $\phi : X \to [0, \infty)$ be an even function satisfying (3.9) and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10) Then, there exists a unique additive mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\| \leq \frac{1}{(1 - L)(1 - |s|)} \phi(x)$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.6. ■

Corollary 3.10. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.13). Then, there exists a unique additive mapping $P : X \to Y$ satisfying (2.19).*

Proof. The proof follows from Theorem 3.7 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ or Theorem 3.9 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. ■

4. Hom-Derivations in Banach Algebras: a Direct Method

In this section, we prove the Hyers-Ulam stability of hom-derivations in Banach algebras, associated with the additive s-functional inequality (0.1) by using the direct method.

Theorem 4.1. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying*

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{4.1}$$

for all $x, y \in A$ and let $f, g : A \to A$ be mappings satisfying

$$\|f(\lambda(x+y)) - \lambda(f(x)+f(y))\| \leq \|s(f(x-y) - f(x) - f(-y))\| + \varphi(x,y), \tag{4.2}$$

$$\|g(\lambda(x+y)) - \lambda(g(x)+g(y))\| \leq \|s(g(x-y) - g(x) - g(-y))\| + \varphi(x,y), \tag{4.3}$$

$$\|g(xy) - g(x)g(y)\| \leq \varphi(x,y), \tag{4.4}$$

$$\|f(xy) - f(x)g(y) - g(x)f(y)\| \leq \varphi(x,y), \tag{4.5}$$

and $f(0) = g(0) = 0$ for all $x, y \in A$ and all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| < 1\}$. Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ such that

$$\|f(x) - F(x)\| \leq \frac{1}{2(1-|s|^2)}\left[\Psi(x,x) + |s|\Psi(x,-x)\right], \tag{4.6}$$

$$\|g(x) - G(x)\| \leq \frac{1}{2(1-|s|^2)}\left[\Psi(x,x) + |s|\Psi(x,-x)\right], \tag{4.7}$$

$$F(xy) = F(x)G(y) + G(x)F(y) \tag{4.8}$$

where

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \tag{4.9}$$

for all $x, y \in A$.

Proof. Let $\lambda = 1$ in (4.2) and (4.3). By Theorem 2.1, there are unique additive mappings $G, F : A \to A$ satisfying (4.6) and (4.7), respectively, which are given by

$$G(x) = \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) \text{ and } F(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Letting $y = 0$ in (4.2), we get $F(\lambda x) = \lambda F(x)$ for all $\lambda \in \mathbb{T}^1$ and all $x \in A$. So, the mapping $F : A \to A$ is $\mathbb{C}$-linear. Similarly, one can show that the additive mapping $G : A \to A$ is $\mathbb{C}$-linear. For each $x, y \in A$, it follows from (4.4) that

$$\begin{aligned}
\|G(xy) - G(x)G(y)\| &= \lim_{n\to\infty} 4^n \left\| g\left(\frac{xy}{2^n \cdot 2^n}\right) - g\left(\frac{x}{2^n}\right) g\left(\frac{y}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0.
\end{aligned}$$

So $G(xy) = G(x)G(y)$ for all $x, y \in A$. Thus, the $\mathbb{C}$-linear mapping $G : A \to A$ is a homomorphism satisfying (4.7). For each $x, y \in A$, it follows from (4.5) that

$$\begin{aligned}
&\|F(xy) - F(x)G(y) - G(x)F(y)\| \\
&= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right) g\left(\frac{y}{2^n}\right) - g\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0.
\end{aligned}$$

Hence, the $\mathbb{C}$-linear mapping $F : A \to A$ is a hom-derivation satisfying (4.6) and (4.8). ∎

Theorem 4.2. *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (4.1) and $\varphi(x, y) = (x, -y)$ for all $x, y \in A$. Let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.2)-(4.5). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and*

$$\|f(x) - F(x)\| \leq \frac{1}{2(1 - |s|)}\Psi(x, x), \tag{4.10}$$

$$\|g(x) - G(x)\| \leq \frac{1}{2(1 - |s|)}\Psi(x, x) \tag{4.11}$$

for all $x \in A$, where the function Ψ is defined as in (4.9).

Proof. The proof is similar to the proof of Theorem 4.1. ∎

Theorem 4.3. *Let $\phi : A \to [0, \infty)$ be a function satisfying*

$$\sum_{j=1}^{\infty} 4^j \phi\left(\frac{x}{2^j}\right) < \infty \tag{4.12}$$

for all $x \in A$ and let $f, g : A \to A$ be mappings satisfying

$$\begin{aligned} \|f(\lambda(x + y)) - \lambda(f(x) + f(y))\| &\leq \|s(f(x - y) - f(x) - f(-y))\| \\ &\quad + \phi(x) + \phi(y), & (4.13)\\ \|g(\lambda(x + y)) - \lambda(g(x) + g(y))\| &\leq \|s(g(x - y) - g(x) - g(-y))\| \\ &\quad + \phi(x) + \phi(y), & (4.14)\\ \|g(xy) - g(x)g(y)\| &\leq \phi(x) + \phi(y), & (4.15)\\ \|f(xy) - f(x)g(y) - g(x)f(y)\| &\leq \phi(x) + \phi(y), & (4.16) \end{aligned}$$

and $f(0) = g(0) = 0$ for all $x, y \in A$ and all $\lambda \in \mathbb{T}^1$. Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and

$$\|f(x) - F(x)\| \leq \frac{1}{2(1 - |s|^2)}\left[(2 + |s|)\Phi(x) + |s|\Phi(-x)\right], \tag{4.17}$$

$$\|g(x) - G(x)\| \leq \frac{1}{2(1 - |s|^2)}\left[(2 + |s|)\Phi(x) + |s|\Phi(-x)\right] \tag{4.18}$$

where

$$\Phi(x) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j}\right) \tag{4.19}$$

for all $x \in A$.

Proof. The proof is similar to the proof of Theorem 4.1. ∎

Theorem 4.4. *Let $\phi : A \to [0, \infty)$ be an even function satisfying (4.12) and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.13)-(4.16). Then, there exist a* *unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and*

$$\|f(x) - F(x)\| \leq \frac{1}{1-|s|}\Phi(x), \tag{4.20}$$

$$\|g(x) - G(x)\| \leq \frac{1}{1-|s|}\Phi(x) \tag{4.21}$$

for all $x \in A$, where the function Φ is defined as in (4.19).

Proof. The proof is similar to the proof of Theorem 4.1. ∎

Corollary 4.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f, g : A \to A$ be mappings satisfying*

$$\begin{aligned} \|f(\lambda(x+y)) - \lambda(f(x)+f(y))\| &\leq \|s(f(x-y) - f(x) - f(-y))\| \\ &\quad +\theta(\|x\|^r + \|y\|^r), \end{aligned} \tag{4.22}$$

$$\begin{aligned} \|g(\lambda(x+y)) - \lambda(g(x)+g(y))\| &\leq \|s(g(x-y) - g(x) - g(-y))\| \\ &\quad +\theta(\|x\|^r + \|y\|^r), \end{aligned} \tag{4.23}$$

$$\|g(xy) - g(x)g(y)\| \leq \theta(\|x\|^r + \|y\|^r), \tag{4.24}$$

$$\|f(xy) - f(x)g(y) - g(x)f(y)\| \leq \theta(\|x\|^r + \|y\|^r), \tag{4.25}$$

and $f(0) = g(0) = 0$ for all $x, y \in A$ and all $\lambda \in \mathbb{T}^1$. Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and

$$\|f(x) - F(x)\| \leq \frac{2\theta}{(1-|s|)(2^r-2)}\|x\|^r, \tag{4.26}$$

$$\|g(x) - G(x)\| \leq \frac{2\theta}{(1-|s|)(2^r-2)}\|x\|^r \tag{4.27}$$

for all $x \in A$.

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in A$ or Theorem 4.4 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in A$. ∎

Theorem 4.6. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty \tag{4.28}$$

for all $x, y \in A$ and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.2)-(4.5). Then, there exist a unique homomorphism $G : A \to A$ satisfying (4.7) and a unique hom-derivation $F : A \to A$ satisfying (4.6) and (4.8).

Proof. The proof is similar to the proof of Theorem 4.1. ∎

Theorem 4.7. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (4.28) and $\varphi(x,y) = (x,-y)$ for all $x, y \in A$. Let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.2)-(4.5). Then, there exist a unique homomorphism $G : A \to A$ satisfying (4.11) and a unique hom-derivation $F : A \to A$ satisfying (4.8) and (4.10).*

Proof. The proof is similar to the proof of Theorem 4.1. ∎

Theorem 4.8. *Let $\phi : A \to [0,\infty)$ be a function satisfying*

$$\Phi(x) := \sum_{j=1}^{\infty} \frac{1}{2^j}\phi\left(2^j x\right) < \infty \tag{4.29}$$

for all $x \in A$ and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.13)-(4.16). Then, there exist a unique homomorphism $G : A \to A$ satisfying (4.18) and a unique hom-derivation $F : A \to A$ satisfying (4.8) and (4.17).

Proof. The proof is similar to the proof of Theorem 4.1. ∎

Theorem 4.9. *Let $\phi : A \to [0,\infty)$ be an even function satisfying (4.29) and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.13)-(4.16). Then, there exist a unique homomorphism $G : A \to A$ satisfying (4.21) and a unique hom-derivation $F : A \to A$ satisfying (4.8) and (4.20).*

Proof. The proof is similar to the proof of Theorem 4.1. ∎

Corollary 4.10. *Let $r < 1$ and θ be positive real numbers, and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.22)-(4.25). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and*

$$\|f(x) - F(x)\| \le \frac{2\theta}{(1-|s|)(2-2^r)}\|x\|^r, \tag{4.30}$$

$$\|g(x) - G(x)\| \le \frac{2\theta}{(1-|s|)(2-2^r)}\|x\|^r \tag{4.31}$$

for all $x \in A$.

Proof. The proof follows from Theorem 4.7 by taking $\varphi(x,y) = \theta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in A$ or Theorem 4.9 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in A$. ∎

5. Hom-Derivations in Banach Algebras: a Fixed Point Method

In this section, we present the Hyers-Ulam stability of hom-derivations in Banach algebras, associated to the additive s-functional inequality (0.1) by using the fixed point method.

Theorem 5.1. *Let $\varphi : A^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y) \le \frac{L}{2}\varphi(x,y) \tag{5.1}$$

for all $x, y \in A$ and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.2)-(4.5). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and

$$\|f(x) - F(x)\| \le \frac{L}{2(1-L)}E(x),$$
$$\|g(x) - G(x)\| \le \frac{L}{2(1-L)}E(x)$$

where the function $E : A \to \mathbb{R}$ is defined as

$$E(x) = \frac{1}{1-|s|^2}\left[\varphi(x,x) + |s|\varphi(x,-x)\right] \tag{5.2}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 3.1 and 4.1. ∎

Theorem 5.2. *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (5.1) and $\varphi(x, y) = (x, -y)$ for all $x, y \in A$. Let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.2)-(4.5). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and*

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{L}{2(1-L)(1-|s|)}\varphi(x, x) \\ \|g(x) - G(x)\| &\leq \frac{L}{2(1-L)(1-|s|)}\varphi(x, x) \end{aligned}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 3.1 and 4.1. ∎

Theorem 5.3. *Let $\phi : A \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{2}\right) \leq \frac{L}{4}\phi(x) \leq \frac{L}{2}\phi(x) \tag{5.3}$$

for all $x \in A$ and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.13)-(4.16). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{L}{2(1-L)}\mathcal{E}(x) \\ \|g(x) - G(x)\| &\leq \frac{L}{2(1-L)}\mathcal{E}(x) \end{aligned}$$

where the function $\mathcal{E} : A \to \mathbb{R}$ is defined as

$$\mathcal{E}(x) = \frac{1}{1-|s|^2}\left[(2+|s|)\phi(x) + |s|\phi(-x)\right] \tag{5.4}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 3.1 and 4.1. ∎

Theorem 5.4. *Let $\phi : A \to [0, \infty)$ be an even function satisfying (5.3) and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.13)-(4.16). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and*

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{L}{(1-L)(1-|s|)}\phi(x) \\ \|g(x) - G(x)\| &\leq \frac{L}{(1-L)(1-|s|)}\phi(x) \end{aligned}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 3.1 and 4.1. ∎

Corollary 5.5. *Let $r > 1$ and θ be nonnegative real numbers,, and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.22)-(4.25). Then, there exist a unique homomorphism $G : A \to A$ satisfying (4.27) and a unique hom-derivation $F : A \to A$ satisfying (4.8) and (4.26).*

Proof. The proof follows from Theorem 5.2 by taking $\varphi(x,y) = \theta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in X$ or Theorem 5.4 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. ■

Theorem 5.6. *Let $\varphi : A^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{5.5}$$

for all $x, y \in A$ and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.2)-(4.5). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{1}{2(1-L)}E(x), \\ \|g(x) - G(x)\| &\leq \frac{1}{2(1-L)}E(x) \end{aligned}$$

for all $x \in A$, where the function $E : A \to \mathbb{R}$ is defined as in (5.2).

Proof. The proof is similar to the proofs of Theorems 3.6 and 4.1. ■

Theorem 5.7. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (5.5) and $\varphi(x,y) = (x,-y)$ for all $x, y \in A$. Let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.2)-(4.5). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and*

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{1}{2(1-L)(1-|s|)}\varphi(x,x) \\ \|g(x) - G(x)\| &\leq \frac{1}{2(1-L)(1-|s|)}\varphi(x,x) \end{aligned}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 3.6 and 4.1. ■

Theorem 5.8. *Let $\phi : A \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x) \leq 2L\phi\left(\frac{x}{2}\right) \tag{5.6}$$

for all $x \in A$ and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.13)-(4.16). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{1}{2(1-L)}\mathcal{E}(x) \\ \|g(x) - G(x)\| &\leq \frac{1}{2(1-L)}\mathcal{E}(x) \end{aligned}$$

for all $x \in A$, where the function $\mathcal{E} : A \to \mathbb{R}$ is defined as in (5.4).

Proof. The proof is similar to the proofs of Theorems 3.6 and 4.1. ■

Theorem 5.9. *Let $\phi : A \to [0, \infty)$ be an even function satisfying (5.6) and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.13)-(4.16). Then, there exist a unique homomorphism $G : A \to A$ and a unique hom-derivation $F : A \to A$ satisfying (4.8) and*

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{1}{(1-L)(1-|s|)}\phi(x) \\ \|g(x) - G(x)\| &\leq \frac{1}{(1-L)(1-|s|)}\phi(x) \end{aligned}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 3.6 and 4.1. ∎

Corollary 5.10. *Let $r < 1$ and θ be positive real numbers, and let $f, g : A \to A$ be mappings satisfying $f(0) = g(0) = 0$ and (4.22)-(4.25). Then, there exist a unique homomorphism $G : A \to A$ satisfying (4.31) and a unique hom-derivation $F : A \to A$ satisfying (4.8) and (4.30).*

Proof. The proof follows from Theorem 5.7 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ or Theorem 5.9 by taking $\phi(x) = \theta\|x\|^r$ for all $x \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. ∎

Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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