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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Strong Convergence Results for G-Asymptotically Nonexpansive Mappings in Hilbert Spaces with Graphs

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Abstract In this paper, by combining two modified Ishikawa iteration processes and two modified S-iteration processes with the shrinking projection method, we propose four new hybrid iteration processes for two G-asymptotically nonexpansive mappings. We also prove some strong convergence theorems for common fixed points of two G-asymptotically nonexpansive mappings in Hilbert spaces with graphs. These theorems are the extension and improvement of certain main results in [H.A. Hammad, W. Cholamjiak, D. Yambangwai, H. Dutta, A modified shrinking projection methods for numerical reckoning fixed points of G-nonexpansive mappings in Hilbert spaces with graphs, Miskolc Math. Notes 20 (2) (2019) 941–956]. In addition, we provide a numerical example for supporting obtained results.

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1. Introduction

In recent times, the Banach contraction principle [1] was improved and extended in many different ways and many fixed point results were obtained [2–6]. In 2008, by combining the concepts in fixed point theory and graph theory, Jachymski [7] generalized the Banach contraction principle in a complete metric space with a directed graph. In 2012, Aleomraninejad *et al.* [8] introduced the notion of G-contractive mapping and G-nonexpansive mapping in a metric space with a directed graph and stated the convergence

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for these mappings. After that, there were many the convergence results of various iteration processes to fixed points and common fixed points of G-nonexpansive mappings in Hilbert spaces and Banach spaces with graphs [9–15].

Motivated by these works, Sangago *et al.* [16] introduced the notion of a G-asymptotically nonexpansive mapping and proved the weak and strong convergence of a modified Noor iteration process to common fixed points of a finite family of G-asymptotically nonexpansive mappings in Banach spaces with graphs. After that, authors in [17, 18] proposed a two-step iteration process for two G-asymptotically nonexpansive mappings and a three-step iteration process for three G-asymptotically nonexpansive mappings. Furthermore, the authors also established the weak and strong convergence results of some proposed iteration processes to common fixed points of G-asymptotically nonexpansive mappings in Banach spaces with graphs.

Observe that the Opial's condition [19] is considered to study the weak convergence theorems, and the semicompact property of the mappings or the condition (A) [20], the condition (B) [21] and the condition (C) [10] are used to state the strong convergence results. In other way, motivated by CQ projection method in [22], some authors proposed some modified shrinking projection methods and proved the strong convergence for common fixed points of G-nonexpansive mappings in Hilbert spaces with graphs [23–25]. The following iteration processes were introduced in [25] by modifying the shrinking projection method with Ishikawa iteration process and S-iteration process.

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)u_n + a_n T_2 v_n \\ \Omega_{n+1} = \big\{ w \in \Omega_n : \|w_n - w\| \le \|u_n - w\| \big\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.1}$$

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)v_n + a_n T_2 v_n \\ \Omega_{n+1} = \big\{ w \in \Omega_n : \|w_n - w\| \le \|u_n - w\| \big\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)T_1 u_n + a_n T_2 v_n \\ \Omega_{n+1} = \big\{ w \in \Omega_n : \|w_n - w\| \le \|u_n - w\| \big\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.3}$$

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)T_1 v_n + a_n T_2 v_n \\ \Omega_{n+1} = \big\{ w \in \Omega_n : \|w_n - w\| \le \|u_n - w\| \big\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$, and $T_1, T_2 : \Omega \longrightarrow \Omega$ are G-nonexpansive mappings, and $P_{\Omega_{n+1}} u_1$ is the metric projection of u_1 onto Ω_{n+1}.

Motivated by the iteration processes (1.1), (1.2), (1.3) and (1.4), we introduce the following iteration processes for two G-asymptotically nonexpansive mappings:

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)u_n + a_n T_2^n v_n \\ \Omega_{n+1} = \left\{ w \in \Omega_n : \|w_n - w\|^2 \leq \|u_n - w\|^2 + \delta_n \right\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.5}$$

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)v_n + a_n T_2^n v_n \\ \Omega_{n+1} = \left\{ w \in \Omega_n : \|w_n - w\|^2 \leq \|u_n - w\|^2 + \sigma_n \right\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.6}$$

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)T_1^n u_n + a_n T_2^n v_n \\ \Omega_{n+1} = \left\{ w \in \Omega_n : \|w_n - w\|^2 \leq \|u_n - w\|^2 + \varepsilon_n \right\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.7}$$

$$\begin{cases} u_1 \in \Omega, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)T_1^n v_n + a_n T_2^n v_n \\ \Omega_{n+1} = \left\{ w \in \Omega_n : \|w_n - w\|^2 \leq \|u_n - w\|^2 + \gamma_n \right\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$, and $T_1, T_2 : \Omega \longrightarrow \Omega$ are G-asymptotically nonexpansive mappings, and $P_{\Omega_{n+1}} u_1$ is the metric projection of u_1 onto Ω_{n+1}, and $\delta_n, \sigma_n, \varepsilon_n, \gamma_n$ are defined as in Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5 in Section 3, respectively. After that, we prove some strong convergence results of the iteration processes (1.5), (1.6), (1.7), (1.8) to the projection of the initial point u_1 onto the set of all common fixed points of T_1 and T_2 in Hilbert spaces with graphs. In addition, we provide a numerical example for supporting obtained results.

2. Preliminaries

Let X be a real normed space and Ω be a nonempty subset of X. Let Δ denote the diagonal of the Cartesian product $\Omega \times \Omega$, that is, $\Delta = \{(u,u) : u \in \Omega\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with Ω, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supset \Delta$. We assume that G has no parallel edges. So we can identify the graph G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G, that is, the graph obtained from G by reversing the direction of edges. Therefore, we obtain

$$E(G^{-1}) = \{(u,v) \in X \times X : (v,u) \in E(G)\}.$$

We recall some basic notions concerning the graphs, for others, see [26, Chapter 8]. If u and v are vertices in a graph G, then a *path* in G from u to v of length N for $N \in \mathbb{N} \cup \{0\}$ is a sequence $\{u_i\}_{i=0}^N$ of $N+1$ vertices such that $u_0 = u, u_N = v$ and $(u_i, u_{i+1}) \in E(G)$

for $i = 0, 1, \dots, N-1$. A graph G is said to be *connected* if there is a path between any two vertices. A directed graph $G = (V(G), E(G))$ is said to be *transitive* if for any $u, v, w \in V(G)$ such that (u, v) and (v, w) are in $E(G)$, then $(u, w) \in E(G)$.

Definition 2.1 ([12], p.4)**.** Let X be a normed space, Ω be a nonempty subset of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = \Omega$. Then Ω is said to have *property* (G) if for any sequence $\{u_n\}$ in Ω such that $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $\{u_n\}$ weakly converging to $u \in \Omega$, there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $(u_{n(k)}, u) \in E(G)$ for all $k \in \mathbb{N}$.

Definition 2.2 ([25], Definition 5)**.** Let X be a normed space, Ω be a nonempty subset of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = \Omega$. The set of edges $E(G)$ is said to be *convex* if for all $(x, y), (u, v) \in E(G)$ and for each $\lambda \in [0, 1]$, we have

$$\lambda(x, y) + (1 - \lambda)(u, v) \in E(G).$$

Note that in [9–13, 15, 17, 18, 23, 25], the authors assumed the directed graph $G = (V(G), E(G))$ such that

(1) $E(G)$ is convex.
(2) $E(G) \supset \{(u, u) : u \in V(G)\}$.

To support the obtained results, the authors gave some numerical examples. Unfortunately, there were some confusions in those examples. Indeed, the authors [27] showed that

(1) In [10, Example 4.5], [13, Example 3.4], [15, Example 1], [18, Example 4.5], the authors considered $X = \mathbb{R}$, $\Omega = [0, 2]$, and $G = (V(G), E(G))$ is a directed graph defined by $V(G) = \Omega$ and $(x, y) \in E(G)$ if and only if $0.5 \leq x \leq y \leq 1.7$. However, $(2, 2) \notin E(G)$ and hence the condition $E(G) \supset \{(u, u) : u \in V(G)\}$ is not satisfied.

(2) In [11, Example 1], the authors considered $X = \mathbb{R}$, $\Omega = [0, 2]$, and $G = (V(G), E(G))$ is a directed graph defined by $V(G) = \Omega$ and $(x, y) \in E(G)$ if and only if $0.8 \leq x, y \leq 1.7$ and $x, y \in \mathbb{Q}$. However, $(2, 2) \notin E(G)$ and hence the condition $E(G) \supset \{(u, u) : u \in V(G)\}$ is not satisfied.

(3) In [17, Example 4.5], the author considered $X = \mathbb{R}$, $\Omega = [0, 2]$, and $G = (V(G), E(G))$ is a directed graph defined by $V(G) = \Omega$ and $(x, y) \in E(G)$ if and only if $0.75 \leq x, y \leq 1.7$. However, $(2, 2) \notin E(G)$ and hence the condition

$$E(G) \supset \{(u, u) : u \in V(G)\}$$

is not satisfied.

(4) In [9, Example 3.2], the author considered $X = \mathbb{R}$, $\Omega = [3, 3.3]$, and $G = (V(G), E(G))$ is a directed graph defined by $V(G) = \Omega$ and

$$E(G) = \{(x, y) : x \in [3, 3.2], y \in [3, 3.3] \text{ with } |x - y| < 1\}.$$

However, $(3.3, 3.3) \notin E(G)$ and hence the condition $E(G) \supset \{(u, u) : u \in V(G)\}$ is not satisfied.

(5) In [12, Example 3.5], the authors considered $X = \mathbb{R}$, $\Omega = [0, \frac{1}{2}]$, and $G = (V(G), E(G))$ is a directed graph defined by $V(G) = \Omega$ and

$$E(G) = \left\{(x, y) : x, y \in \left[0, \frac{3}{8}\right] \text{ with } |x - y| < \frac{1}{8}\right\}.$$

However, $(\frac{1}{2}, \frac{1}{2}) \notin E(G)$ and hence the condition $E(G) \supset \{(u, u) : u \in V(G)\}$ is not satisfied.

(6) In [23, Example 4.1], the authors considered $X = \mathbb{R}$, $\Omega = [0,3]$, and $G = (V(G), E(G))$ is a directed graph defined by $V(G) = \Omega$ and $(x,y) \in E(G)$ if and only if $0 \leq x, y \leq 2$ or $x = y \in [0,3]$. By choosing $(2,0),(3,3) \in E(G)$, we have $0.5(2,0) + 0.5(3,3) = (2.5, 1.5) \notin E(G)$. Therefore, $E(G)$ is not convex.

Moreover, in [25, Example 1], we see that the authors considered $X = \mathbb{R}^3$, $\Omega = [-2,0]^3$, and $G = (V(G), E(G))$ is a directed graph defined by $V(G) = \Omega$ and $(x,y) \in E(G)$ if and only if $-1.5 \leq x_i, y_i \leq -0.5$ or $x = y \in \Omega$ for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \Omega$. By choosing $x = (-1,-1,-1.5), y = (-1.5,-1.5,-1), u = v = (-2,-2,-2)$, we have $(x,y),(u,v) \in E(G)$, and $0.2(x,y)+0.8(u,v) = (z,w) \notin E(G)$ with $z = (-1.8,-1.8,-1.9)$, $w = (-1.9,-1.9,-1.8)$. Therefore, $E(G)$ is not convex.

Motivated by these confusions, the authors [27] introduced the coordinate-convexity such as.

Definition 2.3 ([27], Definition 3.1)**.** Let X be a normed space, Ω be a nonempty subset of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = \Omega$. The set of edges $E(G)$ is said to be *coordinate-convex* if for all $(p,u),(p,v),(u,p),(v,p) \in E(G)$ and for all $t \in [0,1]$, we have

$$t(p,u) + (1-t)(p,v) \in E(G) \text{ and } t(u,p) + (1-t)(v,p) \in E(G).$$

Remark 2.4 ([27], Remark 3.2)**.** If $E(G)$ is convex, then $E(G)$ is coordinate-convex. However, the set $E(G)$ defined as in [23, Example 4.1] is a coordinate-convex set which is not convex.

Definition 2.5. Let X be a normed space, Ω be a nonempty subset of $X, G = (V(G), E(G))$ be a directed graph such that $V(G) = \Omega$ and $T : \Omega \longrightarrow \Omega$ be a mapping. Then

(1) ([14], Definition 2.1) T is said to be *G-nonexpansive* if
 (a) T is *edge-preserving*, that is, for all $(u,v) \in E(G)$, we have
$$(Tu, Tv) \in E(G).$$
 (b) $\|Tu - Tv\| \leq \|u - v\|$ for all $(u,v) \in E(G)$.

(2) ([16], Definition 3.1) T is said to be *asymptotically G-nonexpansive* if
 (a) T is edge-preserving.
 (b) There exists a sequence $\{\lambda_n\} \subset [1,\infty)$ with $\sum\limits_{n=1}^{\infty}(\lambda_n - 1) < \infty$ such that for all $(u,v) \in E(G)$, we have $\|T^n u - T^n v\| \leq \lambda_n \|u - v\|$, where $\{\lambda_n\}$ is said to be an *asymptotic coefficient sequence.*

Remark 2.6. Every G-nonexpansive mapping is a G-asymptotically nonexpansive mapping with the asymptotic coefficient sequence $\lambda_n = 1$ for all $n \in \mathbb{N}$.

Lemma 2.7 ([16], Theorem 3.3)**.** *Suppose that*

(1) X is a Banach space.
(2) Ω is a nonempty closed and convex subset of X and Ω has property (G).
(3) $G = (V(G), E(G))$ is a directed graph such that $V(G) = \Omega$.
(4) $T : \Omega \longrightarrow \Omega$ is a G-asymptotically nonexpansive mapping with the asymptotic coefficient sequence $\{\lambda_n\} \subset [1,\infty)$ satisfying $\sum\limits_{n=1}^{\infty}(\lambda_n - 1) < \infty$.
(5) $\{u_n\}$ weakly converges to $u \in \Omega$, $(u_n, u_{n+1}) \in E(G)$ and $\lim\limits_{n\to\infty}\|Tu_n - u_n\| = 0$.

Then $Tu = u$.

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, Ω be a nonempty, closed and convex subset of a Hilbert space H. Now, we recall some basic notions of Hilbert spaces which we will use in next section.

The nearest point projection of H onto Ω is denoted by P_Ω, that is, for all $u \in H$, $\|u - P_\Omega u\| = \inf\{\|u - v\| : v \in \Omega\}$. Then P_Ω is called *the metric projection* of H onto Ω. It is known that for each $u \in H$, $q = P_\Omega u$ is equivalent to $\langle u - q, q - v\rangle \geq 0$ for all $v \in \Omega$.

Lemma 2.8 ([28], p. 5)**.** *Let H be a real Hilbert space and Ω be a nonempty, closed and convex subset of H, and P_Ω be a metric projection of H onto Ω. Then for all $u \in H$ and $v \in \Omega$, we have*

$$\|v - P_\Omega u\|^2 + \|u - P_\Omega u\|^2 \leq \|u - v\|^2.$$

Lemma 2.9 ([29], Corollary 2.14)**.** *Let H be a real Hilbert space. Then for all $u, v \in H$ and $\lambda \in [0, 1]$, we have*

$$\|\lambda u + (1-\lambda)v\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2 - \lambda(1-\lambda)\|u - v\|^2.$$

The following result will be used in next section. The proof of this lemma is easy and is omitted.

Lemma 2.10. *Let H be a real Hilbert space. Then for all $u, v, w \in H$, we have*

$$\|u - v\|^2 = \|u - w\|^2 + \|w - v\|^2 + 2\langle u - w, w - v\rangle.$$

Lemma 2.11 ([30], Lemma 1.3)**.** *Let H be a real Hilbert space and Ω be a nonempty, closed and convex subset of H. Then for $x, y, z \in H$ and $a \in \mathbb{R}$, the following set is convex and closed.*

$$\big\{w \in \Omega : \|y - w\|^2 \leq \|x - w\|^2 + \langle z, w\rangle + a\big\}.$$

3. Main Results

First, we denote by $F(T) = \{u \in \Omega : Tu = u\}$ the set of fixed points of the mapping $T : \Omega \longrightarrow \Omega$. The following result is a sufficient condition for the closedness and convexity of the set $F(T)$ in real Hilbert spaces, where T is a G-asymptotically nonexpansive mapping.

Proposition 3.1. *Assume that*

(1) H is a real Hilbert space.
(2) Ω is a nonempty closed and convex subset of H.
(3) $G = (V(G), E(G))$ is a directed graph such that $V(G) = \Omega$.
(4) $T : \Omega \longrightarrow \Omega$ is a G-asymptotically nonexpansive mapping with an asymptotic coefficient sequence $\{\lambda_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(\lambda_n - 1) < \infty$, and

$$F(T) \times F(T) \subset E(G).$$

Then

(1) If Ω has property (G), then $F(T)$ is closed.
(2) If the graph G is transitive, $E(G)$ is coordinate-convex, then $F(T)$ is convex.

Proof. (1). Suppose that $F(T) \neq \emptyset$. Let $\{p_n\}$ be a sequence in $F(T)$ such that

$$\lim_{n\to\infty} \|p_n - p\| = 0 \tag{3.1}$$

for some $p \in \Omega$. Due to the fact that $F(T) \times F(T) \subset E(G)$, we obtain $(p_n, p_{n+1}) \in E(G)$. Since Ω has property (G), there exists a subsequence $\{p_{n(k)}\}$ of $\{p_n\}$ such that $(p_{n(k)}, p) \in E(G)$ for $k \in \mathbb{N}$. Since T is a G-asymptotically nonexpansive mapping, we obtain

$$\begin{aligned} \|p - Tp\| &\leq \|p - p_{n(k)}\| + \|p_{n(k)} - Tp\| \\ &= \|p - p_{n(k)}\| + \|Tp_{n(k)} - Tp\| \\ &\leq \|p - p_{n(k)}\| + \lambda_1\|p_{n(k)} - p\| \\ &= (1+\lambda_1)\|p - p_{n(k)}\|. \end{aligned} \tag{3.2}$$

It follows from (3.1) and (3.2) that $Tp = p$ and hence $p \in F(T)$. Therefore, $F(T)$ is closed.

(2). Let $p_1, p_2 \in F(T)$. For $t \in [0,1]$, we put $p = tp_1 + (1-t)p_2$. Since $p_1, p_2 \in F(T)$ and $F(T) \times F(T) \subset E(G)$, we obtain $(p_1,p_1), (p_1,p_2), (p_2,p_1)(p_2,p_2) \in E(G)$. Since $E(G)$ is coordinate-convex, we conclude that $(p_1, p) = t(p_1,p_1) + (1-t)(p_1,p_2) \in E(G)$, $(p, p_1) = t(p_1,p_1) + (1-t)(p_2,p_1) \in E(G)$ and $(p_2,p) = t(p_2,p_1) + (1-t)(p_2,p_2) \in E(G)$. Due to the fact that T is a G-asymptotically nonexpansive mapping, we get

$$\|p_1 - T^n p\| = \|T^n p_1 - T^n p\| \leq \lambda_n \|p_1 - p\| \tag{3.3}$$

and

$$\|p_2 - T^n p\| = \|T^n p_2 - T^n p\| \leq \lambda_n \|p_2 - p\|. \tag{3.4}$$

Furthermore, by using Lemma 2.10, we get

$$\|p_1 - T^n p\|^2 = \|p_1 - p\|^2 + \|p - T^n p\|^2 + 2\langle p_1 - p, p - T^n p\rangle \tag{3.5}$$

and

$$\|p_2 - T^n p\|^2 = \|p_2 - p\|^2 + \|p - T^n p\|^2 + 2\langle p_2 - p, p - T^n p\rangle. \tag{3.6}$$

It follows from (3.3) and (3.5) that

$$\|p - T^n p\|^2 \leq (\lambda_n^2 - 1)\|p_1 - p\|^2 - 2\langle p_1 - p, p - T^n p\rangle \tag{3.7}$$

Moreover, we conclude from (3.4) and (3.6) that

$$\|p - T^n p\|^2 \leq (\lambda_n^2 - 1)\|p_2 - p\|^2 - 2\langle p_2 - p, p - T^n p\rangle. \tag{3.8}$$

By multiplying t on the both sides of (3.7), and multiplying $(1-t)$ on the both sides of (3.8), we get

$$\begin{aligned} \|p - T^n p\|^2 &\leq t(\lambda_n^2 - 1)\|p_1 - p\|^2 + (1-t)(\lambda_n^2 - 1)\|p_2 - p\|^2 \\ &\quad -2t\langle p_1 - p, p - T^n p\rangle - 2(1-t)\langle p_2 - p, p - T^n p\rangle \\ &= t(\lambda_n^2 - 1)\|p_1 - p\|^2 + (1-t)(\lambda_n^2 - 1)\|p_2 - p\|^2 \\ &\quad -2\langle tp_1 + (1-t)p_2 - p, p - T^n p\rangle \\ &= t(\lambda_n^2 - 1)\|p_1 - p\|^2 + (1-t)(\lambda_n^2 - 1)\|p_2 - p\|^2 \\ &\quad -2\langle 0, p - T^n p\rangle \\ &= t(\lambda_n^2 - 1)\|p_1 - p\|^2 + (1-t)(\lambda_n^2 - 1)\|p_2 - p\|^2. \end{aligned} \tag{3.9}$$

Since $\sum_{n=1}^{\infty}(\lambda_n - 1) < \infty$, we have $\lim_{n\to\infty} \lambda_n = 1$. Therefore, we conclude from (3.9) that

$$\lim_{n\to\infty} \|p - T^n p\| = 0. \tag{3.10}$$

Furthermore, since $(p_1, p) \in E(G)$ and T is edge-preserving, we have $(p_1, T^n p) \in E(G)$. Then, from the transitive property of G and $(p, p_1), (p_1, T^n p) \in E(G)$, we conclude that $(p, T^n p) \in E(G)$. Since T is a G-asymptotically nonexpansive mapping, we obtain

$$\|Tp - p\| \le \|Tp - T^{n+1}p\| + \|T^{n+1}p - p\| \le \lambda_1\|p - T^n p\| + \|T^{n+1}p - p\|. \tag{3.11}$$

Taking the limit in (3.11) as $n \to \infty$ and using (3.10), we find that $Tp = p$, that is, $p \in F(T)$. Therefore, $F(T)$ is convex. ∎

Let $T_1, T_2 : \Omega \longrightarrow \Omega$ be two G-asymptotically nonexpansive mappings with the asymptotic coefficient sequences $\{\xi_n\}, \{\eta_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(\xi_n - 1) < \infty$, $\sum_{n=1}^{\infty}(\eta_n - 1) < \infty$, respectively. Putting $\lambda_n = \max\{\xi_n, \eta_n\}$, we have $\{\lambda_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(\lambda_n - 1) < \infty$ and by Definition 2.5, we obtain $\|T_1^n u - T_1^n v\| \le \lambda_n\|u - v\|$, $\|T_2^n u - T_2^n v\| \le \lambda_n\|u - v\|$ for all $(u, v) \in E(G)$.

In the following theorems, we will assume that the set $\mathcal{F} := F(T_1) \cap F(T_2)$ is nonempty and bounded in Ω, that is, there exists a positive number κ such that

$$\mathcal{F} \subset \{u \in \Omega : \|u\| \le \kappa\}.$$

The following theorem shows the convergence of iteration (1.5) to common fixed points of two G-asymptotically nonexpansive mappings in Hilbert spaces with directed graphs.

Theorem 3.2. *Assume that*

(1) H is a real Hilbert space.
(2) Ω is a nonempty closed, convex subset of H and Ω has property (G).
(3) $G = (V(G), E(G))$ is a directed and transitive graph, $V(G) = \Omega$ and $E(G)$ is coordinate-convex.
(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ are two G-asymptotically nonexpansive mappings such that $F(T_i) \times F(T_i) \subset E(G)$ for all $i = 1, 2$.
(5) $\{u_n\}$ is the sequence generated by (1.5) such that $\{a_n\}, \{b_n\} \subset [0, 1]$, and $\liminf_{n\to\infty} a_n > 0$, $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$, and $(u_n, p), (p, u_n) \in E(G)$ for all $p \in \mathcal{F}$, and $\delta_n = (\lambda_n^2 - 1)a_n(1 + b_n\lambda_n^2)(\|u_n\| + \kappa)^2$. .

Then the sequence $\{u_n\}$ strongly converges to $P_{\mathcal{F}}u_1$.

Proof. The proof of Theorem 3.2 is divided into six steps.

Step 1. We claim that $P_{\mathcal{F}}u_1$ is well-defined.

By Proposition 3.1, we conclude that $F(T_1)$ and $F(T_2)$ are closed and convex. Therefore, $\mathcal{F} = F(T_1) \cap F(T_2)$ is closed and convex. Note that $\mathcal{F}$ is nonempty by the assumption. This fact ensures that $P_{\mathcal{F}}u_1$ is well-defined.

Step 2. We claim that $P_{\Omega_{n+1}}u_1$ is well-defined.

We first prove by mathematical induction that Ω_n is closed and convex for $n \in \mathbb{N}$. Obviously, $\Omega_1 = \Omega$ is closed and convex. Now we suppose that Ω_k is closed and convex for some $k \in \mathbb{N}$. Then by the definition of Ω_{k+1} and Lemma 2.11, we conclude that Ω_{k+1} is closed and convex. Therefore, Ω_n is closed and convex for $n \in \mathbb{N}$.

Next, we will prove by mathematical induction that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Obviously, for all $p \in \mathcal{F}$, we have $T_1 p = T_2 p = p \in \Omega = \Omega_1$. Therefore, $\mathcal{F} \subset \Omega_1$. We suppose that $\mathcal{F} \subset \Omega_k$ for some $k \in \mathbb{N}$. We will show that $\mathcal{F} \subset \Omega_{k+1}$. Indeed, for all $p \in \mathcal{F}$, since $(u_k, p) \in E(G)$ and T_1 is edge-preserving, we have $(T_1^k u_k, p) \in E(G)$. Due to the fact that $E(G)$ is coordinate-convex, we have

$$(v_k, p) = \big((1-b_k)u_k + b_k T_1^k u_k, p\big) = (1-b_k)(u_k, p) + b_k(T_1^k u_k, p) \in E(G). \quad (3.12)$$

Then, using Lemma 2.9 and noting that T_1, T_2 are G-asymptotically nonexpansive mappings, we obtain that

$$\begin{aligned} \|w_k - p\|^2 &= \|(1-a_k)(u_k - p) + a_k(T_2^k v_k - p)\|^2 \\ &= (1-a_k)\|u_k - p\|^2 + a_k\|T_2^k v_k - p\|^2 - a_k(1-a_k)\|T_2^k v_k - u_k\|^2 \\ &\leq (1-a_k)\|u_k - p\|^2 + a_k\lambda_k^2\|v_k - p\|^2 \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \|v_k - p\|^2 &= \|(1-b_k)(u_k - p) + b_k(T_1^k u_k - p)\|^2 \\ &= (1-b_k)\|u_k - p\|^2 + b_k\|T_1^k u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\leq (1-b_k)\|u_k - p\|^2 + b_k\lambda_k^2\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &= \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\leq \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2. \end{aligned} \quad (3.14)$$

By substituting (3.14) into (3.13), we have

$$\begin{aligned} \|w_k - p\|^2 &\leq (1-a_k)\|u_k - p\|^2 + a_k\lambda_k^2\big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2 \\ &= \|u_k - p\|^2 + (\lambda_k^2 - 1)a_k(1 + b_k\lambda_k^2)\|u_k - p\|^2 \\ &\leq \|u_k - p\|^2 + (\lambda_k^2 - 1)a_k(1 + b_k\lambda_k^2)(\|u_k\| + \|p\|)^2 \\ &\leq \|u_k - p\|^2 + (\lambda_k^2 - 1)a_k(1 + b_k\lambda_k^2)(\|u_k\| + \kappa)^2 \\ &= \|u_k - p\|^2 + \delta_k. \end{aligned} \quad (3.15)$$

This implies that $p \in \Omega_{k+1}$ and hence $\mathcal{F} \subset \Omega_{k+1}$. Therefore, we conclude that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F} \neq \emptyset$, we have $\Omega_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, we conclude that $P_{\Omega_{n+1}} u_1$ is well-defined.

Step 3. We claim that $\lim_{n\to\infty} \|u_n - u_1\|$ *exists.* Indeed, since $u_n = P_{\Omega_n} u_1$, we have

$$\|u_n - u_1\| \leq \|x - u_1\| \quad \text{for all} \quad x \in \Omega_n. \quad (3.16)$$

Since $u_{n+1} = P_{\Omega_{n+1}} u_1 \in \Omega_{n+1} \subset \Omega_n$, from (3.16), by taking $x = u_{n+1}$, we obtain

$$\|u_n - u_1\| \leq \|u_{n+1} - u_1\|. \quad (3.17)$$

Since $\mathcal{F}$ is nonempty, closed and convex subset of H, there exists a unique $q = P_{\mathcal{F}} u_1$ and hence $q \in \mathcal{F} \subset \Omega_n$. Thus, from (3.16), by taking $x = q$, we get

$$\|u_n - u_1\| \leq \|q - u_1\|. \quad (3.18)$$

It follows from (3.17) and (3.18) that the sequence $\{\|u_n - u_1\|\}$ is bounded and nondecreasing. Therefore, $\lim_{n\to\infty} \|u_n - u_1\|$ exists.

Step 4. We claim that $\lim_{n\to\infty} u_n = u$ *for some* $u \in \Omega$. Indeed, since $u_n = P_{\Omega_n} u_1$, from Lemma 2.8, we have

$$\|v - u_n\|^2 + \|u_1 - u_n\|^2 \leq \|v - u_1\|^2 \quad \text{for all} \quad v \in \Omega_n. \quad (3.19)$$

For $m > n$, we see that $u_m = P_{\Omega_m} u_1 \in \Omega_m \subset \Omega_n$. Therefore, from (3.19), by taking $v = u_m$, we have

$$\|u_m - u_n\|^2 + \|u_1 - u_n\|^2 \le \|u_m - u_1\|^2.$$

This implies that

$$\|u_m - u_n\|^2 \le \|u_m - u_1\|^2 - \|u_n - u_1\|^2. \tag{3.20}$$

Since $\lim_{n\to\infty} \|u_n - u_1\|$ exists, we conclude from (3.20) that $\lim_{m,n\to\infty} \|u_m - u_n\| = 0$ and hence $\{u_n\}$ is a Cauchy sequence. Therefore, there exists $u \in \Omega$ such that $\lim_{n\to\infty} u_n = u$. Furthermore, we also have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.21}$$

Step 5. We claim that $u \in \mathcal{F}$. Indeed, since $u_{n+1} \in \Omega_{n+1}$, by the definition of Ω_{n+1}, we have

$$\|w_n - u_{n+1}\|^2 \le \|u_n - u_{n+1}\|^2 + \delta_n. \tag{3.22}$$

Since $\{u_n\}$ is bounded, there exists $A_1 > 0$ such that

$$0 \le \delta_n = (\lambda_n^2 - 1)a_n(1 + b_n\lambda_n^2)(\|u_n\| + \kappa)^2 \le A_1(\lambda_n^2 - 1)(1 + \lambda_n^2). \tag{3.23}$$

Taking the limit in (3.23) as $n \to \infty$ and using $\lim_{n\to\infty} \lambda_n = 1$, we get $\lim_{n\to\infty} \delta_n = 0$. Then, from (3.21) and (3.22), we obtain

$$\lim_{n\to\infty} \|w_n - u_{n+1}\| = 0. \tag{3.24}$$

It follows from (3.21), (3.24) and $\|w_n - u_n\| \le \|w_n - u_{n+1}\| + \|u_{n+1} - u_n\|$ that

$$\lim_{n\to\infty} \|w_n - u_n\| = 0. \tag{3.25}$$

Furthermore, we have

$$\|w_n - u_n\| = \|(1 - a_n)u_n + a_n T_2^n v_n - u_n\| = a_n\|T_2^n v_n - u_n\|. \tag{3.26}$$

Thus, by (3.25), (3.26) and noting that $\liminf_{n\to\infty} a_n > 0$, we get

$$\lim_{n\to\infty} \|T_2^n v_n - u_n\| = 0. \tag{3.27}$$

Next, for $p \in \mathcal{F}$, by using similar arguments as in the proof of (3.13), (3.14) and (3.15), we obtain

$$\begin{aligned}
& \|w_n - p\|^2 \\
\le\; & (1 - a_n)\|u_n - p\|^2 + a_n\lambda_n^2\|v_n - p\|^2 \\
=\; & (1 - a_n)\|u_n - p\|^2 + a_n\lambda_n^2\Big(\big(1 + b_n(\lambda_n^2 - 1)\big)\|u_n - p\|^2 - b_n(1 - b_n)\|T_1^n u_n - u_n\|^2\Big) \\
\le\; & \|u_n - p\|^2 + \delta_n - a_n b_n(1 - b_n)\lambda_n^2\|T_1^n u_n - u_n\|^2 \\
\le\; & \|u_n - p\|^2 + \delta_n - a_n b_n(1 - b_n)\|T_1^n u_n - u_n\|^2.
\end{aligned} \tag{3.28}$$

Furthermore, it follows from the boundedness property of $\{u_n\}$ and $\{w_n\}$ that there exists $A_2 > 0$ such that $\|u_n\| + \|w_n\| \le A_2$ for all $n \in \mathbb{N}$. In this way, we conclude from

(3.28) that

$$\begin{aligned} a_nb_n(1-b_n)\|T_1^nu_n-u_n\|^2 &\leq \|u_n-p\|^2-\|w_n-p\|^2+\delta_n \\ &= \|u_n\|^2-\|w_n\|^2+2\langle w_n-u_n,p\rangle+\delta_n \\ &\leq (\|u_n\|-\|w_n\|)(\|u_n\|+\|w_n\|)+2\|w_n-u_n\|.\|p\|+\delta_n \\ &\leq \|u_n-w_n\|(\|u_n\|+\|w_n\|)+2\|w_n-u_n\|.\|p\|+\delta_n \\ &\leq A_2\|u_n-w_n\|+2\|w_n-u_n\|.\|p\|+\delta_n. \end{aligned} \tag{3.29}$$

Then, by combining (3.29) with (3.25) and using $\lim_{n\to\infty}\delta_n=0$, $\liminf_{n\to\infty}a_nb_n(1-b_n)>0$, we get

$$\lim_{n\to\infty}\|T_1^nu_n-u_n\|=0. \tag{3.30}$$

We also have

$$\begin{aligned} \|v_n-u_n\| &= \|(1-b_n)u_n+b_nT_1^nu_n-u_n\| \\ &= b_n\|T_1^nu_n-u_n\|. \end{aligned} \tag{3.31}$$

It follows from (3.30) and (3.31) that

$$\lim_{n\to\infty}\|v_n-u_n\|=0. \tag{3.32}$$

For $p\in\mathcal{F}$, we conclude from (3.12) and the assumption (5) that $(v_n,p),(p,u_n)\in E(G)$. Thus, by combining this and the transitive property of G, we have $(v_n,u_n)\in E(G)$. Due to the fact that T_2 is a G-asymptotically nonexpansive mapping and $(v_n,u_n)\in E(G)$, we obtain

$$\begin{aligned} \|T_2^nu_n-u_n\| &\leq \|T_2^nu_n-T_2^nv_n\|+\|T_2^nv_n-u_n\| \\ &\leq \lambda_n\|v_n-u_n\|+\|T_2^nv_n-u_n\|. \end{aligned} \tag{3.33}$$

Therefore, we conclude from (3.27), (3.32) and (3.33) that

$$\lim_{n\to\infty}\|T_2^nu_n-u_n\|=0. \tag{3.34}$$

Since $(p,u_n)\in E(G)$ for all $p\in\mathcal{F}$ and $n\in\mathbb{N}$, we have $(p,u_{n+1})\in E(G)$. By combining $(u_n,p),(p,u_{n+1})\in E(G)$ and the transitive property of G, we get $(u_n,u_{n+1})\in E(G)$. For each $i=1,2$, due to the fact that T_i is a G-asymptotically nonexpansive mapping and $(u_n,u_{n+1})\in E(G)$, we obtain

$$\begin{aligned} \|u_{n+1}-T_i^nu_{n+1}\| &\leq \|u_{n+1}-u_n\|+\|u_n-T_i^nu_n\|+\|T_i^nu_n-T_i^nu_{n+1}\| \\ &\leq \|u_{n+1}-u_n\|+\|u_n-T_i^nu_n\|+\lambda_n\|u_n-u_{n+1}\| \\ &= (1+\lambda_n)\|u_{n+1}-u_n\|+\|u_n-T_i^nu_n\|. \end{aligned} \tag{3.35}$$

It follows from (3.21), (3.30), (3.34) and (3.35) that

$$\lim_{n\to\infty}\|u_{n+1}-T_i^nu_{n+1}\|=0. \tag{3.36}$$

Since $(p,u_{n+1})\in E(G)$ for $p\in\mathcal{F}$ and T_i is edge-preserving, we have $(p,T_i^nu_{n+1})\in E(G)$. By combining this with $(u_{n+1},p)\in E(G)$ and using the transitive property of G, we obtain $(u_{n+1},T_i^nu_{n+1})\in E(G)$. Due to the fact that T_i is a G-asymptotically nonexpansive mapping, we get

$$\begin{aligned} \|u_{n+1}-T_iu_{n+1}\| &\leq \|u_{n+1}-T_i^{n+1}u_{n+1}\|+\|T_iu_{n+1}-T_i^{n+1}u_{n+1}\| \\ &\leq \|u_{n+1}-T_i^{n+1}u_{n+1}\|+\lambda_1\|u_{n+1}-T_i^nu_{n+1}\|. \end{aligned} \tag{3.37}$$

Taking the limit in (3.37) as $n \to \infty$ and using (3.30), (3.34), (3.36), we find that

$$\lim_{n\to\infty} \|T_i u_n - u_n\| = 0. \tag{3.38}$$

Therefore, by Lemma 2.7 and (3.38), we find that $T_1 u = T_2 u = u$ and hence $u \in \mathcal{F}$.

Step 6. We claim that $u = q = P_{\mathcal{F}} u_1$. Indeed, since $u_n = P_{\Omega_n} u_1$, we have

$$\langle u_1 - u_n, u_n - y\rangle \geq 0 \quad \text{for all} \quad y \in \Omega_n. \tag{3.39}$$

Let $p \in \mathcal{F}$. Since $\mathcal{F} \subset \Omega_n$, we have $p \in \Omega_n$. Then, from (3.39), we obtain

$$\langle u_1 - u_n, u_n - p\rangle \geq 0. \tag{3.40}$$

Taking the limit in (3.40) as $n \to \infty$ and using $\lim_{n\to\infty} u_n = u$, we find that

$$\langle u_1 - u, u - p\rangle \geq 0.$$

This implies that $u = P_{\mathcal{F}} u_1$. ■

The following theorem shows the convergence of iteration (1.6) to common fixed points of two G-asymptotically nonexpansive mappings in Hilbert spaces with directed graphs.

Theorem 3.3. *Assume that*

(1) H is a real Hilbert space.
(2) Ω is a nonempty closed, convex subset of H and Ω has property (G).
(3) $G = (V(G), E(G))$ is a directed and transitive graph, $V(G) = \Omega$ and $E(G)$ is coordinate-convex.
(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ are two G-asymptotically nonexpansive mappings such that $F(T_i) \times F(T_i) \subset E(G)$ for all $i = 1, 2$.
(5) $\{u_n\}$ is the sequence generated by (1.6) such that $\{a_n\}, \{b_n\} \subset [0, 1]$, and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, $(u_n, p), (p, u_n) \in E(G)$ for all $p \in \mathcal{F}$, and $\sigma_n = (\lambda_n^2 - 1)(a_n + b_n - a_n b_n + a_n b_n \lambda_n^2)(\|u_n\| + \kappa)^2$.

Then the sequence $\{u_n\}$ strongly converges to $P_{\mathcal{F}} u_1$.

Proof. The proof of Theorem 3.3 is divided into six steps.

Step 1. We claim that $P_{\mathcal{F}} u_1$ is well-defined. By using similar arguments as in the proof of Step 1 in Theorem 3.2, we conclude that $P_{\mathcal{F}} u_1$ is well-defined.

Step 2. We claim that $P_{\Omega_{n+1}} u_1$ is well-defined.

First, by using similar arguments as in the proof of Step 2 in Theorem 3.2, we find that Ω_n is closed and convex for $n \in \mathbb{N}$.

Next, we will prove by mathematical induction that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Obviously, for all $p \in \mathcal{F}$, we have $T_1 p = T_2 p = p \in \Omega = \Omega_1$. Therefore, $\mathcal{F} \subset \Omega_1$. We suppose that $\mathcal{F} \subset \Omega_k$ for some $k \in \mathbb{N}$. We will prove that $\mathcal{F} \subset \Omega_{k+1}$. Indeed, for all $p \in \mathcal{F}$, since $(u_k, p) \in E(G)$ and T_1 is edge-preserving, we have $(T_1^k u_k, p) \in E(G)$. Therefore, since $E(G)$ is coordinate-convex, we get

$$(v_k, p) = \big((1 - b_k)u_k + b_k T_1^k u_k, p\big) = (1 - b_k)(u_k, p) + b_k(T_1^k u_k, p) \in E(G). \tag{3.41}$$

Then, using Lemma 2.9 and noting that T_1, T_2 are G-asymptotically nonexpansive mappings, we obtain that

$$\begin{aligned} \|w_k - p\|^2 &= \|(1-a_k)(v_k - p) + a_k(T_2^k v_k - p)\|^2 \\ &= (1-a_k)\|v_k - p\|^2 + a_k\|T_2^k v_k - p\|^2 - a_k(1-a_k)\|T_2^k v_k - v_k\|^2 \\ &\le (1-a_k)\|v_k - p\|^2 + a_k\lambda_k^2\|v_k - p\|^2 \\ &= \big(1 + a_k(\lambda_k^2 - 1)\big)\|v_k - p\|^2 \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} \|v_k - p\|^2 &= \|(1-b_k)(u_k - p) + b_k(T_1^k u_k - p)\|^2 \\ &= (1-b_k)\|u_k - p\|^2 + b_k\|T_1^k u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\le (1-b_k)\|u_k - p\|^2 + b_k\lambda_k^2\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &= \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\le \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2. \end{aligned} \tag{3.43}$$

By substituting (3.43) into (3.42), we obtain

$$\begin{aligned} \|w_k - p\|^2 &\le \big(1 + a_k(\lambda_k^2 - 1)\big)\big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2 \\ &= \|u_k - p\|^2 + (\lambda_k^2 - 1)(a_k + b_k - a_k b_k + a_k b_k \lambda_k^2)\|u_k - p\|^2 \\ &\le \|u_k - p\|^2 + (\lambda_k^2 - 1)(a_k + b_k - a_k b_k + a_k b_k \lambda_k^2)(\|u_k\| + \|p\|)^2 \\ &\le \|u_k - p\|^2 + (\lambda_k^2 - 1)(a_k + b_k - a_k b_k + a_k b_k \lambda_k^2)(\|u_k\| + \kappa)^2 \\ &= \|u_k - p\|^2 + \sigma_k. \end{aligned} \tag{3.44}$$

This implies that $p \in \Omega_{k+1}$ and hence $\mathcal{F} \subset \Omega_{k+1}$. Therefore, we conclude that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F} \neq \emptyset$, we have $\Omega_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, we conclude that $P_{\Omega_{n+1}} u_1$ is well-defined.

Step 3. We claim that $\lim_{n\to\infty} \|u_n - u_1\|$ *exists.* Indeed, by using similar arguments as in the proof of Step 3 in Theorem 3.2, we find that there exists a unique $q = P_{\mathcal{F}} u_1$ and $\lim_{n\to\infty} \|u_n - u_1\|$ exists.

Step 4. We claim that $\lim_{n\to\infty} u_n = u$ *for some* $u \in \Omega$. Indeed, by using similar arguments as in the proof of Step 4 in Theorem 3.2, we find that there exists $u \in \Omega$ such that $\lim_{n\to\infty} u_n = u$ and

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.45}$$

Step 5. We claim that $u \in \mathcal{F}$. Indeed, since $u_{n+1} \in \Omega_{n+1}$, by the definition of Ω_{n+1}, we have

$$\|w_n - u_{n+1}\|^2 \le \|u_n - u_{n+1}\|^2 + \sigma_n. \tag{3.46}$$

Since $\{u_n\}$ is bounded, there exists $B_1 > 0$ such that

$$0 \le \sigma_n = (\lambda_n^2 - 1)(a_n + b_n - a_n b_n + a_n b_n \lambda_n^2)(\|u_n\| + \kappa)^2 \le B_1(\lambda_n^2 - 1)(\lambda_n^2 + 2). \tag{3.47}$$

Taking the limit in (3.47) as $n \to \infty$ and using $\lim_{n\to\infty} \lambda_n = 1$, we get $\lim_{n\to\infty} \sigma_n = 0$. Then, from (3.45) and (3.46), we obtain

$$\lim_{n\to\infty} \|w_n - u_{n+1}\| = 0. \tag{3.48}$$

Therefore, from (3.45), (3.48) and $\|w_n - u_n\| \leq \|w_n - u_{n+1}\| + \|u_{n+1} - u_n\|$, we obtain

$$\lim_{n\to\infty} \|w_n - u_n\| = 0. \tag{3.49}$$

Next, for $p \in \mathcal{F}$, by using similar arguments as in the proof of (3.42), (3.43) and (3.44), we obtain

$$\begin{aligned}
\|w_n - p\|^2 &\leq \big(1 + a_n(\lambda_n^2 - 1)\big)\|v_n - p\|^2 \\
&\leq \big(1+a_n(\lambda_n^2 - 1)\big)\Big(\big(1 + b_n(\lambda_n^2 - 1)\big)\|u_n - p\|^2 - b_n(1 - b_n)\|T_1^n u_n - u_n\|^2\Big) \\
&= \|u_n - p\|^2 + (\lambda_n^2 - 1)(a_n + b_n - a_n b_n + a_n b_n \lambda_n^2)\|u_n - p\|^2 \\
&\quad -(1 - a_n + a_n\lambda_n^2)b_n(1 - b_n)\|T_1^n u_n - u_n\|^2 \\
&\leq \|u_n - p\|^2 + \sigma_n - (1 - a_n + a_n\lambda_n^2)b_n(1 - b_n)\|T_1^n u_n - u_n\|^2 \\
&\leq \|u_n - p\|^2 + \sigma_n - (1 - a_n)b_n(1 - b_n)\|T_1^n u_n - u_n\|^2. \qquad (3.50)
\end{aligned}$$

Moreover, by the boundedness property of $\{u_n\}$ and $\{w_n\}$, we conclude that there exists $B_2 > 0$ such that $\|u_n\| + \|w_n\| \leq B_2$ for all $n \in \mathbb{N}$. It follows from (3.50) that

$$\begin{aligned}
&(1 - a_n)b_n(1 - b_n)\|T_1^n u_n - u_n\|^2 \\
\leq\ & \|u_n - p\|^2 - \|w_n - p\|^2 + \sigma_n \\
=\ & \|u_n\|^2 - \|w_n\|^2 + 2\langle w_n - u_n, p\rangle + \sigma_n \\
\leq\ & (\|u_n\| - \|w_n\|)(\|u_n\| + \|w_n\|) + 2\|w_n - u_n\|.\|p\| + \sigma_n \\
\leq\ & \|u_n - w_n\|(\|u_n\| + \|w_n\|) + 2\|w_n - u_n\|.\|p\| + \sigma_n \\
\leq\ & B_2\|u_n - w_n\| + 2\|w_n - u_n\|.\|p\| + \sigma_n. \qquad (3.51)
\end{aligned}$$

Then, by combining (3.49) with (3.51) and using $\lim\limits_{n\to\infty} \sigma_n = 0$, $\liminf\limits_{n\to\infty}(1-a_n)b_n(1-b_n) > 0$, we conclude that

$$\lim_{n\to\infty} \|T_1^n u_n - u_n\| = 0. \tag{3.52}$$

Moreover, we have

$$\begin{aligned}
\|v_n - u_n\| &= \|(1 - b_n)u_n + b_n T_1^n u_n - u_n\| \\
&= b_n\|T_1^n u_n - u_n\|. \qquad (3.53)
\end{aligned}$$

It follows from (3.52) and (3.53), we find that

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.54}$$

We also have

$$\|v_n - w_n\| \leq \|v_n - u_n\| + \|u_n - w_n\|. \tag{3.55}$$

It follows from (3.49), (3.54) and (3.55) that

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{3.56}$$

We have

$$\begin{aligned}
\|w_n - v_n\| &= \|(1 - a_n)v_n + a_n T_2^n v_n - v_n\| \\
&= a_n\|T_2^n v_n - v_n\|. \qquad (3.57)
\end{aligned}$$

Therefore, from (3.56), (3.57) and $\liminf\limits_{n\to\infty} a_n > 0$, we obtain

$$\lim_{n\to\infty} \|T_2^n v_n - v_n\| = 0. \tag{3.58}$$

For $p \in \mathcal{F}$, from (3.41) and the assumption (5), we have $(v_n, p), (p, u_n) \in E(G)$. Then, by the transitive property of G, we get that $(v_n, u_n) \in E(G)$. Since T_2 is a G-asymptotically nonexpansive mapping and $(v_n, u_n) \in E(G)$, we obtain

$$\begin{aligned} \|T_2^n u_n - u_n\| &\leq \|T_2^n u_n - T_2^n v_n\| + \|T_2^n v_n - v_n\| + \|v_n - u_n\| \\ &\leq \lambda_n \|u_n - v_n\| + \|T_2^n v_n - v_n\| + \|v_n - u_n\| \\ &= (1 + \lambda_n)\|v_n - u_n\| + \|T_2^n v_n - v_n\|. \end{aligned} \tag{3.59}$$

Thus, we conclude from (3.54), (3.58) and (3.59) that

$$\lim_{n\to\infty} \|T_2^n u_n - u_n\| = 0. \tag{3.60}$$

Next, by (3.52), (3.60), and using similar arguments as in the proof of Step 5 in Theorem 3.2, we find that $T_1 u = T_2 u = u$ and hence $u \in \mathcal{F}$.

Step 6. We claim that $u = q = P_{\mathcal{F}} u_1$. Indeed, by using similar arguments as in the proof of Step 6 in Theorem 3.2, we conclude that $u = P_{\mathcal{F}} u_1$. ∎

The following theorem shows the convergence of iteration (1.7) to common fixed points of two G-asymptotically nonexpansive mappings in Hilbert spaces with directed graphs.

Theorem 3.4. *Assume that*

(1) H is a real Hilbert space.
(2) Ω is a nonempty closed, convex subset of H and Ω has property (G).
(3) $G = (V(G), E(G))$ is a directed and transitive graph, $V(G) = \Omega$ and $E(G)$ is coordinate-convex.
(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ are two G-asymptotically nonexpansive mappings such that $F(T_i) \times F(T_i) \subset E(G)$ for all $i = 1, 2$.
(5) $\{u_n\}$ is the sequence generated by (1.7) such that $\{a_n\}, \{b_n\} \subset [0, 1]$, and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, $(u_n, p), (p, u_n) \in E(G)$ for all $p \in \mathcal{F}$, and $\varepsilon_n = (\lambda_n^2 - 1)(1 + a_n b_n \lambda_n^2)(\|u_n\| + \kappa)^2$.

Then the sequence $\{u_n\}$ strongly converges to $P_{\mathcal{F}} u_1$.

Proof. The proof of Theorem 3.4 is divided into six steps.

Step 1. We claim that $P_{\mathcal{F}} u_1$ is well-defined. Indeed, by using similar arguments as in the proof of Step 1 in Theorem 3.2, we find that $P_{\mathcal{F}} u_1$ is well-defined.

Step 2. We claim that $P_{\Omega_{n+1}} u_1$ is well-defined.

First, by using similar arguments as in the proof of Step 2 in Theorem 3.2, we find that Ω_n is closed and convex for $n \in \mathbb{N}$.

Next, we will prove by mathematical induction that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Obviously, for all $p \in \mathcal{F}$, we have $T_1 p = T_2 p = p \in \Omega = \Omega_1$. Therefore, $\mathcal{F} \subset \Omega_1$. We suppose that $\mathcal{F} \subset \Omega_k$ for some $k \in \mathbb{N}$. We will prove that $\mathcal{F} \subset \Omega_{k+1}$. Indeed, for all $p \in \mathcal{F}$, since $(u_k, p) \in E(G)$ and T_1 is edge-preserving, we have $(T_1^k u_k, p) \in E(G)$. Due to the fact that $E(G)$ is coordinate-convex, we obtain

$$\begin{aligned} (v_k, p) &= \big((1 - b_k)u_k + b_k T_1^k u_k, p\big) \\ &= (1 - b_k)(u_k, p) + b_k (T_1^k u_k, p) \in E(G). \end{aligned} \tag{3.61}$$

Then, using Lemma 2.9 and noting that T_1, T_2 are G-asymptotically nonexpansive mappings, we get

$$\begin{aligned}\|w_k - p\|^2 &= \|(1-a_k)(T_1^k u_k - p) + a_k(T_2^k v_k - p)\|^2 \\ &= (1-a_k)\|T_1^k u_k - p\|^2 + a_k\|T_2^k v_k - p\|^2 - a_k(1-a_k)\|T_2^k v_k - T_1^k u_k\|^2 \\ &\le (1-a_k)\lambda_k^2\|u_k - p\|^2 + a_k\lambda_k^2\|v_k - p\|^2 - a_k(1-a_k)\|T_2^k v_k - T_1^k u_k\|^2 \\ &\le (1-a_k)\lambda_k^2\|u_k - p\|^2 + a_k\lambda_k^2\|v_k - p\|^2 \qquad (3.62)\end{aligned}$$

and

$$\begin{aligned}\|v_k - p\|^2 &= \|(1-b_k)(u_k - p) + b_k(T_1^k u_k - p)\|^2 \\ &= (1-b_k)\|u_k - p\|^2 + b_k\|T_1^k u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\le (1-b_k)\|u_k - p\|^2 + b_k\lambda_k^2\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &= \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\le \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2. \qquad (3.63)\end{aligned}$$

By substituting (3.63) into (3.62), we get

$$\begin{aligned}\|w_k - p\|^2 &\le (1-a_k)\lambda_k^2\|u_k - p\|^2 + a_k\lambda_k^2[1 + b_k(\lambda_k^2 - 1)]\|u_k - p\|^2 \\ &= \big(\lambda_k^2 + a_k b_k \lambda_k^2(\lambda_k^2 - 1)\big)\|u_k - p\|^2 \\ &\le \|u_k - p\|^2 + (\lambda_k^2 - 1)(1 + a_k b_k \lambda_k^2)(\|u_k\| + \|p\|)^2 \\ &\le \|u_k - p\|^2 + (\lambda_k^2 - 1)(1 + a_k b_k \lambda_k^2)(\|u_k\| + \kappa)^2 \\ &= \|u_k - p\|^2 + \varepsilon_k. \qquad (3.64)\end{aligned}$$

This implies that $p \in \Omega_{k+1}$ and hence $\mathcal{F} \subset \Omega_{k+1}$. Therefore, we conclude that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F} \neq \emptyset$, we have $\Omega_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, we conclude that $P_{\Omega_{n+1}} u_1$ is well-defined.

Step 3. We claim that $\lim_{n\to\infty} \|u_n - u_1\|$ *exists.* Indeed, by using similar arguments as in the proof of Step 3 in Theorem 3.2, we find that there exists a unique $q = P_{\mathcal{F}} u_1$ and $\lim_{n\to\infty} \|u_n - u_1\|$ exists.

Step 4. We claim that $\lim_{n\to\infty} u_n = u$ *for some* $u \in \Omega$. Indeed, by using similar arguments as in the proof of Step 4 in Theorem 3.2, we conclude that there exists $u \in \Omega$ such that $\lim_{n\to\infty} u_n = u$ and

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \qquad (3.65)$$

Step 5. We claim that $u \in \mathcal{F}$. Indeed, since $u_{n+1} \in \Omega_{n+1}$, by the definition of Ω_{n+1}, we obtain

$$\|w_n - u_{n+1}\|^2 \le \|u_n - u_{n+1}\|^2 + \varepsilon_n. \qquad (3.66)$$

Since $\{u_n\}$ is bounded, there exists $C_1 > 0$ such that

$$0 \le \varepsilon_n = (\lambda_n^2 - 1)(1 + a_n b_n \lambda_n^2)(\|u_n\| + \kappa)^2 \le C_1(\lambda_n^2 - 1)(1 + \lambda_n^2). \qquad (3.67)$$

Taking the limit in (3.67) as $n \to \infty$ and using $\lim_{n\to\infty} \lambda_n = 1$, we get $\lim_{n\to\infty} \varepsilon_n = 0$. Then, from (3.65) and (3.66), we obtain

$$\lim_{n\to\infty} \|w_n - u_{n+1}\| = 0. \qquad (3.68)$$

Then, we conclude from (3.65), (3.68) and $\|w_n - u_n\| \leq \|w_n - u_{n+1}\| + \|u_{n+1} - u_n\|$ that

$$\lim_{n\to\infty} \|w_n - u_n\| = 0. \tag{3.69}$$

Next, for $p \in \mathcal{F}$, by using similar arguments as in the proof of (3.62), (3.63) and (3.64), we obtain

$$\begin{aligned}
\|w_n - p\|^2 &\leq (1-a_n)\lambda_n^2\|u_n - p\|^2 + a_n\lambda_n^2\|v_n - p\|^2 \\
&\leq (1-a_n)\lambda_n^2\|u_n - p\|^2 + a_n\lambda_n^2\Big([1 + b_n(\lambda_n^2 - 1)]\|u_n - p\|^2 \\
&\quad -b_n(1-b_n)\|T_1^n u_n - u_n\|^2\Big) \\
&= \|u_n - p\|^2 + (\lambda_n^2 - 1)(1 + a_n b_n \lambda_n^2)\|u_n - p\|^2 - \lambda_n^2 a_n b_n(1-b_n)\|T_1^n u_n - u_n\|^2 \\
&\leq \|u_n - p\|^2 + \varepsilon_n - \lambda_n^2 a_n b_n(1-b_n)\|T_1^n u_n - u_n\|^2 \\
&\leq \|u_n - p\|^2 + \varepsilon_n - a_n b_n(1-b_n)\|T_1^n u_n - u_n\|^2.
\end{aligned} \tag{3.70}$$

Moreover, by the boundedness property of $\{u_n\}$ and $\{w_n\}$, we find that there exists $C_2 > 0$ such that $\|u_n\| + \|w_n\| \leq C_2$ for all $n \in \mathbb{N}$. It follows from (3.70) that

$$\begin{aligned}
a_n b_n(1-b_n)\|T_1^n u_n - u_n\|^2 &\leq \|u_n - p\|^2 - \|w_n - p\|^2 + \varepsilon_n \\
&= \|u_n\|^2 - \|w_n\|^2 + 2\langle w_n - u_n, p\rangle + \varepsilon_n \\
&\leq (\|u_n\| - \|w_n\|)(\|u_n\| + \|w_n\|) + 2\|w_n - u_n\|.\|p\| + \varepsilon_n \\
&\leq \|u_n - w_n\|(\|u_n\| + \|w_n\|) + 2\|w_n - u_n\|.\|p\| + \varepsilon_n \\
&\leq C_2\|u_n - w_n\| + 2\|w_n - u_n\|.\|p\| + \varepsilon_n.
\end{aligned} \tag{3.71}$$

Then, by combining (3.71) with (3.69) and using $\lim_{n\to\infty} \varepsilon_n = 0$, $\liminf_{n\to\infty} a_n b_n(1-b_n) > 0$, we get

$$\lim_{n\to\infty} \|T_1^n u_n - u_n\| = 0. \tag{3.72}$$

By using similar arguments as in the proof of (3.62), (3.63) and (3.64), we obtain

$$\begin{aligned}
\|w_n - p\|^2 &\leq (1-a_n)\lambda_n^2\|u_n - p\|^2 + a_n\lambda_n^2\|v_n - p\|^2 - a_n(1-a_n)\|T_2^n v_n - T_1^n u_n\|^2 \\
&\leq (1-a_n)\lambda_n^2\|u_n - p\|^2 + a_n\lambda_n^2[1 + b_n(\lambda_n^2 - 1)]\|u_n - p\|^2 \\
&\quad -a_n(1-a_n)\|T_2^n v_n - T_1^n u_n\|^2 \\
&= \|u_n - p\|^2 + \varepsilon_n - a_n(1-a_n)\|T_2^n v_n - T_1^n u_n\|^2.
\end{aligned} \tag{3.73}$$

Therefore, by the boundedness property of $\{u_n\}$, $\{w_n\}$ and using similar arguments as in the proof of (3.71), from (3.73), we find that

$$\begin{aligned}
a_n(1-a_n)\|T_2^n v_n - T_1^n u_n\|^2 &\leq \|u_n - p\|^2 - \|w_n - p\|^2 + \varepsilon_n \\
&\leq C_2\|u_n - w_n\| + 2\|w_n - u_n\|.\|p\| + \varepsilon_n.
\end{aligned} \tag{3.74}$$

Then, by combining (3.74) with (3.69) and using $\lim_{n\to\infty} \varepsilon_n = 0$, $\liminf_{n\to\infty} a_n(1-a_n) > 0$, we get

$$\lim_{n\to\infty} \|T_2^n v_n - T_1^n u_n\| = 0. \tag{3.75}$$

We also have

$$\|v_n - u_n\| = \|(1-b_n)u_n + b_n T_1^n u_n - u_n\| = b_n\|T_1^n u_n - u_n\|. \tag{3.76}$$

It follows from (3.72) and (3.76) that

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.77}$$

We have

$$\|T_2^n v_n - v_n\| \le \|T_2^n v_n - T_1^n u_n\| + \|T_1^n u_n - u_n\| + \|u_n - v_n\|. \tag{3.78}$$

Therefore, we conclude from (3.72), (3.75), (3.77) and (3.78) that

$$\lim_{n\to\infty} \|T_2^n v_n - v_n\| = 0. \tag{3.79}$$

For $p \in \mathcal{F}$, from (3.61) and the assumption (5), we get $(v_n, p), (p, u_n) \in E(G)$. Then, by transitive property of G, we obtain $(v_n, u_n) \in E(G)$. Since T_2 is a G-asymptotically nonexpansive mapping and $(v_n, u_n) \in E(G)$, we have

$$\begin{aligned}\|T_2^n u_n - u_n\| &\le \|T_2^n u_n - T_2^n v_n\| + \|T_2^n v_n - v_n\| + \|v_n - u_n\| \\ &\le \lambda_n \|v_n - u_n\| + \|T_2^n v_n - v_n\| + \|v_n - u_n\| \\ &= (1+\lambda_n)\|v_n - u_n\| + \|T_2^n v_n - v_n\|. \end{aligned} \tag{3.80}$$

It follows from (3.77), (3.79) and (3.80) that

$$\lim_{n\to\infty} \|T_2^n u_n - u_n\| = 0. \tag{3.81}$$

Next, by (3.72), (3.81), and using similar arguments as in the proof of Step 5 in Theorem 3.2, we find that $T_1 u = T_2 u = u$ and hence $u \in \mathcal{F}$.

Step 6. We claim that $u = q = P_{\mathcal{F}} u_1$. Indeed, by using similar arguments as in the proof of Step 6 in Theorem 3.2, we conclude that $u = P_{\mathcal{F}} u_1$. ■

The following theorem shows the convergence of iteration (1.8) to common fixed points of two G-asymptotically nonexpansive mappings in Hilbert spaces with directed graphs.

Theorem 3.5. *Assume that*

(1) H is a real Hilbert space.

(2) Ω is a nonempty closed, convex subset of H and Ω has property (G).

(3) $G = (V(G), E(G))$ is a directed and transitive graph, $V(G) = \Omega$ and $E(G)$ is coordinate-convex.

(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ are two G-asymptotically nonexpansive mappings such that $F(T_i) \times F(T_i) \subset E(G)$ for all $i = 1, 2$.

(5) $\{u_n\}$ is the sequence generated by (1.8) such that $\{a_n\}, \{b_n\} \subset [0, 1]$, and $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$, $(u_n, p), (p, u_n) \in E(G)$ for all $p \in \mathcal{F}$, and $\gamma_n = (\lambda_n^2 - 1)(1 + b_n\lambda_n^2)(\|u_n\| + \kappa)^2$.

Then the sequence $\{u_n\}$ strongly converges to $P_{\mathcal{F}} u_1$.

Proof. The proof of Theorem 3.5 is divided into six steps.

Step 1. We claim that $P_{\mathcal{F}} u_1$ is well-defined. Indeed, by using similar arguments as in the proof of Step 1 in Theorem 3.2, we conclude that $P_{\mathcal{F}} u_1$ is well-defined.

Step 2. We claim that $P_{\Omega_{n+1}} u_1$ is well-defined.

First, by using similar arguments as in the proof of Step 2 in Theorem 3.2, we find that Ω_n is closed and convex for $n \in \mathbb{N}$.

Next, we will prove by mathematical induction that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Obviously, for all $p \in \mathcal{F}$, we have $T_1 p = T_2 p = p \in \Omega = \Omega_1$. Therefore, $\mathcal{F} \subset \Omega_1$. We suppose that $\mathcal{F} \subset \Omega_k$ for some $k \in \mathbb{N}$. We will prove that $\mathcal{F} \subset \Omega_{k+1}$. Indeed, for all $p \in \mathcal{F}$, since

$(u_k, p) \in E(G)$ and T_1 is edge-preserving, we have $(T_1^k u_k, p) \in E(G)$. Due to the fact that $E(G)$ is coordinate-convex, we get

$$\begin{aligned} (v_k, p) &= \big((1-b_k)u_k + b_k T_1^k u_k, p\big) \\ &= (1-b_k)(u_k, p) + b_k(T_1^k u_k, p) \in E(G). \end{aligned} \tag{3.82}$$

Then, using Lemma 2.9 and noting that T_1, T_2 are G-asymptotically nonexpansive mappings, we find that

$$\begin{aligned} \|w_k - p\|^2 &= \|(1-a_k)(T_1^k v_k - p) + a_k(T_2^k v_k - p)\|^2 \\ &= (1-a_k)\|T_1^k v_k - p\|^2 + a_k\|T_2^k v_k - p\|^2 - a_k(1-a_k)\|T_2^k v_k - T_1^k v_k\|^2 \\ &\leq (1-a_k)\lambda_k^2\|v_k - p\|^2 + a_k\lambda_k^2\|v_k - p\|^2 - a_k(1-a_k)\|T_2^k v_k - T_1^k v_k\|^2 \\ &= \lambda_k^2\|v_k - p\|^2 - a_k(1-a_k)\|T_2^k v_k - T_1^k v_k\|^2 \\ &\leq \lambda_k^2\|v_k - p\|^2 \end{aligned} \tag{3.83}$$

and

$$\begin{aligned} \|v_k - p\|^2 &= \|(1-b_k)(u_k - p) + b_k(T_1^k u_k - p)\|^2 \\ &= (1-b_k)\|u_k - p\|^2 + b_k\|T_1^k u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\leq (1-b_k)\|u_k - p\|^2 + b_k\lambda_k^2\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &= \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2 - b_k(1-b_k)\|T_1^k u_k - u_k\|^2 \\ &\leq \big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2. \end{aligned} \tag{3.84}$$

By substituting (3.84) into (3.83), we obtain

$$\begin{aligned} \|w_k - p\|^2 &\leq \lambda_k^2\|v_k - p\|^2 \\ &= \lambda_k^2\big(1 + b_k(\lambda_k^2 - 1)\big)\|u_k - p\|^2 \\ &= \|u_k - p\|^2 + (\lambda_k^2 - 1)(1 + b_k\lambda_k^2)\|u_k - p\|^2 \\ &\leq \|u_k - p\|^2 + (\lambda_k^2 - 1)(1 + b_k\lambda_k^2)(\|u_k\| + \|p\|)^2 \\ &\leq \|u_k - p\|^2 + (\lambda_k^2 - 1)(1 + b_k\lambda_k^2)(\|u_k\| + \kappa)^2 \\ &= \|u_k - p\|^2 + \gamma_k. \end{aligned} \tag{3.85}$$

This implies that $p \in \Omega_{k+1}$ and hence $\mathcal{F} \subset \Omega_{k+1}$. Therefore, we conclude that $\mathcal{F} \subset \Omega_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F} \neq \emptyset$, we have $\Omega_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, we conclude that $P_{\Omega_{n+1}} u_1$ is well-defined.

Step 3. We claim that $\lim_{n\to\infty} \|u_n - u_1\|$ *exists.* Indeed, by using similar arguments as in the proof of Step 3 in Theorem 3.2, we conclude that there exists a unique $q = P_{\mathcal{F}} u_1$ and $\lim_{n\to\infty} \|u_n - u_1\|$ exists.

Step 4. We claim that $\lim_{n\to\infty} u_n = u$ *for some* $u \in \Omega$. Indeed, by using similar arguments as in the proof of Step 4 in Theorem 3.2, we find that there exists $u \in \Omega$ such that $\lim_{n\to\infty} u_n = u$ and

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.86}$$

Step 5. We claim that $u \in \mathcal{F}$. Indeed, since $u_{n+1} \in \Omega_{n+1}$, by the definition of Ω_{n+1}, we have

$$\|w_n - u_{n+1}\|^2 \leq \|u_n - u_{n+1}\|^2 + \gamma_n. \tag{3.87}$$

Since $\{u_n\}$ is bounded, there exists $D_1 > 0$ such that

$$0 \leq \gamma_n = (\lambda_n^2 - 1)(1 + b_n\lambda_n^2)(\|u_n\| + \kappa)^2 \leq D_1(\lambda_n^2 - 1)(1 + \lambda_n^2). \tag{3.88}$$

Taking the limit in (3.88) as $n \to \infty$ and using $\lim_{n\to\infty} \lambda_n = 1$, we get $\lim_{n\to\infty} \gamma_n = 0$. Then, from (3.86) and (3.87), we obtain

$$\lim_{n\to\infty} \|w_n - u_{n+1}\| = 0. \tag{3.89}$$

Therefore, from (3.86), (3.89) and $\|w_n - u_n\| \leq \|w_n - u_{n+1}\| + \|u_{n+1} - u_n\|$, we find that

$$\lim_{n\to\infty} \|w_n - u_n\| = 0. \tag{3.90}$$

Next, for $p \in \mathcal{F}$, by using similar arguments as in the proof of (3.83), (3.84) and (3.85), we obtain

$$\begin{aligned} \|w_n - p\|^2 &\leq \lambda_n^2\|v_n - p\|^2 \\ &\leq \lambda_n^2\big(1 + b_n(\lambda_n^2 - 1)\big)\|u_n - p\|^2 - \lambda_n^2 b_n(1 - b_n)\|T_1^n u_n - u_n\|^2 \\ &\leq \|u_n - p\|^2 + (\lambda_n^2 - 1)(1 + b_n\lambda_n^2)\|u_n - p\|^2 - b_n(1 - b_n)\|T_1^n u_n - u_n\|^2 \\ &\leq \|u_n - p\|^2 + \gamma_n - b_n(1 - b_n)\|T_1^n u_n - u_n\|^2. \end{aligned} \tag{3.91}$$

Moreover, by the boundedness property of $\{u_n\}$ and $\{w_n\}$, we conclude that there exists $D_2 > 0$ such that $\|u_n\| + \|w_n\| \leq D_2$ for all $n \in \mathbb{N}$. It follows from (3.91) that

$$\begin{aligned} b_n(1 - b_n)\|T_1^n u_n - u_n\|^2 &\leq \|u_n - p\|^2 - \|w_n - p\|^2 + \gamma_n \\ &= \|u_n\|^2 - \|w_n\|^2 + 2\langle w_n - u_n, p\rangle + \gamma_n \\ &\leq (\|u_n\| - \|w_n\|)(\|u_n\| + \|w_n\|) + 2\|w_n - u_n\|.\|p\| + \gamma_n \\ &\leq \|u_n - w_n\|(\|u_n\| + \|w_n\|) + 2\|w_n - u_n\|.\|p\| + \gamma_n \\ &\leq D_2\|u_n - w_n\| + 2\|w_n - u_n\|.\|p\| + \gamma_n. \end{aligned} \tag{3.92}$$

Therefore, by combining (3.92) with (3.90) and using $\lim_{n\to\infty} \gamma_n = 0$, $\liminf_{n\to\infty} b_n(1 - b_n) > 0$, we get

$$\lim_{n\to\infty} \|T_1^n u_n - u_n\| = 0. \tag{3.93}$$

By using similar arguments as in the proof of (3.83), (3.84) and (3.85), we have

$$\begin{aligned} \|w_n - p\|^2 &\leq \lambda_n^2\|v_n - p\|^2 - a_n(1 - a_n)\|T_2^n v_n - T_1^n v_n\|^2 \\ &\leq \lambda_n^2[1 + b_n(\lambda_n^2 - 1)]\|u_n - p\|^2 - a_n(1 - a_n)\|T_2^n v_n - T_1^n v_n\|^2 \\ &= \|u_n - p\|^2 + (\lambda_n^2 - 1)(1 + b_n\lambda_2^n)\|u_n - p\|^2 - a_n(1 - a_n)\|T_2^n v_n - T_1^n v_n\|^2 \\ &\leq \|u_n - p\|^2 + \gamma_n - a_n(1 - a_n)\|T_2^n v_n - T_1^n v_n\|^2. \end{aligned} \tag{3.94}$$

Furthermore, by the boundedness property of $\{u_n\}$, $\{w_n\}$ and using similar arguments as in the proof of (3.92), from (3.94), we find that

$$\begin{aligned} a_n(1 - a_n)\|T_2^n v_n - T_1^n v_n\|^2 &\leq \|u_n - p\|^2 - \|w_n - p\|^2 + \gamma_n \\ &\leq D_2\|u_n - w_n\| + 2\|w_n - u_n\|.\|p\| + \gamma_n. \end{aligned} \tag{3.95}$$

Then, by combining (3.90) with (3.95) and using $\lim_{n\to\infty} \gamma_n = 0$, $\liminf_{n\to\infty} a_n(1 - a_n) > 0$, we get

$$\lim_{n\to\infty} \|T_2^n v_n - T_1^n v_n\| = 0. \tag{3.96}$$

We have

$$\|v_n - u_n\| = \|(1-b_n)u_n + b_n T_1^n u_n - u_n\| = b_n\|T_1^n u_n - u_n\|. \tag{3.97}$$

Therefore, we conclude from (3.93) and (3.97) that

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.98}$$

For $p \in \mathcal{F}$, from (3.82) and the assumption (5), we have $(v_n, p), (p, u_n) \in E(G)$. Then, by the transitive property of G, we obtain $(v_n, u_n) \in E(G)$. Since T_1 is a G-asymptotically nonexpansive mapping and $(v_n, u_n) \in E(G)$, we get

$$\begin{aligned} \|T_1^n v_n - v_n\| &\le \|T_1^n v_n - T_1^n u_n\| + \|T_1^n u_n - u_n\| + \|u_n - v_n\| \\ &\le \lambda_n\|v_n - u_n\| + \|T_1^n u_n - u_n\| + \|u_n - v_n\| \\ &= (1+\lambda_n)\|v_n - u_n\| + \|T_1^n u_n - u_n\| \end{aligned} \tag{3.99}$$

It follows from (3.98), (3.93) and (3.99) that

$$\lim_{n\to\infty} \|T_1^n v_n - v_n\| = 0. \tag{3.100}$$

We have

$$\|T_2^n v_n - v_n\| \le \|T_2^n v_n - T_1^n v_n\| + \|T_1^n v_n - v_n\|. \tag{3.101}$$

Therefore, from (3.96), (3.100) and (3.101), we get

$$\lim_{n\to\infty} \|T_2^n v_n - v_n\| = 0. \tag{3.102}$$

By combining $(v_n, p), (p, u_n) \in E(G)$ and the transitive property of G, we conclude that $(v_n, u_n) \in E(G)$. Since T_2 is a G-asymptotically nonexpansive mapping and $(v_n, u_n) \in E(G)$, we have

$$\begin{aligned} \|T_2^n u_n - u_n\| &\le \|T_2^n u_n - T_2^n v_n\| + \|T_2^n v_n - v_n\| + \|v_n - u_n\| \\ &\le \lambda_n\|v_n - u_n\| + \|T_2^n v_n - v_n\| + \|v_n - u_n\| \\ &= (1+\lambda_n)\|v_n - u_n\| + \|T_2^n v_n - v_n\|. \end{aligned} \tag{3.103}$$

Then, we conclude from (3.98), (3.102) and (3.103) that

$$\lim_{n\to\infty} \|T_2^n u_n - u_n\| = 0. \tag{3.104}$$

Next, by (3.93), (3.104), and using similar arguments as in the proof of Step 5 in Theorem 3.2, we find that $T_1 u = T_2 u = u$ and hence $u \in \mathcal{F}$.

Step 6. We claim that $u = q = P_{\mathcal{F}} u_1$. Indeed, by using similar arguments as in the proof of Step 6 in Theorem 3.2, we find that $u = P_{\mathcal{F}} u_1$. ■

Since every G-nonexpansive mapping is a G-asymptotically nonexpansive mapping with the asymptotic coefficient $\lambda_n = 1$ for all $n \in \mathbb{N}$, by Theorem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5 we get the following corollaries. Note that these corollaries are the improvement of [25, Theorem 1], [25, Theorem 2], [25, Theorem 3] and [25, Theorem 4] in the sense that the convexity of $E(G)$ is replaced by coordinate-convexity.

Corollary 3.6. *Assume that*

(1) H is a real Hilbert space.

(2) Ω is a nonempty closed, convex subset of H and Ω has property (G).

(3) $G = (V(G), E(G))$ is a directed and transitive graph, $V(G) = \Omega$ and $E(G)$ is coordinate-convex.

(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ *are two G-nonexpansive mappings such that* $\mathcal{F} = F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_i) \times F(T_i) \subset E(G)$ *for all* $i = 1, 2$.

(5) $\{u_n\}$ *is the sequence generated by* (1.1) *such that* $\{a_n\}, \{b_n\} \subset [0,1]$, *and* $\liminf_{n\to\infty} a_n > 0$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and* $(u_n, p), (p, u_n) \in E(G)$ *for all* $p \in \mathcal{F}$.

Then the sequence $\{u_n\}$ *strongly converges to* $P_{\mathcal{F}} u_1$.

Corollary 3.7. *Assume that*

(1) H *is a real Hilbert space.*

(2) Ω *is a nonempty closed, convex subset of* H *and* Ω *has property* (G).

(3) $G = (V(G), E(G))$ *is a directed and transitive graph,* $V(G) = \Omega$ *and* $E(G)$ *is coordinate-convex.*

(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ *are two G-nonexpansive mappings such that* $\mathcal{F} = F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_i) \times F(T_i) \subset E(G)$ *for all* $i = 1, 2$.

(5) $\{u_n\}$ *is the sequence generated by* (1.2) *such that* $\{a_n\}, \{b_n\} \subset [0,1]$, *and* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, $(u_n, p), (p, u_n) \in E(G)$ *for all* $p \in \mathcal{F}$.

Then the sequence $\{u_n\}$ *strongly converges to* $P_{\mathcal{F}} u_1$.

Corollary 3.8. *Assume that*

(1) H *is a real Hilbert space.*

(2) Ω *is a nonempty closed, convex subset of* H *and* Ω *has property* (G).

(3) $G = (V(G), E(G))$ *is a directed and transitive graph,* $V(G) = \Omega$ *and* $E(G)$ *is coordinate-convex.*

(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ *are two G-nonexpansive mappings such that* $\mathcal{F} = F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_i) \times F(T_i) \subset E(G)$ *for all* $i = 1, 2$.

(5) $\{u_n\}$ *is the sequence generated by* (1.3) *such that* $\{a_n\}, \{b_n\} \subset [0,1]$, *and* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, $(u_n, p), (p, u_n) \in E(G)$ *for all* $p \in \mathcal{F}$.

Then the sequence $\{u_n\}$ *strongly converges to* $P_{\mathcal{F}} u_1$.

Corollary 3.9. *Assume that*

(1) H *is a real Hilbert space.*

(2) Ω *is a nonempty closed, convex subset of* H *and* Ω *has property* (G).

(3) $G = (V(G), E(G))$ *is a directed and transitive graph,* $V(G) = \Omega$ *and* $E(G)$ *is coordinate-convex.*

(4) $T_1, T_2 : \Omega \longrightarrow \Omega$ *are two G-nonexpansive mappings such that* $\mathcal{F} = F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_i) \times F(T_i) \subset E(G)$ *for all* $i = 1, 2$.

(5) $\{u_n\}$ *is the sequence generated by* (1.4) *such that* $\{a_n\}, \{b_n\} \subset [0,1]$, *and* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, $(u_n, p), (p, u_n) \in E(G)$ *for all* $p \in \mathcal{F}$.

Then the sequence $\{u_n\}$ *strongly converges to* $P_{\mathcal{F}} u_1$.

Finally, we give a numerical example to illustrate for the convergence of the proposed iteration processes to common fixed points two asymptotically G-nonexpansive mappings.

In addition, the example also shows that the convergence of the proposed iteration processes to common fixed points of given mappings are faster than known others.

Example 3.10. Let $H = \mathbb{R}$, $\Omega = [0.5, 2]$, and $G = (V(G), E(G))$ be a directed graph defined by $V(G) = \Omega, (u, v) \in E(G)$ if and only if $0.5 \leq u \neq v \leq 1.7$ or $u = v \in \Omega$. Then $E(G)$ is coordinate-convex and $\{(u, u) : u \in V(G)\} \subset E(G)$. Define two mappings $T_1, T_2 : \Omega \longrightarrow \Omega$ by

$$T_1 u = \frac{20}{31}\arcsin(u-1) + 1, T_2 u = u^{\ln u} \text{ for all } u \in \Omega.$$

Consider $a_n = \dfrac{n+2}{4n+5}$ and $b_n = \dfrac{n+1}{3n+7}$ for all $n \in \mathbb{N}$. Then T_1, T_2 are two asymptotically G-nonexpansive mappings with $\lambda_n = 1$ for all $n \in \mathbb{N}$. Indeed, for all $(u, v) \in E(G)$, we only consider $0.5 \leq u, v \leq 1.7$. Thus, for each $i = 1, 2$ we get $0.5 \leq T_i u, T_i v \leq 1.7$ and hence $(T_i u, T_i v) \in E(G)$. This implies that T_1, T_2 are edge-preserving. Moreover, by calculating directly, we conclude that $\|T_i^n u - T_i^n v\| \leq \|u - v\|$ for all $(u, v) \in E(G)$ and for each $i = 1, 2$. Therefore, T_1, T_2 are two G-asymptotically nonexpansive mappings. We also have $F(T_1) \cap F(T_2) = \{1\} \neq \emptyset$. By choosing $u_1 = 1.4$. From the iteration processes $\{u_n\}$ generated by (1.1)–(1.8), we obtain the following iteration processes.

Iter.(1.1): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)u_n + a_n T_2 v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Iter.(1.2): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)v_n + a_n T_2 v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Iter.(1.3): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)T_1 u_n + a_n T_2 v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Iter.(1.4): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)T_1 v_n + a_n T_2 v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Iter.(1.5): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)u_n + a_n T_2^n v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Iter.(1.6): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)v_n + a_n T_2^n v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Iter.(1.7): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)T_1^n u_n + a_n T_2^n v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Iter.(1.8): $$\begin{cases} u_1 = 1.4, \Omega_1 = \Omega \\ v_n = (1-b_n)u_n + b_n T_1^n u_n \\ w_n = (1-a_n)T_1^n v_n + a_n T_2^n v_n \\ \Omega_{n+1} = [0.5, \frac{u_n+w_n}{2}] \\ u_{n+1} = \frac{u_n+w_n}{2}. \end{cases}$$

Numerical results of the iteration processes (1.1)–(1.8) are presented by the following table and figure.

TABLE 1. Comparison of the rate of convergences of iteration processes (1.1)–(1.8).

n	(1.1)	(1.2)	(1.3)	(1.4)	(1.5)	(1.6)	(1.7)	(1.8)
1	1.4000000	1.4000000	1.4000000	1.4000000	1.4000000	1.4000000	1.4000000	1.4000000
2	1.3509601	1.3419931	1.306125	1.2998519	1.3509601	1.3419931	1.306125	1.2998519
3	1.3095293	1.2920305	1.2323242	1.2217892	1.2977383	1.2744698	1.1986696	1.1884618
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
23	1.0229087	1.0082666	1.0005835	1.0002762	1.0168919	1.0011666	1.0000003	1.0000003
24	1.0200088	1.0068644	1.0004315	1.0001969	1.0147151	1.0008796	1.0000002	1.0000001
25	1.0174733	1.0056995	1.0003191	1.0001403	1.0128211	1.000663	1.0000001	1.0000001
26	1.0152572	1.0047318	1.000236	1.0001	1.0111727	1.0004997	1.	1.
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
48	1.000768	1.0000786	1.0000003	1.0000001	1.0005551	1.000001	1.	1.
49	1.0006706	1.0000653	1.0000002	1.	1.0004847	1.0000007	1.	1.
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
54	1.0003405	1.0000257	1.0000001	1.	1.000246	1.0000002	1.	1.
55	1.0002974	1.0000214	1.	1.	1.0002149	1.0000001	1.	1.
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
58	1.0001981	1.0000122	1.	1.	1.0001431	1.0000001	1.	1.
59	1.0001731	1.0000102	1.	1.	1.000125	1.	1.	1.
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
87	1.0000039	1.0000001	1.	1.	1.0000029	1.	1.	1.
88	1.0000035	1.	1.	1.	1.0000025	1.	1.	1.
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
117	1.0000001	1.	1.	1.	1.0000001	1.	1.	1.
118	1.0000001	1.	1.	1.	1.	1.	1.	1.
119	1.0000001	1.	1.	1.	1.	1.	1.	1.
120	1.	1.	1.	1.	1.	1.	1.	1.

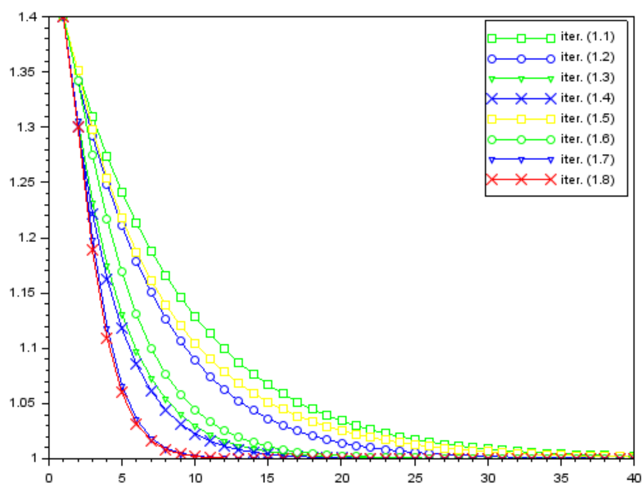

FIGURE 1. Comparison of the convergence of iteration processes (1.1)–(1.8).

The above table and figure show that the iteration processes (1.1)–(1.8) converge to 1. Furthermore, the convergence of the iteration process (1.7) and the iteration process (1.8) to 1 are faster than the iteration processes (1.1)–(1.4) in [25].

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