Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 977–995

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

On Highly Robust Approximate Solutions for Nonsmooth Convex Optimizations with Data Uncertainty

Jutamas Kerdkaew[1,*], Rabian Wangkeeree[1,2] and Gue Myung Lee[3]

[1] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*
email: jutamas.jk85@gmail.com (J. Kerdkaew)

[2] *Research center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok 65000, Thailand*
email: rabianw@nu.ac.th (R. Wangkeeree)

[3] *Department of Applied Mathematics, Pukyong National University, Busan 48513, Republic of Korea*
email: gmlee@pknu.ac.kr (G.M. Lee)

Abstract In this paper, we investigate a convex optimization problem in the face of data uncertainty in both objective and constraint functions. The notion of an ε-quasi highly robust solution (one sort of approximate solutions) for the convex optimization problem with data uncertainty is introduced. The highly robust approximate optimality theorems for ε-quasi highly robust solutions of uncertain convex optimization problem are established by means of a robust optimization approach (worst-case approach). Furthermore, the highly robust approximate duality theorems in terms of Wolfe type on ε-quasi highly robust solutions for the uncertain convex optimization problem are obtained. Moreover, to illustrate the obtained results or support this study, some examples are presented.

MSC: 90C25; 90C46; 49N15

Keywords: convex optimization problems with uncertain data; robust optimization problems; highly robust solutions; approximate quasi solutions; approximate optimality conditions; approximate duality theorems

Submission date: 04.02.2020/ Acceptance date: 28.04.2020

1. Introduction

In these days, the robust optimization technique has been recognized as one of the powerful deterministic methodologies that investigates an optimization problem with data uncertainty within the objective or constraint functions. Employing this methodology,

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*Corresponding author.

Published by The Mathematical Association of Thailand.

many interesting results are obtained for various sorts of uncertain optimization problems, see, e.g., [1–10]. There have been proposed numerous ways to define robust solutions for uncertain programming problems. Among the first one in all such notions is that the so-called *strictly robust solution* also called *minimax robust solution*, which was introduced by Soyster [11]. This concept is to have a solution that is feasible for all possible scenarios and is obtained composed by minimizing the objective function within the worst-case scenario. The notion of the strictly robust solution has been studied extensively from different aspects (see, e.g., [12–16]). Another solution concept is that of a *highly robust solution* which was introduced to study the various uncertain multiobjective programming problems; see, e.g., [17–19]. To the best of our knowledge, this notion for an uncertain (single-objective) programming problem has been not shown so far. It is worth noting that the notion of a strictly robust solution coincides therewith of the highly robust solution if the objective function of single-objective programming problems is uncertainty-free; see, e.g., [3, 8, 20, 21]. The notion of a highly robust solution is stricter than that of the strictly robust solution when the objective function is in the face of data uncertainty. *Nevertheless, in many cases, it is enough to study the highly robust solution for an uncertain single-objective programming problem*; see, e.g., [18, 19, 22].

On the other hand, finding minimizers of optimization problems might not be always possible, and then it leads to the notion of approximate solutions that play a crucial role in the algorithmic study of optimization problems. Among such approximate solutions, the notion of ε-quasi solution first introduced by Loridan [23]. Since then many researchers have studied the approximate solutions in optimization programming problems and approximate necessary conditions under different suitable constrained qualifications have been established, see [21, 24–28] and also the references therein, for example.

To the best of our knowledge, there are only a few papers to deal with approximate optimal solutions of optimization problems with data uncertainty in both objective and constraint functions, for example, [29, 30]. More precisely, by virtue of the epigraphs of the conjugates of the constraint functions, Sun et. al. [30] obtained some approximate optimality conditions for the robust quasi approximate optimal solution of an uncertain semi-infinite optimization problem. The notion of their obtained approximate solutions is given to approximate the strictly robust solutions to the problems. However, as far as we are concerned, the notion of approximate solutions to approximate the highly robust solutions for uncertain optimization problems has been not presented so far. A natural question is: *"How about the study of approximate optimality conditions and approximate duality theorems for an approximate solution that approximates the highly robust solutions to an uncertain convex optimizaion problem?"*. This paper is an effort in this direction.

In this paper, we propose and analyse ε-quasi highly robust solutions of single-objective convex optimization problems with data uncertainty in both the objective and the constraint functions. Firstly, we introduce the concept of an ε-quasi highly robust solution for the problem. Then we establish highly robust approximate optimality theorems for the problem under a robust characteristic cone constraint qualification, introduced in [3]. Furthermore, for such ε-quasi highly robust solutions of the primal convex uncertain optimization, we formulate a Wolfe type dual problem for the primal one. Then we propose a highly robust approximate weak duality and a highly robust approximate strong duality between the primal problem and its Wolfe type dual problem, and also give an example to illustrate the approximate duality theorems.

The organization of this paper is as follows. Section 2 collects some notations and existing results for their subsequent use. Section 3 establishes some highly robust approximate optimality theorems for ε-quasi highly robust solutions of convex optimization problems with data uncertainty. In Section 4, highly robust approximate duality theorems in terms of Wolfe type on ε-quasi highly robust solutions of the problems are presented. Section 5 devotes to the conclusion.

2. Preliminary

Let us first recall some notation and preliminary results which will be used throughout this paper. Throughout the paper, let $\mathbb{R}^n, n \in \mathbb{N}$, be the n-dimensional Euclidean space, and the inner product and the norm of $\mathbb{R}^n$ are denoted respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. The symbol $B(x, r)$ stands for the open ball centered at $x \in \mathbb{R}^n$ with the radius $r > 0$ while the $\mathbf{B}$ stands for the closed unit ball in $\mathbb{R}^n$. For a nonempty subset $A \subseteq \mathbb{R}^n$, we denote the notations of the closure, boundary and convex hull of A by cl$A$, bd$A$, and co$A$, respectively. Specially, when $\lambda x \in A \subseteq \mathbb{R}^n$ for every $\lambda \geq 0$ and every $x \in A$, the set A is said to be a cone. A dual cone A^* of the cone A is given as $A^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in A\}$. Observe that the dual cone A^* is always closed and convex (regardless of A).

For any extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, +\infty]$ the following notations stand, respectively, for its effective domain and epigraph:

$$\text{dom}f := \{x \in \mathbb{R}^n : f(x) < +\infty\},$$

and

$$\text{epi}f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}.$$

The function h is said to be a proper function if and only if $f(x) > -\infty$ for every $x \in \mathbb{R}^n$ and dom$f$ is nonempty. Further, it is said to be a convex function if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

or equivalently, epi$f$ is convex. On the other hand, the function f is said to be a concave function if and only if $-f$ is a convex function. Simultaneously, the function f is called a lower semicontinuous at $x \in \mathbb{R}^n$ if for every sequence $\{x_k\} \subseteq \mathbb{R}^n$ converging to x,

$$f(x) \leq \liminf_{k \to \infty} f(x_k).$$

Equivalently,

$$f(x) \leq \liminf_{y \to x} f(y),$$

where the term on the right-hand side of the inequality denotes the lower limit of the function h defined as

$$\liminf_{y \to x} f(y) = \lim_{r \downarrow 0} \inf_{y \in B(x,r)} f(y).$$

For any proper and convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the subdifferential of f at $\bar{x} \in \text{dom}f$, is defined by

$$\partial f(\bar{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \forall x \in \mathbb{R}^n\}.$$

More generally, for each $\varepsilon \geq 0$, the ε-subdifferential of f at $\bar{x} \in \mathrm{dom} f$, is defined by

$$\partial_\varepsilon f(\bar{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \varepsilon, \forall x \in \mathbb{R}^n\}.$$

It is obvious that for $\varepsilon \geq \varepsilon'$, we have $\partial_{\varepsilon'} f(\bar{x}) \subseteq \partial_\varepsilon f(\bar{x})$. Specially, if f is a proper lower semicontinuous convex function, then for every $\bar{x} \in \mathrm{dom} f$, the ε-subdifferential $\partial_\varepsilon f(\bar{x})$ is a nonempty closed convex set and

$$\partial f(\bar{x}) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(\bar{x}).$$

If $x \notin \mathrm{dom} f$, then we set $\partial f(x) = \emptyset$.

The Legendre-Fenchel conjugate function of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - f(x)\}$$

for all $x \in \mathbb{R}^n$. The function f^* is lower semicontinuous convex irrespective of the nature of f but for f^* to be proper, we need f to be a proper convex function.

Now, we collect the following propositions and a constraint qualification which useful in our later analysis.

Proposition 2.1. [31] *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function and $a \in dom f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. Then*

$$epi\ f^* = \bigcup_{\epsilon \geq 0} \{(v, \langle v, a\rangle + \epsilon - f(a)) : v \in \partial_\epsilon f(a)\}.$$

Proposition 2.2. [32] *Let$f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper lower semicontinuous convex functions. If $dom f \cap dom g \neq \emptyset$, then*

$$epi\ (f + g)^* = cl(epi\ f^* + epi\ g^*).$$

Moreover, if one of the functions f and g is continuous, then

$$epi\ (f + g)^* = cl(epi\ f^* + epi\ g^*).$$

Proposition 2.3. [3] *Let $g_i : R^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \ldots m$ be continuous functions. Suppose that each $\mathcal{V}_i \subseteq \mathbb{R}^q, i = 1, \ldots, m$, is convex, for all $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is a convex function, and for each $x \in \mathbb{R}^n, g_i(x, \cdot)$ is concave on $\mathcal{V}_i$. Then the cone $\bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} epi \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i)\right)^*$ is convex.*

Proposition 2.4. [3] *Let $g_i : R^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \ldots m$ be continuous functions. Suppose that each $\mathcal{V}_i \subseteq \mathbb{R}^q, i = 1, \ldots, m$, is compact and convex, for all $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is a convex function, and there exists $y \in \mathbb{R}^n$ such that $g_i(y, v_i) < 0, \forall v_i \in \mathcal{V}_i, i \in I$. Then the cone $\bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} epi \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i)\right)^*$ is closed.*

Proposition 2.5. [33] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ be continuous functions such that for each $v_i \in \mathbb{R}^q, g(\cdot, v_i)$ is convex. Let $\mathcal{V}_i \subseteq \mathbb{R}^q, i \in I$ be compact and let $K := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i \in I\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $K \subseteq \{x \in \mathbb{R}^n : f(x) \geq 0\}$;

$$\textit{(ii)}\ (0,0) \in epi\ f^* + cl\left(co \bigcup_{\substack{v_i \in \mathcal{V}_i,\\ \lambda_i \geq 0}} epi\ \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^*\right).$$

The following constraint qualification, which was introduced in [3], plays a key role in obtaining results in the paper.

Definition 2.6. [3] Let $g_i : R^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \ldots m$ be functions such that for all $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is convex. Then the *robust characteristic cone constraint qualification (RCCCQ)* is satisfied if the cone $\bigcup_{\substack{v_i \in \mathcal{V}_i,\\ \lambda_i \geq 0}} \text{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^*$ is closed and convex.

To conclude this section, we recall concepts of a convex optimization problem with data uncertainty and notions of its robust solutions as well as introduce a new concept of approximate solution for the highly robust solutions of the problem. Firstly, we begin by considering the following deterministic convex program:

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0, i = 1, \ldots, m, \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, is a convex function and for each $i \in I := \{1, \ldots, m\}, g_i : \mathbb{R}^n \to R$ is a convex function. The following parameterized convex program is an analogue of the deterministic convex program (P) if the objective as well as the constraints are uncertain:

$$\text{Minimize } f(x, u) \text{ subject to } g_i(x, v_i) \leq 0, i \in I. \tag{UP}$$

Here u is an uncertain parameter belonging to a compact convex uncertainty set $\mathcal{U} \subseteq \mathbb{R}^p$, for each $u \in \mathcal{U}, f(\cdot, u) : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and for each $i \in I, v_i$ belongs to a compact convex set $\mathcal{V}_i \subseteq \mathbb{R}^q, g_i(\cdot, v_i)$ is convex. By enforcing the constraints for all possible uncertainty within $\mathcal{V}_i, i \in I$, the problem (UP) becomes an uncertain convex semi-infinite program:

$$\text{Minimize } f(x, u) \text{ subject to } g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i \in I. \tag{2.1}$$

In other words, we study the uncertain convex programming problem (UP) by examining its robust (worst-case) counterpart. Let $K := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i \in I\}$, then it is termed as the robust feasible set of (UP). To avoid triviality in (2.1), we always assume that $K \neq \emptyset$.

In the literature, there are multiple ways of defining robust solutions for (UP). In the following, we recall two concepts of the robust solutions of the uncertain program (UP). The first notion commonly referred to as *strictly robust solution* or *robust minimax solution*, can be found in [2, 11]. This concept has been studied extensively by many authors, see, e.g., [7, 10, 13, 19].

Definition 2.7. A feasible point $\bar{x} \in K$ is said to be a *strictly robust solution* for (UP) if for each $x \in K$,

$$\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u).$$

The second one called *highly robust solution* can be found in Bitran [17]. This concept was also investigated for different uncertain multiobjective optimization problems, see, e.g., [18, 19].

Definition 2.8. A feasible point $\bar{x} \in K$ is said to be a *highly robust solution* for (UP) if for each $u \in \mathcal{U}$ and $x \in K$,

$$f(x, u) \geq f(\bar{x}, u).$$

The following notion is a concept of approximate solution that approximates the strictly robust solutions. It was investigated in a few papers, see, e.g. [30].

Definition 2.9. Let $\varepsilon \geq 0$ be given. A feasible point $\bar{x} \in K$ is said to be *an ε-quasi strictly robust solution (or a robust quasi ε optimal solution)* for (UP) if for each $x \in K$,

$$\max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\| \geq \max_{u\in\mathcal{U}} f(\bar{x}, u).$$

Clearly, if $\varepsilon = 0$, then an ε-quasi strictly robust solution for (UP) reduces to be a strictly robust solution for (UP).

Now, we introduce a new concept of solution to approximate the highly robust solutions for (UP).

Definition 2.10. Let $\varepsilon \geq 0$ be given. A feasible point $\bar{x} \in K$ is said to be an *ε-quasi highly robust solution (or a highly robust quasi ε-optimal solution)* for (UP) if for each $u \in \mathcal{U}$ and $x \in K$,

$$f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\| \geq f(\bar{x}, u).$$

Clearly, if $\varepsilon = 0$, then an ε-quasi highly robust solution for (UP) reduces to be a highly robust solution for (UP).

Remark 2.11.

(i) It is evident from Definition 2.7 and Definition 2.8 that a highly robust solution for (UP) is a strictly robust solution for (UP), but the converse does not hold. This means the highly robust solution is more immune to data uncertainty than the strictly robust solution.

(ii) Also, it is evident from Definition 2.9 and Definition 2.10 that an ε-highly robust solution for (UP) is an ε-strictly robust solution for (UP), but the converse does not hold. Hence, the ε-quasi highly robust solution is more immune to data uncertainty than the ε-quasi strictly robust solution.

The highly robust solution is more immune to data uncertainty than the strictly robust solution and the ε-quasi highly robust solution can reduce to be the highly robust solution. Therefore, the ε-quasi highly robust solution, which is more immune to data uncertainty than the ε-quasi strictly robust solution, is different from the strictly robust solution. The following example sheds some light onto this fact.

Example 2.12. Consider an uncertain convex program with an uncertain objective and uncertainty-free constraints:

$$\text{Minimize } ux + |x + 1| \quad \text{subject to} \quad x \in \mathbb{R}, \tag{2.2}$$

where $u \in [-1, 1]$. Following the robust optimization of [2], the robust counterpart of (2.2) reads

$$\text{Minimize } \max_{u\in[-1,1]} ux + \min |x + 1| \quad \text{subject to} \quad x \in \mathbb{R},$$

which is equivalent to $\min\{|x| + |x + 1| : x \in \mathbb{R}\}$. Then it is easy to check that the set of strictly robust solutions, denoted by S^{SR}, for (2.2) is $[-1, 0]$ while the set of solutions for (2.2) is $(-\infty,-1]$ if $u=1$, $\{-1\}$ if $u\in(-1,1)$, and $[-1,\infty)$ if $u=-1$. So, the set of highly robust solutions, denoted by S^{HR}, for (2.2) is $\{-1\}$. Consider $\bar{x}:=-\frac{3}{2}\in(-\infty,-1]$ with $\bar{\varepsilon}:=4>0$. We can see that for any $x\in\mathbb{R}$ and $u\in[-1,1]$,

$$ux+|x+1|+\sqrt{\bar{\varepsilon}}\|x-\bar{x}|\geq u\bar{x}+|\bar{x}-1|.$$

Thus, $\bar{x}=-\frac{3}{2}$ is an $\bar{\varepsilon}$-quasi highly robust solution of (2.2). Notice that $\bar{x}\notin[-1,0]=S^{SR}$, so, the ε-quasi highly robust solution for (2.2) is different from strictly robust solutions of (2.2), making it valuable to study the ε-quasi highly robust solutions.

3. Highly Robust Approximate Optimality Conditions for ε-Quasi Highly Robust Solutions

In this section, we focus on highly robust approximate optimality conditions for an ε-quasi highly robust solution of (UP). Our desired results are established using a robust optimization approach (worst-case approach).

Lemma 3.1. *Let $\bar{x}\in K$ and let $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}, g_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}, i\in I$ be continuous functions such that for each $u\in\mathbb{R}^p, f(\cdot,u)$ is convex on $\mathbb{R}^n$ and for each $v_i\in\mathbb{R}^q, g_i(\cdot,v_i)$ is convex on $\mathbb{R}^n$ and let $A:=\bigcup\limits_{\substack{v_i\in\mathcal{V}_i,\\ \lambda_i\geq 0}}\text{epi}\left(\sum_{i=1}^m\lambda_i g_i(\cdot,v_i)\right)^*$. Suppose that the constraint qualification (RCCCQ), defined in Definition 2.6, holds. Then the following statements are equivalent:*

(i) $\bar{x}$ is an ε-quasi highly robust solution for (UP);
(ii) there exist $\hat{\lambda}_i\geq 0$ and $\hat{v}_i\in\mathcal{V}_i, i\in I$ such that for any $x\in\mathbb{R}^n$ and $u\in\mathcal{U}$,

$$f(\bar{x},u)\leq f(x,u)+\sum_{i=1}^m\hat{\lambda}_i g_i(x,\hat{v}_i)+\sqrt{\varepsilon}\|x-\bar{x}\|.$$

Proof. [(i) ⇒ (ii)] Assume that $\bar{x}$ is an ε-quasi highly robust solution for (UP). So for any $x\in K, f(x,u)+\sqrt{\varepsilon}\|x-\bar{x}\|\geq f(\bar{x},u)$ for all $u\in\mathcal{U}$. Hence we obtain the inclusion $K\subseteq\{x\in\mathbb{R}^n: h(x,u)\geq 0\}$ where $h(x,u)=f(x,u)+\sqrt{\varepsilon}\|x-\bar{x}\|-f(\bar{x},u)$ for $u\in\mathcal{U}$. Due to the Lemma 2.5,

$$(0,0)\in\text{epi } h^*(\cdot,u)+\text{cl }(\text{co } A),\ \text{where } u\in\mathcal{U}.$$

Since A is closed and convex,

$$(0,0)\in\text{epi } h^*(\cdot,u)+A,\ \text{where } u\in\mathcal{U}.$$

Hence, there exist $\bar{\lambda}_i\geq 0,\bar{v}_i\in\mathcal{V}_i$ such that

$$(0,0)\in\text{epi } h^*(\cdot,u)+\text{epi}\left(\sum_{i=1}^m\bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^*,\ \text{where } u\in\mathcal{U}. \tag{3.1}$$

Let us prove that for any $u\in\mathcal{U}$,

$$\text{epi } h^*(\cdot,u)=\text{epi } f^*(\cdot,u)+\sqrt{\varepsilon}\mathbf{B}+\Big[f(\bar{x},u)+\|\bar{x}\|,+\infty\Big). \tag{3.2}$$

From Proposition 2.2, we have

$$\begin{aligned}\text{epi } h^*(\cdot,u) &= \text{epi } \Big[f(\cdot,u)+\sqrt{\varepsilon}\|\cdot-\bar{x}\|-f(\bar{x},u)\Big]^* \\ &= \text{epi } f^*(\cdot,u)+\text{epi } \Big[\sqrt{\varepsilon}\|\cdot-\bar{x}\|-f(\bar{x},u)\Big]^*, \end{aligned} \tag{3.3}$$

where $u\in\mathbb{R}^p$. Observe that for any $u\in\mathcal{U}$,

$$\Big[\sqrt{\varepsilon}\|\cdot-\bar{x}\|-f(\bar{x},u)\Big]^*(z)=\begin{cases} f(\bar{x},u)+\sqrt{\varepsilon}\|\bar{x}\|; & \|z\|\leq\sqrt{\varepsilon}, \\ +\infty; & \|z\|>\sqrt{\varepsilon}. \end{cases}$$

By dealing with (3.3), we obtain

$$\text{epi } h^*(\cdot,u)=\text{epi } f^*(\cdot,u)+\sqrt{\varepsilon}\mathbf{B}\times\Big[f(\bar{x},u)+\sqrt{\varepsilon}\|\bar{x}\|,+\infty\Big),$$

where $u\in\mathcal{U}$. Hence, it follows from (3.1) that for $u\in\mathcal{U}$,

$$(0,0)\in\text{epi } f^*(\cdot,u)+\sqrt{\varepsilon}\mathbf{B}\times\Big[f(\bar{x},u)+\sqrt{\varepsilon}\|\bar{x}\|,+\infty\Big)+\text{epi }\left(\sum_{i=1}^m\bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^*.$$

This yields

$$(0,-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|)\in\text{epi } f^*(\cdot,u)+\text{epi }\left(\sum_{i=1}^m\bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^*+\sqrt{\varepsilon}\mathbf{B}\times\mathbb{R}_+,$$

where $u\in\mathcal{U}$. Therefore, for each $u\in\mathcal{U}$, there exist $u^*\in\mathbb{R}^n,\alpha\geq 0,v_i^*\in\mathbb{R}^n,\beta_i\geq 0,i\in I,w^*\in\mathbf{B}$ and $\eta\in\mathbb{R}_+$ such that

$$\begin{aligned}(0,-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|)\in(u^*,f^*(u^*,u)+\alpha)+\left(\sum_{i=1}^m\bar{\lambda}_i(v_i^*,g_i^*(v_i^*,\bar{v}_i)+\beta_i\right) \\ +(\sqrt{\varepsilon}w^*,\eta).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&0=u^*+\sum_{i=1}^m\bar{\lambda}_i v_i^*+\sqrt{\varepsilon}w^*, \text{ and} \\ &-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|=f^*(u^*,u)+\alpha+\sum_{i=1}^m\bar{\lambda}_i(g_i^*(v_i^*,\bar{v}_i)+\beta_i)+\eta,\ u\in\mathcal{U}.\end{aligned}$$

Thus, for any $x \in \mathbb{R}^n, u \in \mathcal{U}$,

$$
\begin{aligned}
f(\bar{x},u) &= -f^*(u^*,u) - \alpha - \sum_{i=1}^{m} \bar{\lambda}_i (g_i^*(v_i^*,\bar{v}_i) + \beta_i) - \eta - \sqrt{\varepsilon}\|\bar{x}\| \\
&\leq -\Big[\langle u^*, x\rangle - f(x,u)\Big] - \sum_{i=1}^{m} \bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i) - \sqrt{\varepsilon}\|\bar{x}\| \\
&= \Big\langle \sum_{i=1}^{m} \bar{\lambda}_i v_i^* + \sqrt{\varepsilon} w^*, x \Big\rangle + f(x,u) - \sum_{i=1}^{m} \bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i) - \sqrt{\varepsilon}\|\bar{x}\| \\
&= \Big\langle \sum_{i=1}^{m} \bar{\lambda}_i v_i^*, x \Big\rangle + \langle \sqrt{\varepsilon} w^*, x\rangle + f(x,u) - \sum_{i=1}^{m} \bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i) - \sqrt{\varepsilon}\|\bar{x}\| \\
&\leq \Big\langle \sum_{i=1}^{m} \bar{\lambda}_i v_i^*, x \Big\rangle + \sqrt{\varepsilon}\|w^*\|\|x - \bar{x} + \bar{x}\| + f(x,u) \\
&\quad - \sum_{i=1}^{m} \bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i) - \sqrt{\varepsilon}\|\bar{x}\| \\
&\leq \Big\langle \sum_{i=1}^{m} \bar{\lambda}_i v_i^*, x \Big\rangle + \sqrt{\varepsilon}\|x - \bar{x}\| + f(x,u) - \sum_{i=1}^{m} \bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i) \\
&\leq \Big\langle \sum_{i=1}^{m} \bar{\lambda}_i v_i^*, x \Big\rangle + \sqrt{\varepsilon}\|x - \bar{x}\| + f(x,u) \\
&\quad - \Big[\Big\langle \sum_{i=1}^{m} \bar{\lambda}_i g_i(x,v_i), x \Big\rangle - \sum_{i=1}^{m} \bar{\lambda}_i g_i(x,\bar{v}_i)\Big] \\
&= f(x,u) + \sqrt{\varepsilon}\|x - \bar{x}\| + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x,\bar{v}_i).
\end{aligned}
$$

Thus, the statement (ii) is satisfied.
[(ii) $\Rightarrow$ (i)] Suppose that there exist $\bar{\lambda}_i, \geq 0, v_i \in \mathcal{V}_i, i \in I$ such that for any $x \in \mathbb{R}^n$ and $u \in \mathcal{U}$,

$$
f(x,u) + \sqrt{\varepsilon}\|x - \bar{x}\| + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x,\bar{v}_i) \geq f(\bar{x},u).
$$

So, for any feasible point $x \in K$ and $u \in \mathcal{U}$,

$$
\begin{aligned}
f(\bar{x},u) &\leq f(x,u) + \sqrt{\varepsilon}\|x - \bar{x}\| + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x,\bar{v}_i) \\
&\leq f(x,u) + \sqrt{\varepsilon}\|x - \bar{x}\|.
\end{aligned}
$$

Therefore, $\bar{x}$ is an ε-quasi highly robust solution of (UP). ■

Lemma 3.2. *Let all assumptions of Lemma 3.2 be satisfied. Then, the following statements are equivalent:*

(i) $\bar{x}$ is an ε-quasi highly robust solution for (UP)*;*

(ii) for any $u \in \mathcal{U}$*,*

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in epi\ f^*(\cdot, u) + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} epi\left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.$$

Proof. Clearly, (i) ⇒ (ii) is true by the proof of Lemma 3.1. Let us show (ii) ⇒ (i) now. Suppose that for any $u \in \mathcal{U}$,

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in \text{ epi } f^*(\cdot, u) + A + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.$$

Then, for any $u \in \mathcal{U}$, we obtain

$$(0, 0) \in \text{ epi } f^*(\cdot, u) + A + \sqrt{\varepsilon}\mathbf{B} \times \Big[f(\bar{x}, u) + \sqrt{\varepsilon}\|\bar{x}\|, +\infty\Big).$$

From the proof of Theorem 3.1, we knew that for any $u \in \mathcal{U}$, epi $f^*(\cdot, u) + \sqrt{\varepsilon}\mathbf{B} \times \Big[f(\bar{x}, u) + \sqrt{\varepsilon}\|\bar{x}\|, +\infty\Big) = \text{epi }\Big(f(\cdot, u) + \sqrt{\varepsilon}\| \cdot - \bar{x}\| - f(\bar{x}, u)\Big)^*$. So, for any $u \in \mathcal{U}$, one has

$$\begin{aligned}(0, 0) &\in \text{epi }\Big(f(\cdot, u) + \sqrt{\varepsilon}\| \cdot - \bar{x}\| - f(\bar{x}, u)\Big)^* + A \\ &= \text{epi }\Big(f(\cdot, u) + \sqrt{\varepsilon}\| \cdot - \bar{x}\| - f(\bar{x}, u)\Big)^* + \text{cl}\,(\text{co } A)\,.\end{aligned}$$

Using the Lemma 2.5, for any $u \in \mathcal{U}$, we arrive

$$K \subseteq \Big\{x \in \mathbb{R}^n : \Big(f(\cdot, u) + \sqrt{\varepsilon}\| \cdot - \bar{x}\| - f(\bar{x}, u)\Big)(x) \geq 0\Big\}.$$

Thus, for any $u \in \mathcal{U}$ and $x \in K$,

$$f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\| - f(\bar{x}, u) \geq 0.$$

Hence, for any $x \in K$ and $u \in \mathcal{U}$,

$$f(\bar{x}, u) \leq f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\|,$$

which means $\bar{x}$ is an ε-quasi highly robust solution of (UP). ■

Theorem 3.3. *(Highly robust approximate optimality theorem) Let* $\bar{x} \in K$ *and let* $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ *be continuous functions such that for each* $u \in \mathbb{R}^p, f(\cdot, u)$ *is convex on* $\mathbb{R}^n$ *and for each* $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ *is convex on* $\mathbb{R}^n$*. Suppose that the constraint qualification (RCCCQ), defined in Definition 2.6, holds. Then the following statements are equivalent:*

(i) $\bar{x}$ *is an* ε*-quasi highly robust solution for* (UP)*;*

(ii) for any $u \in \mathcal{U}$

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in epi\ f^*(\cdot, u) + \bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} epi\left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^* + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.$$

(iii) there exist $\bar{v}_i \in \mathcal{V}_i$ *and* $\bar{\lambda}_i \geq 0, i \in I$ *such that for any* $u \in \mathcal{U}$*,*

$$0 \in \partial f(\cdot, u)(\bar{x}) + \sum_{i=1}^{m} \partial(\bar{\lambda}_i g_i(\cdot, v_i))(\bar{x}) + \sqrt{\varepsilon}\boldsymbol{B} \ \ and \ \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.$$

Proof. By Lemma 3.1 and Lemma 3.2, the statement [(i) $\Leftrightarrow$ (ii)] is proved.
[(ii) $\Rightarrow$ (iii)] Suppose that the statement (ii) holds, i.e., for any $u \in \mathcal{U}$,

$$\begin{aligned}&(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|)\\ &\in \text{ epi } f^*(\cdot, u) + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \text{epi } \left(\sum_{i=1}^m \lambda_i g_i(\cdot, \bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.\end{aligned}$$

Therefore, for any $u \in \mathcal{U}$ there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, i \in I$ such that

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in \text{epi } f^*(\cdot, u) + \text{epi } \left(\sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.$$

By the continuity of $g_i(\cdot, v_i), i \in I$ and Proposition 2.2, equivalently, for any $u \in \mathcal{U}$ there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, i \in I$ such that

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in \text{epi } f^*(\cdot, u) + \sum_{i=1}^m \text{epi } \left(\bar{\lambda}_i g_i(\cdot, \bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.$$

By Proposition 2.1, equivalently, for any $u \in \mathcal{U}$ there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, i \in I$ and $\varepsilon_i \geq 0, i = 0, 1, \ldots, m$ such that

$$\begin{aligned}&(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|)\\ &\in \bigcup_{\varepsilon_0 \geq 0} \Big\{(w_0, \langle w_0, \bar{x}\rangle + \varepsilon_0 - f(\bar{x}, u)) : w_0 \in \partial_{\varepsilon_0} f(\cdot, u)(\bar{x})\Big\}\\ &\quad + \sum_{i=1}^m \bigcup_{\varepsilon_i \geq 0} \Big\{(w_i, \langle w_i, \bar{x}\rangle + \varepsilon_i - \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)) : w_i \in \partial_{\varepsilon_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x})\Big\}\\ &\quad + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.\end{aligned}$$

Hence, for any $u \in \mathcal{U}$, there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, w_i \in \partial_{\varepsilon_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x}), i \in I, w_0 \in \partial_{\varepsilon_0} f(\cdot, u)(\bar{x}), w^* \in \mathbf{B}, \eta \in \mathbb{R}_+$ and $\varepsilon_i \geq 0, i = 0, 1, \ldots, m$ such that

$$\begin{aligned}&(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|)\\ &= (w_0, \langle w_0, \bar{x}\rangle + \varepsilon_0 - f(\bar{x}, u)) + \sum_{i=1}^m (w_i, \langle w_i, \bar{x}\rangle + \varepsilon_i - \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i))\\ &\quad + (\sqrt{\varepsilon} w^*, \eta).\end{aligned}$$

It follows that, for any $u \in \mathcal{U}$, there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, w_i \in \partial_{\varepsilon_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x}), i \in I, w_0 \in \partial_{\varepsilon_0} f(\cdot, u)(\bar{x}), w^* \in \mathbf{B}, \eta \in \mathbb{R}_+$ and $\varepsilon_i \geq 0, i = 0, 1, \ldots, m$ such that

$$0 = \sum_{i=0}^m w_i + \sqrt{\varepsilon} w^* \quad \text{and}$$

$$-\sqrt{\varepsilon}\|\bar{x} - f(\bar{x}, u) = \sum_{i=0}^m (\langle w_i, \bar{x}\rangle + \varepsilon_i) - f(\bar{x}, u) - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) + \eta.$$

Equivalently, for any $u \in \mathcal{U}$, there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, w_i \in \partial_{\varepsilon_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x}), i \in I, w_0 \in \partial_{\varepsilon_0} f(\cdot, u)(\bar{x}), w^* \in \mathbf{B}, \eta \in \mathbb{R}_+$ and $\varepsilon_i \geq 0, i = 0, 1, \ldots, m$ such that

$$\begin{aligned}
0 &\geq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \\
&= \sqrt{\varepsilon}\|\bar{x}\| + \sum_{i=0}^{m} (\langle w_i, \bar{x}\rangle + \varepsilon_i) + \eta \\
&= \sqrt{\varepsilon}\|\bar{x}\| + \sum_{i=0}^{m} \varepsilon_i - \langle \sqrt{\varepsilon} w^*, \bar{x}\rangle + \eta \\
&\geq \sqrt{\varepsilon}\|\bar{x}\| + \sum_{i=0}^{m} \varepsilon_i - \sqrt{\varepsilon}\|w^*\|\|\bar{x}\| + \eta \\
&\geq \sum_{i=0}^{m} \varepsilon_i \geq 0.
\end{aligned}$$

Hence, the statement (iii) holds.
[(iii) $\Rightarrow$ (ii)] Suppose that the statement (iii) holds. Then for any $u \in \mathcal{U}$, there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, w_i \in \partial(\bar{\lambda}_i g_i)(\cdot, \bar{v}_i)(\bar{x}), i \in I, w_0 \in \partial f(\cdot, u)(\bar{x})$ and $w^* \in \mathbf{B}$ such that

$$0 = w_0 + \sum_{i=1}^{m} \bar{\lambda}_i w_i + \sqrt{\varepsilon} w^* \quad \text{and} \quad \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.$$

This, together with the definition of the subdifferential of $f(\cdot, u)$, yields that for any $x \in K, u \in \mathcal{U}$,

$$\begin{aligned}
f(x, u) - f(\bar{x}, u) &\geq \langle w_0, x - \bar{x}\rangle \\
&= \left\langle -\sum_{i=1}^{m} \bar{\lambda}_i w_i - \sqrt{\varepsilon} w^*, x - \bar{x} \right\rangle \\
&= -\left\langle \sum_{i=1}^{m} \bar{\lambda}_i w_i, x - \bar{x} \right\rangle - \langle \sqrt{\varepsilon} w^*, x - \bar{x}\rangle \\
&\geq -\sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) - \sqrt{\varepsilon}\|w^*\|\|x - \bar{x}\| \\
&\geq -\sqrt{\varepsilon}\|x - \bar{x}\|.
\end{aligned}$$

Therefore, for any $x \in K, u \in \mathcal{U}$,

$$f(\bar{x}, u) \leq f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\|,$$

which means $\bar{x}$ is an ε-quasi highly robust solution of (UP). Thus, by Lemma 3.2, the statement (ii) holds. ∎

Corollary 3.4. *Let $\bar{x} \in K$ and let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ be continuous functions such that for each $u \in \mathbb{R}^p, f(\cdot, u)$ is convex on $\mathbb{R}^n$ and for each $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is convex on $\mathbb{R}^n$. Suppose that for each $x \in \mathbb{R}^n, g_i(x, \cdot)$ is concave on $\mathcal{V}_i, i \in I$ and there exists $y \in \mathbb{R}^n$ such that $g_i(y, v_i) < 0, \forall v_i \in \mathcal{V}_i, i \in I$. Then the following statements are equivalent:*

(i) $\bar{x}$ is an ε-quasi highly robust solution for (UP);

(ii) for any $u \in \mathcal{U}$,

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in \textit{epi } f^*(\cdot, u) + \bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} \textit{epi } \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^* + \sqrt{\varepsilon}\mathbf{B} \times \mathbb{R}_+.$$

(iii) there exist $\bar{v}_i \in \mathcal{V}_i$ *and* $\bar{\lambda}_i \geq 0, i \in I$ *such that for any* $u \in \mathcal{U}$,

$$0 \in \partial f(\cdot, u)(\bar{x}) + \sum_{i=1}^{m} \partial(\bar{\lambda}_i g_i(\cdot, v_i))(\bar{x}) + \sqrt{\varepsilon}\boldsymbol{B} \quad \textit{and} \quad \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.$$

Proof. It follows from Proposition (2.4) and Proposition (2.3) that the constraint qualification (RCCCQ), defined in Definition 2.6, holds. Then, all conditions of Theorem 3.3 are satisfied and so we finish this proof. ■

4. Highly Robust Approximate Duality Theorems for ε-Quasi Highly Robust Solutions

In this section, we formulate a Wolfe type dual problem (UD) for the primal uncertain convex optimization problem (UP). Then we propose a highly robust approximate weak duality theorem and a highly robust approximate strong duality between the primal problem and its Wolfe type dual problem.

Now we formulate a Wolfe dual problem (UD) for (UP) as follows:

$$\begin{aligned} \text{Maximize} \quad & f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\ \text{subject to} \quad & 0 \in \partial f(\cdot, u)(y) + \sum_{i=1}^{m} \partial\left(\lambda_i g_i\right)(\cdot, v_i)(y) + \sqrt{\varepsilon}\mathbf{B}, \\ & u \in \mathcal{U}, \lambda_i \geq 0, v_i \in \mathcal{V}_i, i \in I, \varepsilon \geq 0. \end{aligned} \tag{UD}$$

Let $K_D := \Big\{(y, v, \lambda) \in \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^m_+ : 0 \in \partial f(\cdot, u)(y) + \sum_{i=1}^{m} \partial\left(\lambda_i g_i\right)(\cdot, v_i)(y) + \sqrt{\varepsilon}\mathbf{B}, \lambda_i \geq 0, v_i \in \mathcal{V}_i, i \in I\Big\}$, then it is termed as the robust feasible set of the dual problem (UD).

Definition 4.1. Let $\varepsilon \geq 0$ be given, then $(\bar{y}, \bar{\lambda}, \bar{v})$ is said to be an *ε-quasi highly robust solution of the dual problem* (UD) if for any robust feasible solution $(y, v, \lambda) \in K_D$ and $u \in \mathcal{U}$,

$$f(\bar{y}, u) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \geq f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\varepsilon}\|\bar{y} - y\|.$$

Let us move on the highly robust approximate weak duality theorem and the highly robust approximate strong duality theorem for highly robust solutions. The following theorem proposes a highly robust approximate weak duality between the primal problem and its Wolfe type dual problem.

Theorem 4.2. *(Highly robust approximate weak duality theorem) Let* $\varepsilon \geq 0$ *be given. For any* $(x, u) \in K \times \mathcal{U}$ *and any* $(y, v, \lambda) \in K_D$,

$$f(x, u) \geq f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\varepsilon}\|x - y\|.$$

Proof. Let $(x,u) \in K \times \mathcal{U}$ and $(y,v,\lambda) \in K_D$, be arbitrary. Then, there exist $w_0 \in \partial f(\cdot,u)(y), w_i \in \partial\left(\lambda_i g_i(\cdot,v_i)\right)(y), i \in I$ and $w^* \in \mathbf{B}$ such that $w_0 + \sum_{i=1}^m w_i + \sqrt{\varepsilon} w^* = 0$. Hence, we obtain

$$\begin{aligned}
&f(x,u) - f(y,u) - \sum_{i=1}^m \lambda_i g_i(y,v_i)\\
&\geq \langle w_0, x-y\rangle - \sum_{i=1}^m \lambda_i g_i(y,v_i)\\
&= \Big\langle -\sum_{i=1}^m w_i - \sqrt{\varepsilon} w^*, x-y\Big\rangle - \sum_{i=1}^m \lambda_i g_i(y,v_i)\\
&= -\Big\langle \sum_{i=1}^m w_i, x-y\Big\rangle - \langle \sqrt{\varepsilon} w^*, x-y\rangle - \sum_{i=1}^m \lambda_i g_i(y,v_i)\\
&\geq -\sum_{i=1}^m \lambda_i g_i(x,v_i) + \sum_{i=1}^m \lambda_i g_i(y,v_i) - \langle \sqrt{\varepsilon} w^*, x-y\rangle - \sum_{i=1}^m \lambda_i g_i(y,v_i)\\
&= -\sum_{i=1}^m \lambda_i g_i(x,v_i) - \sqrt{\varepsilon}\|x-y\|\\
&\geq -\sqrt{\varepsilon}\|x-y\|.
\end{aligned}$$

Thus, one has $f(x,u) \geq f(y,u) + \sum_{i=1}^m \lambda_i g_i(y,v_i) - \sqrt{\varepsilon}\|x-y\|$ as desired. ■

The following highly robust approximate strong duality theorem holds under the constraint qualification (RCCCQ).

Theorem 4.3. *(Highly robust approximate strong duality theorem) Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ be continuous functions such that for each $u \in \mathbb{R}^p, f(\cdot,u)$ is convex on $\mathbb{R}^n$ and for each $v_i \in \mathbb{R}^q, g_i(\cdot,v_i)$ is convex on $\mathbb{R}^n$. Suppose that the constraint qualification (RCCCQ), defined in Definition 2.6, holds. If $\bar{x} \in K$ is an ε-quasi highly robust solution of the primal problem (UP), then there exist $\bar{\lambda} \in \mathbb{R}^m_+$ and $\bar{v} \in \mathbb{R}^q$ such that $(\bar{x},\bar{v},\bar{\lambda})$ is an ε-quasi highly robust solution of the dual problem (UD).*

Proof. Let $\bar{x} \in K$ be an ε-quasi highly robust solution of (UP). Hence, by Theorem 3.3, for any $u \in \mathcal{U}$, there exist $\bar{v}_i \in \mathcal{V}_i, \bar{\lambda}_i \geq 0, i \in I$ such that

$$0 \in \partial f(\cdot,u)(\bar{x}) + \sum_{i=1}^m \partial(\bar{\lambda}_i g_i(\cdot,v_i))(\bar{x}) + \sqrt{\varepsilon}\mathbf{B} \quad \text{and} \quad \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x},\bar{v}_i) = 0.$$

This means $(\bar{x},\bar{v},\bar{\lambda})$ is a feasible solution of (UD), i.e., $(\bar{x},\bar{v},\bar{\lambda}) \in K_D$. By Theorem 4.2, for any $u \in \mathcal{U}$ and $(y,v,\lambda) \in K_D$, we have

$$f(\bar{x},u) \geq f(y,u) + \sum_{i=1}^m \lambda_i g_i(y,v_i) - \sqrt{\varepsilon}\|x-y\|.$$

It follows that for any $u \in \mathcal{U}$ and $(y, v, \lambda) \in K_D$,

$$\begin{aligned}
&f(\bar{x}, u) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) - \Big[f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)\Big] \\
&\geq -\sqrt{\varepsilon}\|x - y\| + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \\
&= -\sqrt{\varepsilon}\|x - y\|.
\end{aligned}$$

It yields, for any $u \in \mathcal{U}$ and $(y, v, \lambda) \in K_D$,

$$f(\bar{x}, u) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \geq f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\varepsilon}\|x - y\|.$$

Therefore, $(\bar{x}, \bar{v}, \bar{\lambda})$ is an ε-quasi highly robust solution of (UD) as desired. ■

The following example illustrates Theorem 4.2 and Theorem 4.3.

Example 4.4. Let $f : \mathbb{R}^2 \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R}^2 \times \mathcal{V} \to \mathbb{R}$ be defined by

$$f(x, u) = ux_1 + x_2^2 \quad \text{and} \quad g(x, v) = x_1^2 - vx_1,$$

where $\mathcal{U} := [-1, 1]$ and $\mathcal{V} := \mathbb{R}$. Consider the following convex optimization problem with uncertainty:

$$\text{Minimize } f(x, u) \;\text{ subject to }\; g(x, v) \leq 0,\; v \in \mathcal{V}. \tag{4.1}$$

Observe that the robust feasible set of (4.1) is the set

$$\begin{aligned}
K :&= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - vx_1 \leq 0,\, v \in \mathcal{V}\} \\
&= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \in \mathbb{R}\},
\end{aligned}$$

while the set of all ε-quasi highly solution of (4.1) is

$$\begin{aligned}
S^{HR} := &\Big\{(x_1, x_2) \in K : ux_1 + x_2^2 \leq uy_1 + y_2^2 + \sqrt{\varepsilon}\|(y_1, y_2) - (x_1, x_2)\|, \\
&\quad (y_1, y_2) \in K, u \in \mathcal{U}\Big\} \\
= &\Big\{(x_1, x_2) \in K : u(0) + x_2^2 \leq u(0) + y_2^2 + \sqrt{\varepsilon}\|(0, y_2) - (0, x_2)\|, y_2 \in \mathbb{R}, u \in \mathcal{U}\Big\} \\
= &\Big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, \frac{-\sqrt{\varepsilon}}{2} \leq x_2 \leq \frac{\sqrt{\varepsilon}}{2}\Big\}.
\end{aligned}$$

We can prove that the (RCCCQ) holds for (4.1). To show the cone $\bigcup_{v \in \mathcal{V}, \lambda \geq 0} \text{epi } (\lambda g(\cdot, v))^*$ is closed and convex, let $v \in \mathcal{V}$ and $\lambda \geq 0$ be given. Then, we have

$$(\lambda g(\cdot, v))^* (x^*) = \begin{cases} 0; & \lambda = 0, \\ \dfrac{(x^* + \lambda v)^2}{4\lambda}; & \lambda > 0. \end{cases}$$

So, it can be seen that

$$\begin{aligned}
\bigcup_{v \in \mathcal{V}, \lambda \geq 0} \text{epi } (\lambda g(\cdot, v))^* &= \Big(\{0\} \times \mathbb{R}_+\Big) \cup \bigcup_{v \in \mathcal{V}, \lambda > 0} \Big\{(x^*, \alpha) : \alpha \geq \frac{(x^* + \lambda v)^2}{4\lambda}\Big\} \\
&= \mathbb{R} \times \mathbb{R}_+.
\end{aligned}$$

Next, we formulate a dual problem for (4.1) as follows:

$$\begin{aligned}&\text{Maximize } f(y_1,y_2,u)+\lambda g(y_1,y_2,v)\\&\text{subjet to } 0\in\partial f(\cdot,u)(y_1,y_2)+\partial\left(\lambda g(\cdot,v)\right)(y_1,y_2)+\sqrt{\varepsilon}\mathbf{B},\\&\qquad u\in\mathcal{U},\lambda\geq 0,v\in\mathcal{V},\varepsilon\geq 0.\end{aligned}\tag{4.2}$$

Then the set $K_D:=\Big\{((y_1,y_2),v,\lambda): y_1\in\mathbb{R},(0,0)\in\partial f(\cdot,u)(y_1,y_2)+\partial\left(\lambda g(\cdot,v)\right)(y_1,y_2)+\sqrt{\varepsilon}\mathbf{B},u\in[-1,1],\lambda\geq 0,v\in\mathbb{R},\varepsilon\geq 0\Big\}$ is the robust feasible set of (4.2). We can calculate the robust feasible set K_D as follows:

$$\begin{aligned}K_D:=&\Big\{((y_1,y_2),v,\lambda):(0,0)\in\partial f(\cdot,u)(y_1,y_2)+\partial\left(\lambda g(\cdot,v)\right)(y_1,y_2)\\&\quad+\sqrt{\varepsilon}\mathbf{B},u\in[-1,1],\lambda\geq 0,v\in\mathbb{R},\varepsilon\geq 0\Big\}\\=&\Big\{((y_1,y_2),v,\lambda):y_1\in\mathbb{R},u+2\lambda y_1-\lambda v+\sqrt{\varepsilon}w_1=0,\\&\quad y_2=-\frac{\sqrt{\varepsilon}}{2},w_1^2+w_2^2\leq 1,u\in[-1,1],\lambda\geq 0,v\in\mathbb{R},\varepsilon\geq 0\Big\}.\end{aligned}$$

Observe that for any $u\in\mathcal{U},(x_1,x_2)\in K$ and $(y_1,y_2,v,\lambda)\in K_D$,

$$\begin{aligned}&f(x_1,x_2,u)-\Big[f(y_1,y_2,u)+\lambda g(y_1,y_2,v)-\sqrt{\varepsilon}\|(x_1,x_2)-(y_1,y_2)\|\Big]\\&=x_2^2-\Big[uy_1+y_2^2+\lambda y_1^2-\lambda v y_1-\sqrt{\varepsilon}\sqrt{y_1^2+(x_2-y_2)^2}\Big]\\&=x_2^2-y_2^2-\lambda y_1^2+(\lambda v-u)y_1+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2-y_2)^2}\\&=x_2^2-\frac{\varepsilon}{4}w_2^2+\lambda y_1^2+\sqrt{\varepsilon}w_1y_1+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2}\\&=(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2+\lambda y_1^2+\sqrt{\varepsilon}\left[w_1y_1-w_2(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)\right]\\&\quad+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2}\\&\geq\sqrt{\varepsilon}\left[w_1y_1-w_2(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)\right]+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2}\\&\geq-\sqrt{\varepsilon}\sqrt{(-w_1)^2+w_2^2}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2}+.\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2}\\&\geq 0.\end{aligned}$$

Hence, for any $u\in\mathcal{U},(x_1,x_2)\in K$ and $(y_1,y_2,v,\lambda)\in K_D$,

$$f(x_1,x_2,u)\geq f(y_1,y_2,u)+\lambda g(y_1,y_2,v)-\sqrt{\varepsilon}\|(x_1,x_2)-(y_1,y_2)\|,$$

and so the conclusion of Theorem 4.2 (The highly robust approximate weak duality theorem) holds. Let $(\bar{x}_1,\bar{x}_2)\in K$ be an ε-quasi highly robust solution for (UP). So, $\bar{x}_1=0$ and $-\frac{\sqrt{\varepsilon}}{2}\leq\bar{x}_2\leq\frac{\sqrt{\varepsilon}}{2}$. By taking $\bar{\lambda}:=\sqrt{\varepsilon}$ and $\bar{v}=\frac{u}{\sqrt{\varepsilon}}+w_1$, we can see that $((\bar{x}_1,\bar{x}_2),\bar{v},\bar{\lambda})\in K_D$. Indeed, $\bar{\lambda}\geq 0,\bar{v}\in\mathbb{R}$ and

$$u+2\bar{\lambda}\bar{x}_1-\bar{\lambda}\bar{v}+\sqrt{\varepsilon}w_1=u-\sqrt{\varepsilon}\left(\frac{u}{\sqrt{\varepsilon}}+w_1\right)+\sqrt{\varepsilon}w_1=0.$$

Besides, for any $u \in \mathcal{U}$ and $(y_1, y_2, v, \lambda) \in K_D$,

$$\begin{aligned} & f(\bar{x}_1, \bar{x}_2, u) + \bar{\lambda} g(\bar{x}_1, \bar{x}_2, \bar{v}) - \Big[f(y_1, y_2, u) + \lambda g(y_1, y_2, v) \Big] \\ & \geq -\sqrt{\varepsilon} \|(x_1, x_2) - (y_1, y_2)\| + \bar{\lambda} g(\bar{x}_1, \bar{x}_2, \bar{v}) \\ & = -\sqrt{\varepsilon} \|(x_1, x_2) - (y_1, y_2)\|. \end{aligned}$$

Therefore, $(\bar{x}_1, \bar{x}_2)$ is an ε-quasi highly robust solution of (4.2), and then the conclusion of Theorem 4.3 (The highly robust approximate strong duality theorem) holds.

5. Conclusion

This paper devotes to the ε-quasi highly robust solution for a robust convex optimization problem in the face of data uncertainty in both objective and constraint functions. The highly robust approximate optimality theorems for an ε-quasi highly robust solution of a robust convex optimization problem are established by using a robust optimization approach (worst-case approach). Furthermore, by employing this approach, we obtain highly robust approximate duality theorems in terms of Wolfe type on ε-quasi highly robust solution for the convex optimization problems with data uncertainty. In addition, to illustrate or support this study, some examples are presented.

Acknowledgments

The authors would like to thank the referees for careful reading and constructive comments. This research is partially supported by the Science Achivement Scholarship of Thailand and Naresuan university. The third author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2017R1E1A1A03069931).

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