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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Stability of a Generalization of Cauchy's and the Quadratic Functional Equations in Quasi-Banach Spaces

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Abstract In this paper, we extend and improve the concept of Almahalebi [M. Almahalebi, Stability of a generalization of Cauchy's and the quadratic functional equations, J. Fixed Point Theory Appl. 20 (12) (2018)] to qusic-Banach spaces by using the fixed point method. Second, we investigate the stability of the following generalization of Cauchys and the quadratic functional equations

$$\sum_{k=0}^{n-1} f(x+b_k y) = nf(x) + nf(y),$$

where $n \in N_2 b_k = \exp(2i\pi k)$ for $0 \leq k \leq n-1$, in quasi-Banach spaces. Moreover, we obtain the hyperstability results of this equation by using the fixed point method.

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1. Introduction

In linear algebra, functional analysis and related areas of mathematics, a **quasinorm** is similar to a norm in that it satisfies the norm axioms, except that the triangle inequality is replaced by

$$\|x+y\| \leq K(\|x\| + \|y\|)$$

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for some constants $K > 0$ (see [1–4]).

Definition 1.1. **([5, 6])** Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:
(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| =\mid \lambda \mid \|x\|$ for all $\lambda \in \mathbb{R}$ and all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $s \geq 1$ such that $\|x + y\| \leq s(\|x\| + \|y\|)$ for all $x, y \in X$.
The pair $(X, \|\cdot\|)$ is called a **quasi-normed apace** if $\|\cdot\|$ is a quasi-norm on X. A quasi-normed $\|\cdot\|$ is called a **p-norm** $(0 < p \leq 1)$ is

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach apace is called a **p-quasi-Banach space**.

Definition 1.2. **([5, 6])** Let X be a quasi-normed space.
(i) A sequence $\{x_n\}$ in X is called a **quasi-convergent** to a point $x \in X$ if and only if $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$;
(ii) A sequence $\{x_n\}$ in X is called a **quasi-Cauchy sequence** if and only if $\|x_n - x_m\| \longrightarrow 0$ as $n, m \longrightarrow \infty$;
(iii) Let $\{x_n\}$ is a sequence in normed space $(X, \|\cdot\|)$. X is **complete** if for any quasi-Cauchy sequence $\{x_n\}$ is quasi-convergent;
(iv) A complete quasi-normed space is called a **quasi-Banach space**.

In 1989, Bakhtin [7] developed the notion of b-metric space and established some fixed point theorems in b-metric spaces. Subsequently, several results appeared in this direction ([8–17]) as follows:

Definition 1.3. [7] A b-metric on a set X is a mapping $d : X \times X \to [0, +\infty)$ satisfying the following conditions: for any $x, y, z \in X$,
(b_1) $d(x, y) = 0$ if and only if $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) there exists $s \geq 1$ such that $d(x, y) \leq s(d(x, z) + d(z, y))$.
Then (X, d) is known as a b-metric space with coefficient s.

Note that every metric space is a b-metric space with $s = 1$. Some examples of **b-metric space** are given below: Let $X = \mathbb{E}$ be a vecter space. Define a mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \|x - y\|$$

for all $x, y \in X$, where $\|\cdot\| : \mathbb{E} \to \mathbb{R}$ in a quasi-norm function. Then (X, d) is a b-metric space with coefficient $s = K$.

Throughout this paper, we will denote the set of natural numbers, nonnegative integers, nonnegative real numbers, the set $X \setminus \{0\}$ and the set of natural numbers greater than or equal to m by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}_+$, X_0, $\mathbb{N}_m$, respectively. And we use the notation B^A denotes the family of all functions mapping a set $A \neq \emptyset$ into a set $B \neq \emptyset$).
Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_+^{X^n}$ and F_1, F_2 be operators mapping from a nonempty set $D \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$F_1\varphi(x_1, ..., x_n) = F_2\varphi(x_1, ..., x_n), \qquad (x_1, ..., x_n \in X) \tag{1.1}$$

is ε–hyperstable provided that every $\varphi_0 \in D$ which satisfies

$$d(F_1\varphi_0(x_1, ..., x_n), F_2\varphi_0(x_1, ..., x_n)) \leq \varepsilon(x_1, ..., x_n), \qquad (x_1, ..., x_n \in X) \tag{1.2}$$

fulfills the (1.2).

Our aim is to prove the stability and hyperstability results for the generalization of Cauchys and the quadratic functional equations

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + nf(y), \tag{1.3}$$

where $n \in \mathbb{N}_2$ and $b_k = exp(\frac{2i\pi k}{n})$ for $0 \leq k \leq n-1$, in Banach spaces using the fixed point method, the general solution and stability of this equation and its generalizations were studied by numerous researchers (see, [18–22]).

Before proceeding to the main results, we state the following theorem which is useful for our purpose.

On the other hand, Brzdek and et al. [23] proved a simple fixed point theorem for some (not necessarily linear) operators and derived from it several quite general results on the stability of a very wide class of functional equations in single variable in 2011 as follows:

Theorem 1.4. *Let X be a nonempty set, (Y, d) be a complete metric space, $f_1, ..., f_s : X \to X$ and $L_1, ..., L_s : X \to \mathbb{R}_+$ be given mappings. Let $\Lambda : \mathbb{R}_+^X \to R_+^X$ be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)), \tag{1.4}$$

for $\delta \in \mathbb{R}_+^X$ and $x \in X$. If $T : Y^X \to Y^X$ is an operator satisfying the inequality

$$d(T\xi(x), T\mu(x)) \leq \textstyle\sum_{i=1}^{s} L_i(x) d(\xi(f_i(x)), \mu(f_i(x))), \qquad \xi, \mu \in Y^X, x \in X,$$

and a function $\varepsilon : X \to \mathbb{R}_+$ and a mapping $\varphi : X \to Y$ satisfy

$$d(T\varphi(x), \varphi(x)) \leq \varepsilon(x), (x \in X),$$

$$\varepsilon^*(x) := \textstyle\sum_{k=0}^{\infty} \Lambda^k \varepsilon(x) < \infty, \qquad (x \in X),$$

then for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} T^n\varphi(x),$$

exists and the function $\psi \in Y^X$ so defined is the unique fixed point of T with

$$d(\varphi(x), \psi(x)) \leq \varepsilon^*(x), (x \in X).$$

In 2018, Almahalebi [24] investigated the stability of the following generalization of Cauchys and the quadratic functional equations (1.4) in Banach spaces. Also, he proved the hyperstability results of this equation by the fixed point method of the Brzdek's [1] results as follows:

Theorem 1.5. *Let X be a $\mathbb{C}-$normed space, Y be a Banach space, $\varepsilon : X_0 \times X_0 \to \mathbb{R}_+$ and*

$$l(X) := \{u \in Aut(X) : u', (u' - b_k u) \in Aut(X),$$

$$\alpha_u := n\lambda(u') + nu' + \sum_{k=1}^{n-1} \lambda(u' - b_k u) < 1\} \neq \emptyset$$

is an infinite set, where

$$\lambda(u) := inf\ \{t \in \mathbb{R}_+ : \varepsilon(ux, uy) \leq t\varepsilon(x, y), \forall x, y \in X_0\}$$

for all $u \in Aut(X)$. Assume that $f : X \to Y$ satisfies the inequality

$$\|f(x+y) - nf(x) - nf(y) + \sum_{k=1}^{n-1} f(x+b_k y)\| \leq \varepsilon(x,y)$$

for all $x, y \in X_0$such that $x + b_k y \neq 0$ for $0 < k < n-1$. Then, for each nonempty subset $\mathcal{U} \subset l(X)$ such that

$$u \circ v = v \circ u, \forall u, v \in \mathcal{U}, \tag{1.5}$$

there exists a unique function $Q : X \to Y$ satisfies the Eq. (1.2) *and*

$$\|f(x) - Q(x)\| \leq \bar{\varepsilon}(x) \quad x \in X_0, \tag{1.6}$$

where

$$\bar{\varepsilon}(x) := inf\ \{\frac{\varepsilon(u'x, ux)}{1-\alpha_u} : u \in \mathcal{U}\} \qquad x \in X_0.$$

In this paper, we extend and improve the concept of Almahalebi [24] to qusic-Banach spaces by using the fixed point method. Second, we investigate the stability of the following generalization of Cauchys and the quadratic functional equations

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + nf(y),$$

where $n \in N_2 b_k = \exp(2i\pi k)$ for $0 \leq k \leq n-1$, in quasi-Banach spaces. Moreover, we obtain the hyperstability results of this equation by using the fixed point method.

2. A Fixed Point Approach to the Stability

For our subsequent results, we take the following four hypotheses.
(H1) X is a nonempty set and (Y, d) is a complete b-metric space.
(H2) $f_1, ..., f_k : X \to X$ and $L_1, ..., L_k : X \to \mathbb{R}_+$ are given maps.
(H3) $\mathscr{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d((\mathscr{T}\xi)(x), (\mathscr{T}\mu)(x)) \leq \sum_{i=1}^{k} L_i(x) d(\xi(f_i(x)), \mu(f_i(x))), \quad \xi, \mu \in Y^X,\ x \in X \tag{2.1}$$

(H4) Λ is a linear operator defined by

$$(\Lambda\delta)(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)) \tag{2.2}$$

for $\delta : X \to \mathbb{R}_+$ and $x \in X$. Obviously, Λ is monotone with respect to the pointwise ordering in $\mathbb{R}_+^X$ (provided that L_i is nonnegative).

The basic tool in this paper is the following theorem which asserts the existence of a unique fixed point of operator $\mathscr{T} : Y^X \to Y^X$.

Theorem 2.1. *Assume that hypotheses (H1)–(H4) are satisfied. Suppose that there are functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that, for all $x \in X$,*

$$d((\mathscr{T}\varphi)(x), \varphi(x)) \leq \varepsilon(x) \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} s^{n+1}(\Lambda^n \varepsilon)(x) =: \varepsilon^*(x) < \infty \tag{2.4}$$

hold. Then, for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} (\mathscr{T}^n \varphi)(x) \tag{2.5}$$

exists and the function $\psi : X \to Y$ so defined is a unique fixed point of $\mathscr{T}$ with

$$d(\varphi(x), \psi(x)) \le \varepsilon^*(x), \quad x \in X. \tag{2.6}$$

Proof. First we show by induction that, for every $n \in \mathbb{N}_+$,

$$d((\mathscr{T}^n\varphi)(x), (\mathscr{T}^{n+1}\varphi)(x)) \le (\Lambda^n\varepsilon)(x), \quad x \in X. \tag{2.7}$$

Clearly, by (2.5), the case $n = 0$ is trivial. Now fix $n \in \mathbb{N}_+$ and suppose that (2.7) is valid. Then, using hypothesis (H3) and the inductive assumption, for every $x \in X$, we get

$$\begin{aligned} d((\mathscr{T}^{n+1}\varphi)(x), (\mathscr{T}^{n+2}\varphi)(x)) &\le \sum_{i=1}^{k} L_i(x) d((\mathscr{T}^n\varphi)(f_i(x)), (\mathscr{T}^{n+1}\varphi)(f_i(x))) \\ &\le \sum_{i=1}^{k} L_i(x)(\Lambda^n\varepsilon)(f_i(x)) = (\Lambda^{n+1}\varepsilon)(x) \end{aligned}$$

completing the proof of (2.7). Therefore, for $n, k \in \mathbb{N}_+, k > 0$,

$$\begin{aligned} d((\mathscr{T}^n\varphi)(x), (\mathscr{T}^{n+k}\varphi)(x)) &\le \sum_{i=0}^{k-1} s^{i+1} d((\mathscr{T}^{n+i}\varphi)(x), (\mathscr{T}^{n+i+1}\varphi)(x)) \\ &\le \sum_{i=n}^{n+k-1} s^{i+1}(\Lambda^i\varepsilon)(x) \le \varepsilon^*(x), \quad x \in X. \end{aligned} \tag{2.8}$$

By the convergence of the series $\sum(\Lambda^n\varepsilon)(x)$, it follows from the above estimate that, for every $x \in X$, $((\mathscr{T}^n\varphi)(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and, as (Y, d) is complete, the limit $\psi(x)$ exists for all $x \in X$. Taking $n = 0$ and letting $k \to \infty$ in (2.8), we obtain that (2.6) holds and, in view of (2.1), $\mathscr{T}(\psi) = \psi$.

For the proof of the uniqueness of ψ, suppose that $\psi_1, \psi_2 \in Y^X$ are two fixed points of $\mathscr{T}$ with $d(\psi(x), \psi_i(x)) \le \varepsilon^*(x)$ for $x \in X$, $i = 1, 2$. We next prove that, for every $m \in \mathbb{N}$,

$$d(\psi_1(x), \psi_2(x)) = d((\mathscr{T}^m\varphi_1)(x), (\mathscr{T}^m\varphi_2)(x)) \le 2s\sum_{i=m}^{\infty}(\Lambda^i\varepsilon)(x), \quad x \in X. \tag{2.9}$$

Clearly (for $m = 0$),

$$d(\psi_2(x), \psi_1(x)) \le s[d(\psi_1(x), \psi(x)) + d(\psi(x), \psi_2(x))] \le 2s\varepsilon^*(x), \quad x \in X.$$

Now assume that (2.9) is valid for some $m \in \mathbb{N}_+$. Then, by (2.1), for every $x \in X$ we obtain the following inequality:

$$\begin{aligned}
d((\mathscr{T}^{m+1}\varphi_2)(x),(\mathscr{T}^{m+1}\varphi_1)(x)) &= d(\mathscr{T}(\mathscr{T}^{m}\varphi_2)(x),\mathscr{T}(\mathscr{T}^{m}\varphi_1)(x)) \\
&\le \sum_{i=1}^{k} L_i(x) d((\mathscr{T}^{m}\varphi_2)(f_i(x)),(\mathscr{T}^{m}\varphi_1(f_i(x))) \\
&\le 2s\sum_{i=1}^{k} L_i(x)\left(\sum_{j=m}^{\infty}(\Lambda^j\varepsilon)(f_i(x))\right) \\
&\le 2s\sum_{j=m}^{\infty}\left(\sum_{i=1}^{k} L_i(x)(\Lambda^j\varepsilon)(f_i(x))\right) \\
&\le 2s\sum_{j=m+1}^{\infty}(\Lambda^j\varepsilon)(x).
\end{aligned}$$

Thus we have proved that (2.9) holds for $m \in \mathbb{N}_+$. Now letting $m \to \infty$, on account of (2.4), we get $\psi_1 = \psi_2$. ■

Directly from Theorem 2.1 we obtain the following corollary.

Corollary 2.2. *Assume that hypotheses (H1)–(H4) are satisfied. Suppose that there exist two functions $\varepsilon : X \to \mathbb{R}_+, \psi : X \to Y$ and a constant $q \in [0, \frac{1}{s})$ such that, for all $x \in X$,* (2.3) *and*

$$(\Lambda\varepsilon)(x) \le q\varepsilon(x) \tag{2.10}$$

hold. Then the limit (2.5) *exists for every $x \in X$ and the function $\psi : X \to Y$ so defined is the unique fixed point of $\mathscr{T}$ with*

$$d(\varphi(x),\psi(x)) \le \frac{s}{1-qs}\varepsilon(x), \quad x \in X.$$

Proof. Iterating inequality (2.10), for $n \in N$ and $x \in X$, it follows that

$$(\Lambda^n\varepsilon)(x) \le (\frac{s}{1-qs})^n\varepsilon(x).$$

Therefore, for all $x \in X$,

$$\varepsilon^*(x) = \sum_{n=0}^{\infty}(\Lambda^n\varepsilon)(x) \le \sum_{n=0}^{\infty} r^n\varepsilon(x) = r\varepsilon(x),$$

where $r = \frac{s}{1-qs}$. Thus, condition (2.4) holds and the statement follows from Theorem 2.1. In fact the statement about the uniqueness is not the consequence of that of Theorem 2.1, however, the argument followed in the proof of Theorem 2.1 can easily be adjusted. ■

3. Stability of a Functional Equation in a Quasi-Banach Space

We will denote by $Aut(X)$ the family of all automorphisms of X. Moreover, for each $u \in X^X$, we write $ux := u(x)$ for $x \in X$ and we define $\acute{u}$ by $\acute{u}x := x - ux$ for $x \in X$.

The following theorem is the main result concerning the stability of the functional in equation .

Theorem 3.1. *Let X be a $\mathbb{C}-$normed space, Y be a b-Banach space, $\varepsilon : X_0 \times X_0 \to \mathbb{R}_+$ and*

$$l(X) := \{u \in Aut(X) : u', (u' - b_k u) \in Aut(X),$$

$$\alpha_u := n\lambda(u') + nu' + \sum_{k=1}^{n-1} \lambda(u' - b_k u) < \frac{1}{s}\} \neq \emptyset \tag{3.1}$$

is an infinite set, where

$$\lambda(u) := inf \ \{t \in \mathbb{R}_+ : \varepsilon(ux, uy) \leq t\varepsilon(x, y), \forall x, y \in X_0\}$$

for all $u \in Aut(X)$. Assume that $f : X \to Y$ satisfies the inequality

$$\|f(x + y) - nf(x) - nf(y) + \sum_{k=1}^{n-1} f(x + b_k y)\| \leq \varepsilon(x, y) \tag{3.2}$$

for all $x, y \in X_0$such that $x + b_k y \neq 0$ for $0 < k < n-1$. Then, for each nonempty subset $\mathcal{U} \subset l(X)$ such that

$$u \circ v = v \circ u, \quad \forall u, v \in \mathcal{U}, \tag{3.3}$$

there exists a unique function $Q : X \to Y$ satisfies the equation (1.3) and

$$\|f(x) - Q(x)\| \leq \bar{\varepsilon}(x) \quad x \in X_0, \tag{3.4}$$

where

$$\bar{\varepsilon}(x) := inf \ \{\frac{s\varepsilon(u'x, ux)}{1 - s\alpha_u} : u \in \mathcal{U}\}, \qquad x \in X_0.$$

Proof. Let us fix $u \in \mathcal{U}$ and replacing x with $u'x$ and y with ux in inequality (3.2), we get

$$\|f(x) - nf(u'x) - nf(ux) + \sum_{k=1}^{n-1} f((u' - b_k u)x)\| \leq \varepsilon(u'x, ux) := \varepsilon_u(x) \tag{3.5}$$

for all $x \in X_0$. Given $u \in \mathcal{U}$, we define the operators $\mathcal{T}_u : Y^{X_0} \to Y^{X_0}$ and $\Lambda_u : \mathbb{R}_+^{X_0} \to \mathbb{R}_+^{X_0}$ by

$$\mathcal{T}_u\xi(x) := n\xi(u'x) + n\xi(ux) - \textstyle\sum_{k=1}^{n-1} \xi((u' - b_k u)x),$$

$$\Lambda_u\delta(x) := n\delta(u'x) + n\delta(ux) - \sum_{k=1}^{n-1} \delta((u' - b_k u)x) \tag{3.6}$$

for all $x \in X_0, \xi \in Y_0^X$ and $\delta \in \mathbb{R}_+^{X_0}$. Then, the inequality (3.5) takes the form

$$\|f(x) - \mathcal{T}_u f(x)\| \leq \varepsilon_u(x)$$

for all $x \in X_0$.

Observe that the operator Λ_u has the form given by (2.3) with $s = n+1$ and $f_{n+1}(x) = ux, f_n(x) = u'x, f_i(x) = (u' - b_i u)x, L_n(x) = sn = L_{n+1}(x), L_i(x) = s, i \in \{1, 2, ..., n-1\}$ for all $x \in X_0$. Furthermore, for each $x \in X_0$ and $\xi, \mu \in Y_0^X$, we obtain

$$\begin{aligned}\|\mathcal{T}_u\xi(x) - \mathcal{T}_u\mu(x)\| &= \|n\xi(u'x) + n\xi(ux) - \textstyle\sum_{k=1}^{n-1} \xi((u' - b_k u)x) \\ &\quad -n\mu(u'x) - n\mu(ux) + \textstyle\sum_{k=1}^{n-1} \mu((u' - b_k u)x)\| \\ &\leq sn\|\xi(u'x) - \mu(u'x)\| + sn\|\xi(ux) - \mu(ux)\| \\ &\quad + \textstyle\sum_{k=1}^{n-1} s\|\xi((u' - b_k u)x) - \mu((u' - b_k u)x)\|\end{aligned}$$

In view of the definition of $\lambda(u)$, we note that

$$\varepsilon(ux, uy) \leq \lambda(u)\varepsilon(x, y), \qquad x, y \in X_0.$$

So, it is easy to show that

$$\Lambda_u^s \varepsilon_u(x) \leq \alpha_u^s(u'x, ux),$$

for all $x \in X_0$ and $s \in \mathbb{N}_0$, where

$$\alpha_u = n\lambda(u') + n\lambda(u) + \textstyle\sum_{k=1}^{n-1} \lambda(u' - b_k u).$$

Hence, we obtain

$$\varepsilon^*(x) := \sum_{r=0}^{\infty} s^{r+1}\Lambda_u^r \varepsilon_u(x) \leq s\varepsilon(u'x, ux) \sum_{r=0}^{\infty} (s\alpha_u)^r = \frac{s\varepsilon(u'x, ux)}{1 - s\alpha_u} < \infty \tag{3.7}$$

for all $x \in X_0$. Therefore, according to Theorem 2.1, there exists a unique solution $Q_u : X \to Y$ of the equation

$$Q_u(x) = nQ_u(u'x) + nQ_u(ux) - \sum_{k=1}^{n-1} Q_u((u' - b_k u)x) \tag{3.8}$$

for all$x \in X_0$, which is the fixed point of $\mathcal{T}_u$ such that

$$\|Q_u(x) - f(x)\| \leq \frac{s\varepsilon(u'x, ux)}{1 - s\alpha_u}, \qquad x \in X_0. \tag{3.9}$$

Moreover,

$$Q_u(x) = \lim_{r\to\infty} \mathcal{T}_u^r f(x)$$

To prove that Q_u satisfies the functional (1.4) on X_0, we just prove the following inequality:

$$\|\mathcal{T}_u^r f(x + y) - n\mathcal{T}_u^r f(x) - n\mathcal{T}_u^r f(y) + \sum_{k=1}^{n-1} \mathcal{T}_u^r f(x + b_k y)\| \leq \alpha_u^r \varepsilon(x, y) \tag{3.10}$$

for all $x, y \in X_0$ and all $r \in \mathbb{N}_0$ such that $x + b_k y \neq 0$ for $0 < k < n - 1$. Since the case $r = 0$ is just (3.2), take $r \in \mathbb{N}$ and assume that (3.10) holds for all $x, y \in X_0$ and $r \in \mathbb{N}$.

Then, using (3.6) and the triangle inequality in p-quasi-Banach, we get

$$\|\mathcal{T}_u^{r+1}f(x+y)-n\mathcal{T}_u^{r+1}f(x)-n\mathcal{T}_u^{r+1}f(y)+\sum_{k=1}^{n-1}\mathcal{T}_u^{r+1}f(x+b_ky)\|^p$$
$$=\|n\mathcal{T}_u^{r}f(u'(x+y))+n\mathcal{T}_u^{r}f(u(x+y))-\sum_{k=1}^{n-1}\mathcal{T}_u^{r}f((u'-b_ku)(x+y))$$
$$-n^2\mathcal{T}_u^{r}f(u'x)-n^2\mathcal{T}_u^{r}f(ux)+n\sum_{k=1}^{n-1}\mathcal{T}_u^{r}f((u'-b_ku)x)$$
$$-n^2\mathcal{T}_u^{r}f(u'y)-n^2\mathcal{T}_u^{r}f(uy)+n\sum_{k=1}^{n-1}\mathcal{T}_u^{r}f((u'-b_ku)y)$$
$$+\sum_{k=1}^{n-1}\{n\mathcal{T}_u^{r}f(u'(x+b_ky))+n\mathcal{T}_u^{r}(u(x+b_ky))$$
$$-\sum_{k=1}^{n-1}\mathcal{T}_u^{r}f((u'-b_ku)(x+b_ky))\}\|^p$$
$$\leq n^p\|\mathcal{T}_u^{r}f(u'(x+y))-n\mathcal{T}_u^{r}f(u'x)-n\mathcal{T}_u^{r}f(u'y)+\sum_{k=1}^{n-1}\mathcal{T}_u^{r}f(u'(x-b_ky)\|^p$$
$$+n^p\|\mathcal{T}_u^{r}f(u(x+y))-n\mathcal{T}_u^{r}f(ux)-n\mathcal{T}_u^{r}f(uy)+\sum_{k=1}^{n-1}\mathcal{T}_u^{r}f(u(x-b_ky)\|$$
$$+\sum_{k=1}^{n-1}\|\mathcal{T}_u^{r}f(u'-b_ku)(x+y))-n\mathcal{T}_u^{r}f((u'-b_ku)x)$$
$$-n\mathcal{T}_u^{r}f((u'-b_ku)y)+\sum_{k=1}^{n-1}\mathcal{T}_u^{r}f(u'-b_ku)(x+b_ky))\|^p$$
$$\leq \alpha_u^{rp}(n^p\varepsilon^p(u'x,u'y)+n^p\varepsilon^p(ux+uy)+\sum_{k=1}^{n-1}\varepsilon^p((u'-b_ku)x),(u'-b_ku)y)))$$
$$\leq \alpha_u^{rp}(n^p\lambda^p(u')+n^p\lambda^p(u)+\sum_{k=1}^{n-1}\lambda^p(u'-b_ku))\varepsilon^p(x,y)$$
$$\leq \alpha_u^{rp}(n\lambda(u')+n\lambda(u)+\sum_{k=1}^{n-1}\lambda(u'-b_ku))^p\varepsilon^p(x,y)$$
$$=\alpha_u^{rp+p}\varepsilon^p(x,y).$$

Since $0<p\leq 1$, we get

$$\|\mathcal{T}_u^{r+1}f(x+y)-n\mathcal{T}_u^{r+1}f(x)-n\mathcal{T}_u^{r+1}f(y)+\sum_{k=1}^{n-1}\mathcal{T}_u^{r+1}f(x+b_ky)\|\leq \alpha_u^{r+1}\varepsilon(x,y).$$

Thus, by the mathematical induction, we have shown that (3.10) holds for all $x,y\in X_0$ and all $r\in\mathbb{N}_0$ such that $x+b_ky\neq 0$ for $0<k<n-1$. Letting $r\to\infty$ in (3.10), we get

$$\sum_{k=0}^{n-1}Q_u(x+b_ky)=nQ_u(x)+nQ_u(y),$$

for all $x,y\in X_0$ such that $x+b_ky=0$ for $0\leq k\leq n-1$. Thus, we have proved that for each $u\in\mathcal{U}$, there exists a function $Q_u:X_0\to Y$ which is a solution of the functional equation (1.4) on X_0 and satisfies

$$\|f(x)-Q_u(x)\|\leq\frac{\varepsilon(u'x,ux)}{1-\alpha_u}$$

for all $x\in X_0$.

Next, we prove that each solution $Q:X\to Y$ of (1.3) satisfying the inequality

$$\|f(x)-Q(x)\|\leq L\varepsilon(v'x,vx)\quad x\in X_0 \tag{3.11}$$

with some $L > 0$ and $v \in \mathcal{U}$ is equal to J_w for each $w \in \mathcal{U}$. So, fix $v, w \in \mathcal{U}$,$L > 0$ and $Q : X \to Y$ a solution of (1.4) satisfying (3.11). Note that, by (3.9) and (3.11), there is $L_0 > 0$ such that

$$\begin{aligned}\|Q(x) - Q_w(x)\| &\leq \|Q(x) - f(x)\| + \|f(x) - Q_w(x)\| \\ &\leq L_0 s(\varepsilon(v'x, vx) + \varepsilon(w'x, wx)) \cdot \sum_{r=0}^{\infty} (\alpha_w s)^r \end{aligned} \tag{3.12}$$

for $x \in X_0$. In other side, Q and Q_w are solutions of (3.8) because they satisfy (1.3). For each $j \in \mathbb{N}$, we show that

$$\|Q(x) - Q_w(x)\| \leq L_0 s(\varepsilon(v'x, vx) + \varepsilon(w'x, wx)) \cdot \sum_{r=j}^{\infty} (\alpha_w s)^r \quad x \in X_0. \tag{3.13}$$

The case $j = 0$ is exactly (3.12). So fix $\gamma \in \mathbb{N}_0$ and assume that (3.13) holds for $j = \gamma$. Then, in view of the definition of $\lambda(u)$, we obtain

$$\begin{aligned}\|Q(x) - Q_w(x)\| &= \|nQ(w'x) + nQ(wy) - \textstyle\sum_{k=1}^{n-1} Q((w' - b_k w)x) \\ &\quad -nQ_w(w'x) - nQ_w(wx) + \textstyle\sum_{k=1}^{n-1} Q_w((w' - b_k w)x)\| \\ &\leq ns\|Q(w'x) - Q_w(w'x)\| + ns\|Q(wx) - Q_w(wx)\| \\ &\quad +s \textstyle\sum_{k=1}^{n-1} \|J((w' - b_k w)x) - J_w((w' - b_k w)x)\| \\ &\leq s^2 n L_0(\varepsilon(v'w'x, vw'x) + \varepsilon(w'w'x, ww'x)) \cdot \textstyle\sum_{r=\gamma}^{\infty} (\alpha_w s)^r \\ &\quad +s^2 n L_0(\varepsilon(v'wx, vwx) + \varepsilon(w'wx, wwx)) \cdot \textstyle\sum_{r=\gamma}^{\infty} (\alpha_w s)^r \\ &\quad +s L_0 \textstyle\sum_{k=1}^{n-1} (\varepsilon(v'(w' - b_k w)x, v(w' - b_k w)x) \\ &\quad +\varepsilon(w'(w' - b_k w)x, w(w' - b_k w)x)) \cdot \textstyle\sum_{r=\gamma}^{\infty} (\alpha_w s)^r \\ &\leq s^2 L_0(\varepsilon(v'x, vx) + \varepsilon(w'x, wx))(n\lambda(w') + n\lambda(w) \\ &\quad + \textstyle\sum_{k=1}^{n-1} \lambda(w' - b_k w)) \cdot \textstyle\sum_{r=\gamma}^{\infty} \alpha_w^r. \\ &= s L_0(\varepsilon(v'x, vx) + \varepsilon(w'x, wx)) \cdot \textstyle\sum_{r=\gamma+1}^{\infty} (\alpha_w s)^r.\end{aligned}$$

Now, letting $j \to \infty$ in (3.13) , we get

$$Q(x) = Q_w(x) \qquad \forall x \in X_0. \tag{3.14}$$

In this way, we also have proved that $Q_u = Q_w$ for each $u \in \mathcal{U}$, which yields

$$\|f(x) - Q_u(x)\| \leq \frac{s\varepsilon(u'x, ux)}{1 - s\alpha_u} \quad x \in X_0, u \in \mathcal{U}.$$

This implies (3.4) with $Q := Q_w$ and the uniqueness of Q is given by (3.14). ■

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