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Regular Transformation Semigroups on Some Dictionary Chains

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Abstract : Denote by $OT(X)$ the full order-preserving transformation semigroup on a poset X. The following results are known. If X is any nonempty subset of $\mathbb{Z}$ with the natural order, then $OT(X)$ is a regular semigroup, that is, for every $\alpha \in OT(X)$, $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(X)$. If $\leq_d$ is the dictionary partial order on $X \times X$ where X is a nonempty subset of $\mathbb{Z}$, then $OT(X \times X, \leq_d)$ is regular if and only if X is finite. By using these two known results, we extend the second one to the semigroup $OT(X \times Y, \leq_d)$ where X and Y are nonempty subsets of $\mathbb{Z}$. It is shown that $OT(X \times Y, \leq_d)$ is regular if and only if $|X| = 1$ or Y is finite.

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1 Introduction

An element a of a semigroup S is called *regular* if $a = aba$ for some $b \in S$, and S is said to be a *regular semigroup* if every element of S is regular.

For a nonempty set X, let $T(X)$ be the full transformation semigroup on X, that is, $T(X)$ is the semigroup, under composition, of all mappings $\alpha : X \to X$. The image of $x \in X$ under $\alpha \in T(X)$ is written by $x\alpha$. The range of $\alpha \in T(X)$ is denoted by $\operatorname{ran}\alpha$. It is well-known that $T(X)$ is a regular semigroup ([1], page 4 or [2], page 63).

The following easy fact which was given in [5] and [6] will be used.

Proposition 1.1 ([5], [6]). *Let X be a nonempty set and $\alpha, \beta \in OT(X)$. If $\alpha = \alpha\beta\alpha$, then $\operatorname{ran}(\beta\alpha) = \operatorname{ran}\alpha$ and $x\beta\alpha = x$ for all $x \in \operatorname{ran}\alpha$.*

A mapping φ from a poset X into a poset Y is said to be *order-preserving* if

$$\text{for all } x, x' \in X, \quad x \leq x' \text{ in } X \ \Rightarrow \ x\varphi \leq x'\varphi \text{ in } Y.$$

A bijection φ from a poset X onto a poset Y is called an *order-isomorphism* if φ and φ^{-1} are order-preserving. The posets X and Y are said to be *order-isomorphic* if there is an order-isomorphism from X onto Y.

For a poset X, let $OT(X)$ be the set of all order-preserving mappings α :

$X \to X$. Then $OT(X)$ is a subsemigroup of $T(X)$ which is called the *full order-preserving transformation semigroup* on X. If φ is an order-isomorphism from a poset X onto a poset Y, then $\theta : OT(X) \to OT(Y)$ defined by

$$\theta(\alpha) = \varphi^{-1}\alpha\varphi \ \text{ for all } \ \alpha \in OT(X)$$

is clearly an isomorphism from $OT(X)$ onto $OT(Y)$.

Proposition 1.2. *If the posets X and Y are order-isomorphic, then $OT(X)$ and $OT(Y)$ are isomorphic.*

It is known from [1, page 203] that if X is a finite chain, then $OT(X)$ is a regular semigroup. In 2000, Kemprasit and Changphas [3] extended this result to any chain which is order-isomorphic to a subset of $\mathbb{Z}$, the set of integers with their natural order. In [4], the authors generalized full order-preserving transformation semigroups by using sandwich multiplication and investigated their regularity. Some isomorphism theorems are also provided.

Proposition 1.3 ([3])**.** *If X is a chain which is order-isomorphic to a subset of $\mathbb{Z}$ with the natural order, then $OT(X)$ is a regular semigroup.*

For chains X and Y, define the *dictionary partial order* $\leq_d$ on $X \times Y$ by

$$(x, y) \leq_d (x', y') \Leftrightarrow \begin{array}{l} \text{(i)} \ \ x < x' \text{ or} \\ \text{(ii)} \ \ x = x' \text{ and } y \leq y'. \end{array}$$

Then $(X \times Y, \leq_d)$ becomes a chain. The following result was proved in [5] and [6].

Theorem 1.4 ([5], [6])**.** *Let X be a nonempty subset of $\mathbb{Z}$ with the natural order. Then $OT(X \times X, \leq_d)$ is a regular semigroup if and only if X is finite.*

Our purpose is to extend Theorem 1.4 by considering the regularity of $OT(X \times Y, \leq_d)$ where X and Y are nonempty subsets of $\mathbb{Z}$. It will be shown that $OT(X \times Y, \leq_d)$ is a regular semigroup if and only if $|X| = 1$ or Y is finite where $|X|$ denotes the cardinality of X.

2 Main Results

Let $\mathbb{Z}^+$ and $\mathbb{Z}^-$ denote respectively the set of all positive integers and the set of all negative integers. It is clear that if $\emptyset \neq X \subseteq \mathbb{Z}$, then with the natural order, X satisfies the following properties:

(I) X is bounded above and bounded below if and only if X is order-isomorphic to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{Z}^+$,

(II) X is not bounded above but bounded below if and only if X is order-isomorphic to $\mathbb{Z}^+$,

(III) X is bounded above but not bounded below if and only if X is order-isomorphic to $\mathbb{Z}^-$ and
(IV) X is neither bounded above nor bounded below if and only if X is order-isomorphic to $\mathbb{Z}$.

Throughout, the partial order on a nonempty subset of $\mathbb{Z}$ always means the natural partial order.

To obtain the main theorem, the following two lemmas are needed. The proof of the first one straightforword.

Lemma 2.1. *If φ_1 is an order-isomorphism from a chain X onto a chain $X^{'}$ and φ_2 is an order-isomorphism from a chain Y onto a chain $Y^{'}$, then $\varphi : X \times Y \to X^{'} \times Y^{'}$ defined by*

$$(x, y)\varphi = (x\varphi_1, y\varphi_2) \text{ for all } x \in X \text{ and } y \in Y$$

is an order-isomorphism from the chain $(X \times Y, \leq_d)$ onto $(X^{'} \times Y^{'}, \leq_d^{'})$ where $\leq_d$ and $\leq_d^{'}$ are the dictionary partial order on $X \times Y$ and $X^{'} \times Y^{'}$, respectively.

Lemma 2.2. *Let X and Y be nonempty subsets of $\mathbb{Z}$. If Y is finite, then $(X \times Y, \leq_d)$ is order-isomorphic to a subchain of $\mathbb{Z}$.*

Proof. Since Y is finite, it follows that Y is order-isomorphic to $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{Z}^+$.

If X is finite, then $(X \times Y, \leq_d)$ is a finite chain, so it order-isomorphic to a finite subchain of $\mathbb{Z}$.

Next, assume that X is infinite. Then X is order-isomorphic to $\mathbb{Z}^+, \mathbb{Z}^-$ or $\mathbb{Z}$. Therefore by Lemma 2.1, $(X \times Y, \leq_d)$ is order-isomorphic to one of the following chains:

$$(\mathbb{Z}^+ \times \{1, 2, \ldots, n\}, \leq_d), (\mathbb{Z}^- \times \{1, 2, \ldots, n\}, \leq_d), (\mathbb{Z} \times \{1, 2, \ldots, n\}, \leq_d).$$

We have that

$$\mathbb{Z} \times \{1, 2, \ldots, n\} = \{(k, i) \mid k \in \mathbb{Z} \text{ and } i \in \{1, 2, \ldots, n\}\}$$

and

$$\ldots <_d (-1, 1) <_d \ldots <_d (-1, n) <_d (0, 1) <_d \ldots <_d (0, n) <_d (1, 1) <_d \ldots <_d (1, n) <_d \ldots$$

Define $\varphi : \mathbb{Z} \times \{1, 2, \ldots, n\} \to \mathbb{Z}$ by

$$(k, i)\varphi = kn + i \text{ for } k \in \mathbb{Z} \text{ and } i \in \{1, 2, \ldots, n\}.$$

It is clearly seen that φ is injective and order-preserving. Consequently, $(\mathbb{Z} \times \{1, 2, \ldots, n\}, \leq_d)$ is order-isomorphic to a subchain of $\mathbb{Z}$. Since $(\mathbb{Z}^+ \times \{1, 2, \ldots, n\}$ $\leq_d)$ and $(\mathbb{Z}^- \times \{1, 2, \ldots, n\}, \leq_d)$ are subchains of $(\mathbb{Z} \times \{1, 2, \ldots, n\}, \leq_d)$, it follows that $(X \times Y, \leq_d)$ is order-isomorphic to a subchain of $\mathbb{Z}$, as desired. ☐

Theorem 2.3. *Let X and Y be nonempty subsets of $\mathbb{Z}$. Then $OT(X \times Y, \leq_d)$ is a regular semigroup if and only if $|X| = 1$ or Y is finite.*

Proof. Assume that $OT(X \times Y, \leq_d)$ is regular and suppose that $|X| > 1$ and Y is infinite. By (I)-(IV) mentioned previously, Theorem 1.4 and Lemma 2.1, $(X \times Y, \leq_d)$ is order-isomorphic to one of the following chains:

$$(\mathbb{Z}^+ \times \mathbb{Z}, \leq_d), (\mathbb{Z}^- \times \mathbb{Z}, \leq_d), (\mathbb{Z} \times \mathbb{Z}^+, \leq_d), (\mathbb{Z} \times \mathbb{Z}^-, \leq_d),$$
$$(\mathbb{Z}^+ \times \mathbb{Z}^-, \leq_d), (\mathbb{Z}^- \times \mathbb{Z}^+, \leq_d), (\{1, 2, \dots, n\} \times \mathbb{Z}, \leq_d),$$
$$(\{1, 2, \dots, n\} \times \mathbb{Z}^+, \leq_d), (\{1, 2, \dots, n\} \times \mathbb{Z}^-, \leq_d) \text{ where } n > 1.$$

Since $OT(X \times Y, \leq_d)$ is regular, by Proposition 1.2, $OT(C)$ is regular if C is one of the above nine chains.

Case 1 : $C = (\mathbb{Z}^+ \times \mathbb{Z}, \leq_d)$. Define $\alpha : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{Z}^+ \times \mathbb{Z}$ by

$$(\{x\} \times \mathbb{Z})\alpha = \{(1, x)\} \text{ for all } x \in \mathbb{Z}^+.$$

Then $\alpha \in OT(\mathbb{Z}^+ \times \mathbb{Z}, \leq_d)$ and $\operatorname{ran} \alpha = \{1\} \times \mathbb{Z}^+$. Since $OT(\mathbb{Z}^+ \times \mathbb{Z}, \leq_d)$ is regular, there exists an element $\beta \in OT(\mathbb{Z}^+ \times \mathbb{Z}, \leq_d)$ such that $\alpha = \alpha\beta\alpha$. By Proposition 1.1,

$$(1, x)\beta\alpha = (1, x) \text{ for all } x \in \mathbb{Z}^+.$$

Since $(1, x) <_d (2, 1)$ for all $x \in \mathbb{Z}^+$, it follows that

$$(1, x) = (1, x)\beta\alpha \leq_d (2, 1)\beta\alpha \text{ for all } x \in \mathbb{Z}^+.$$

Since $(2, 1)\beta\alpha \in \operatorname{ran} \alpha = \{1\} \times \mathbb{Z}^+$, we have that $(2, 1)\beta\alpha = (1, k)$ for some $k \in \mathbb{Z}^+$, and hence $x \leq k$ for all $x \in \mathbb{Z}^+$, a contradiction.

Case 2 : $C = (\mathbb{Z}^- \times \mathbb{Z}, \leq_d)$. Define $\alpha : \mathbb{Z}^- \times \mathbb{Z} \to \mathbb{Z}^- \times \mathbb{Z}$ by

$$(\{x\} \times \mathbb{Z})\alpha = \{(-1, x)\} \text{ for all } x \in \mathbb{Z}^-.$$

Then $\alpha \in OT(\mathbb{Z}^- \times \mathbb{Z}, \leq_d)$ and $\operatorname{ran} \alpha = \{-1\} \times \mathbb{Z}^-$. Since $OT(\mathbb{Z}^- \times \mathbb{Z}, \leq_d)$ is regular, $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(\mathbb{Z}^- \times \mathbb{Z}, \leq_d)$. By Proposition 1.1,

$$(-1, x)\beta\alpha = (-1, x) \text{ for all } x \in \mathbb{Z}^-.$$

But $(-2, -1) <_d (-1, x)$ for all $x \in \mathbb{Z}^-$, so

$$(-1, x) = (-1, x)\beta\alpha \geq_d (-2, -1)\beta\alpha = (-1, l) \text{ for some } l \in \mathbb{Z}^-.$$

since $(-2, -1)\beta\alpha \in \operatorname{ran} \alpha = \{-1\} \times \mathbb{Z}^-$. Hence $x \geq l$ for all $x \in \mathbb{Z}^-$ which is a contradiction.

Case 3 : $C = (\mathbb{Z} \times \mathbb{Z}^+, \leq_d)$. Define $\alpha : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Z} \times \mathbb{Z}^+$ by

$$(\{x\} \times \mathbb{Z}^+)\alpha = \{(1, 1)\} \text{ if } x \in \mathbb{Z}^- \cup \{0\}$$

and

$$(\{x\} \times \mathbb{Z}^+)\alpha = \{(1, x)\} \text{ if } x \in \mathbb{Z}^+.$$

Then $\alpha \in OT(\mathbb{Z} \times \mathbb{Z}^+, \leq_d)$ and $\operatorname{ran}\alpha = \{1\} \times \mathbb{Z}^+$. By the same proof of Case 1, we have that $\mathbb{Z}^+$ is bounded above, a contradiction.

Case 4 : $C = (\mathbb{Z} \times \mathbb{Z}^-, \leq_d)$. Let $\alpha : \mathbb{Z} \times \mathbb{Z}^- \to \mathbb{Z} \times \mathbb{Z}^-$ be defined by

$$(\{x\} \times \mathbb{Z}^-)\alpha = \{(-1, -1)\} \text{ if } x \in \mathbb{Z}^+ \cup \{0\}$$

and

$$(\{x\} \times \mathbb{Z}^-)\alpha = \{(-1, x)\} \text{ if } x \in \mathbb{Z}^-.$$

Then $\alpha \in OT(\mathbb{Z} \times \mathbb{Z}^-, \leq_d)$ and $\operatorname{ran}\alpha = \{-1\} \times \mathbb{Z}^-$. We can see from the proof of Case 2 that $\mathbb{Z}^-$ is bounded below, a contradiction.

Case 5 : $C = (\mathbb{Z}^+ \times \mathbb{Z}^-, \leq_d)$ or $C = (\{1, 2, \ldots, n\} \times \mathbb{Z}^-, \leq_d)$ where $n > 1$. Let $\alpha : C \to C$ be defined by

$$(x, y)\alpha = \begin{cases} (2, y) & \text{if } x = 1 \text{ and } y \in \mathbb{Z}^-, \\ (2, -1) & \text{otherwise.} \end{cases}$$

Then $\alpha \in OT(C)$ and $\operatorname{ran}\alpha = \{2\} \times \mathbb{Z}^-$. Since $OT(C)$ is regular, we have that $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(C)$. By Proposition 1.1,

$$(2, x)\beta\alpha = (2, x) \text{ for all } x \in \mathbb{Z}^-.$$

Since $(1, -1) <_d (2, x)$ for all $x \in \mathbb{Z}^-$, it follows that

$$(2, x) = (2, x)\beta\alpha \geq_d (1, -1)\beta\alpha \text{ for all } x \in \mathbb{Z}^-.$$

Since $(1, -1)\beta\alpha \in \operatorname{ran}\alpha = \{2\} \times \mathbb{Z}^-$, there is an element $r \in \mathbb{Z}^-$ such that $(1, -1)\beta\alpha = (2, r)$, and hence $x \geq r$ for all $x \in \mathbb{Z}^-$ which is a contradiction.

Case 6 : $C = (\mathbb{Z}^- \times \mathbb{Z}^+, \leq_d)$. Define $\alpha : \mathbb{Z}^- \times \mathbb{Z}^+ \to \mathbb{Z}^- \times \mathbb{Z}^+$ by

$$(x, y)\alpha = \begin{cases} (-2, y) & \text{if } x = -1 \text{ and } y \in \mathbb{Z}^+, \\ (-2, 1) & \text{otherwise.} \end{cases}$$

Then $\alpha \in OT(\mathbb{Z}^- \times \mathbb{Z}^+, \leq_d)$ and $\operatorname{ran}\alpha = \{-2\} \times \mathbb{Z}^+$. Since $OT(\mathbb{Z}^- \times \mathbb{Z}^+, \leq_d)$ is regular, $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(\mathbb{Z}^- \times \mathbb{Z}^+, \leq_d)$. By Proposition 1.1,

$$(-2, x)\beta\alpha = (-2, x) \text{ for all } x \in \mathbb{Z}^+.$$

But $(-2, x) <_d (-1, 1)$ for all $x \in \mathbb{Z}^+$, so

$$(-2, x) = (-2, x)\beta\alpha \leq_d (-1, 1)\beta\alpha = (-2, s) \text{ for some } s \in \mathbb{Z}^+$$

since $(-1,1)\beta\alpha \in \operatorname{ran}\alpha = \{-2\} \times \mathbb{Z}^+$. Hence $x \leq s$ for all $x \in \mathbb{Z}^+$ which is a contradiction.

Case 7 : $C = (\{1,\ldots,n\} \times \mathbb{Z}, \leq_d)$ or $C = (\{1,\ldots,n\} \times \mathbb{Z}^+, \leq_d)$ where $n > 1$. Let $\alpha : C \to C$ be defined by

$$(i,x)\alpha = \begin{cases} (n-1,x) & \text{if } i = n \text{ and } x \in \mathbb{Z}^+, \\ (n-1,1) & \text{otherwise.} \end{cases}$$

It is easy to see that $\alpha \in OT(C)$ and $\operatorname{ran}\alpha = \{n-1\} \times \mathbb{Z}^+$. Since $OT(C)$ is regular, we have that $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(C)$. By Proposition 1.1,

$$(n-1,x)\beta\alpha = (n-1,x) \text{ for all } x \in \mathbb{Z}^+.$$

Since $(n-1,x) <_d (n,1)$ and $\operatorname{ran}\alpha = \{n-1\} \times \mathbb{Z}^+$, it follows that

$$(n-1,x) = (n-1,x)\beta\alpha \leq_d (n,1)\beta\alpha = (n-1,k) \text{ for some } k \in \mathbb{Z}^+.$$

We deduce that $x \leq k$ for all $x \in \mathbb{Z}^+$ which is a contradiction.

Hence it is shown that if $OT(X \times Y, \leq_d)$ is regular, then $|X| = 1$ or Y is finite.
For the converse, if $|X| = 1$, $(X \times Y, \leq_d)$ is clearly order-isomorphic to Y, then by Proposition 1.3, $OT(X \times Y, \leq_d)$ is regular. If Y is finite, then by Proposition 1.3 and Lemma 2.2, $OT(X \times Y, \leq_d)$ is regular. □

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