Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 951–961

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Best Proximity Point Results for G-Proximal Geraghty Mappings

Phakdi Charoensawan[1,2,3] **and Tanadon Chaobankoh**[1,2,3,*]

[1] *Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai 50200, Thailand*

[2] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*

[3] *Data Science Research Center, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : phakdi@hotmail.com (P. Charoensawan); tanadon.c@cmu.ac.th (T. Chaobankoh)

Abstract Let (X, d) be a complete metric space endowed with a graph G. We introduce a new type of Geraghty contractions which are G-proximal. Best proximity theorems for these mappings in X are given as well as an example supporting the main result. Moreover, we obtain several consequences which generalize other results in the literature.

MSC: 47H04; 47H10

Keywords: best proximity point; G-proximal; Geraghty contraction

Submission date: 03.02.2020 / Acceptance date: 20.03.2020

1. Introduction

Let A and B be non-empty subsets of a metric space (X, d). It is known that for a map $T : A \to B$, the equation $Tx = x$ does not always have a solution, and it clearly has no solution when A and B are disjoint. Nonetheless, it is possible to determine an approximate solution x^* such that the error is exactly $d(x^*, Tx^*) = d(A, B)$. Such point x^* is called a best proximity point of T. In the case that T is a self-mapping, a best proximity point is a fixed point of T.

The famous Banach contraction principle [1] states that if $T : A \to A$ is a contraction and A is complete, then T has a unique fixed point in A. A large number of generalizations and applications in various contexts have been studied since then. Investigation of the existence and uniqueness of a fixed point is one of the key study areas in this field. Moreover, many authors studied fixed points and best proximity points through iteration schemes which have been rapidly developed.

---

*Corresponding author.

Published by The Mathematical Association of Thailand.

Fixed point theorems concerning a metric space endowed with a graph G, which is also a generalization of the Banach contraction principle, were proposed by Jachymski [2]. Then there have been many research papers dealing with this concept. Some recent works in the aforementioned areas are [3–9].

The best proximity point theorem was first studied by [10]. Many researchers then have studied and generalized the result in many aspects, see [11–16]. Biligili et al.[17] obtained a best proximity point theorem for a pair (A, B) satisfying the P-property while some best proximity point results for proximal weak contractions in metric spaces were studied in [18]. See also, [19–24]. In 2017, Klanarong and Suantai [25] presented the notion of a G-proximal generalized contraction in a metric space X endowed with a graph G, which is a development of well-known mappings by Banach, Kannan and Chatterjeaand. They obtained some best proximity point results for these mappings.

One of the well-known generalizations of the Banach contraction principle is the result given by Geraghty [26] which enriches the principle by considering the class of mappings $\theta : [0, \infty) \to [0, 1)$ such that

$$\theta(t_n) \to 1 \quad \Longrightarrow \quad t_n \to 0.$$

In 2019, by including 1 in the ranges of those θ, Ayari [27] provided a new result on the existence and uniqueness of best proximity point for α-proximal Geraghty non-self mappings T.

In this work, by using a class of functions in [27], we introduce a new type of Geraghty contractions called G-proximal Geraghty mappings. These mappings defined on closed subsets of a complete metric space which is endowed with a graph G. Then we establish new results on the existence and uniqueness of best proximity points for these mappings. Our results generalizes other existing results in the literature. We also give an example as well as list some interesting consequences. Subsequently, by applying the main result, we obtain a best proximity point theorem in a metric space endowed with a binary relation.

2. Preliminaries and Definitions

Throughout this work, let $X := (X, d)$ be a metric space, and let A and B be non-empty closed subsets of X. For convenience, we require the following notations:

$$\begin{aligned} d(A,B) &:= \inf\{d(a,b) : a \in A, b \in B\}; \\ A_0 &:= \{a \in A : \text{ there exists } b \in B \text{ such that } d(a,b) = d(A,B)\}; \\ B_0 &:= \{b \in B : \text{ there exists } a \in A \text{ such that } d(a,b) = d(A,B)\}. \end{aligned}$$

Definition 2.1 ([13])**.** Let $T : A \to B$ be a mapping. An element $x^* \in A$ is said to be a *best proximity point* of T if $d(x^*, Tx^*)$ is precisely $d(A, B)$. We denote the set of all best proximity points of T by $\mathrm{BP}(T)$.

Definition 2.2 ([23])**.** Let A_0 be nonempty. Then the pair (A, B) is said to have the *P-property* if $d(x_1, y_1) = d(x_2, y_2) = d(A, B) \Longrightarrow d(x_1, x_2) = d(y_1, y_2)$, where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.3. A metric space X is said to be *endowed with a directed graph* $G = (V_G, E_G)$, if the following hold:

(i) the set of vertices, V_G, coincides with X;
(ii) the set of edges, E_G, contains the diagonal of $X \times X$, i.e., $\{(x, x) : x \in X\}$;
(iii) E_G contains no parallel edges.

We say that G is *transitive* if for all $x, y, z \in X$,

$$(x, z) \text{ and } (z, y) \in E_G \Rightarrow (x, y) \in E_G.$$

Definition 2.4 ([2]). Let $x \in X$. A map $T : X \to X$ is called *G-continuous* at x if for a sequence $\{x_n\}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E_G$ for all n, $Tx_n \to Tx$.

Definition 2.5 ([25]). A mapping $T : A \to B$ is said to be *G-proximal* if $(x_1, x_2) \in E_G$ and $d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B) \Longrightarrow (u_1, u_2) \in E_G$ for all $x_1, x_2, u_1, u_2 \in A$.

3. Main Results

If no otherwise specified, we assume that T is a non-self mapping and X is endowed with a directed graph G for the rest of the paper.

We also require the class of mappings

$$\mathcal{B} := \{\beta : [0, \infty) \to [0, 1] : \beta(t_n) \to 1 \text{ implies } t_n \to 0\},$$

which is an important tool in [27]. This class is a generalization of the well-known class of $[0, 1)$-valued functions introduced by Geraghty [26].

Some examples of these mappings are listed as follows.

(1) $\beta(t) = e^{-kt}$, where $k > 0$.

(2) $\beta(t) = \dfrac{1}{t+1}$.

(3) $\beta(t) = \begin{cases} 1, & , t = 0; \\ \dfrac{\ln(1+t)}{t} & , t > 0. \end{cases}$

We now introduce a new type of Geragthy contractions.

Definition 3.1. A mapping $T : A \to B$ is said to be a *G-proximal Geraghty mapping* if the following hold:

(i) T is G-proximal;

(ii) there exists $\beta \in \mathcal{B}$ such that for all $x, y, u, v \in A$ if $d(u, Tx) = d(v, Ty) = d(A, B)$ and $(x, y) \in E_G$,

$$d(Tx, Ty) \leq \beta(d(x, y))M(x, y, u, v) \tag{3.1}$$

where $M(x, y, u, v) = \max\left\{d(x, y), d(x, u), d(y, v), \dfrac{d(x, v) + d(y, u)}{2}\right\}$.

Theorem 3.2. *Let $T : A \to B$ be a G-proximal Geraghty mapping. Suppose that X is complete, G is transitive and $A_0 \neq \emptyset$. If the following conditions hold:*

(i) T is G-continuous on A and $T(A_0) \subseteq B_0$;

(ii) there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $(x_0, x_1) \in E_G$;

(iii) the pair (A, B) satisfies the P-property,

then $\mathrm{BP}(T) \neq \emptyset$. Moreover, if $(x, y) \in E_G$ for all $x, y \in \mathrm{BP}(T)$, T has a unique best proximity point.

Proof. By $T(A_0) \subset B_0$ and (ii), there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B) = d(x_1, Tx_0)$. Since T is G-proximal, we have $(x_1, x_2) \in E_G$. Continuing in this way, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ and } (x_n, x_{n+1}) \in E_G \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

From (3.2), we have that $d(x_n, Tx_{n-1}) = d(A, B)$ and $d(x_{n+1}, Tx_n) = d(A, B)$. Using the P-property, it follows that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n). \tag{3.3}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then from (3.2), we have that

$$d(x_{n_0+1}, Tx_{n_0}) = d(x_{n_0}, Tx_{n_0}) = d(A, B).$$

Now suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. We shall show that $\{x_n\}$ is a Cauchy sequence. However, we need to prove that $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$ first.

Since $(x_{n-1}, x_n) \in E_G$, (3.3) and T is a G-proximal Geraghty mapping, then

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq \beta(d(x_{n-1}, x_n))M(x_{n-1}, x_n, x_n, x_{n+1}) \\ &\leq M(x_{n-1}, x_n, x_n, x_{n+1}), \text{ for all } n \geq 1, \end{aligned} \tag{3.4}$$

where

$$M(x_{n-1}, x_n, x_n, x_{n+1}) = \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\}.$$

Next, we consider each case of $M(x_{n-1}, x_n, x_n, x_{n+1})$.

If $M(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_{n-1}, x_n)$, from (3.4), we have that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n), \text{ for all } n \geq 1. \end{aligned} \tag{3.5}$$

This means that $d(x_{n-1}, x_n)$ is non-increasing. Thus $\lim_{n\to\infty} d(x_n, x_{n-1}) = r \geq 0$. Suppose that $r > 0$ and let $n \to \infty$ in (3.5). Then

$$1 \leq \lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) \leq 1.$$

It follows that $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$. By the definition of β, $\lim_{n\to\infty} d(x_n, x_{n-1}) = r = 0$ which is a contradiction. Thus $\lim_{n\to\infty} d(x_{n-1}, x_n)$ must be 0.

If $M(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_n, x_{n+1})$, by (3.4) we have

$$d(x_{n+1}, x_n) = d(Tx_{n-1}, Tx_n) \leq \beta(d(x_{n-1}, x_n))d(x_n, x_{n+1}). \tag{3.6}$$

Since $d(x_{n+1}, x_n) > 0$, we have $1 \leq \beta(d(x_{n-1}, x_n))$. Using the fact that $\beta(d(x_{n-1}, x_n)) \leq 1$, then $\beta(d(x_{n-1}, x_n)) = 1$. It follows by the definition of β that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

If $M(x_{n-1}, x_n, x_n, x_{n+1}) = \dfrac{d(x_{n-1}, x_{n+1})}{2}$, by (3.4), we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_{n-1}, Tx_n) &\leq \beta(d(x_{n-1}, x_n))\frac{d(x_{n-1}, x_{n+1})}{2} \\ &\leq \beta(d(x_{n-1}, x_n))\left[\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right] \\ &\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \text{ for all } n \geq 1. \end{aligned} \tag{3.7}$$

Then by (3.7),

$$d(x_{n+1}, x_n) \leq d(x_{n-1}, x_n), \text{ for all } n \geq 1.$$

Thus $d(x_{n-1}, x_n)$ is non-increasing. It follows that

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = r \geq 0. \tag{3.8}$$

Then by (3.8),

$$\lim_{n\to\infty} \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} = r \geq 0. \tag{3.9}$$

Suppose that $r > 0$ and let $n \to \infty$ in (3.7). Using (3.8) and (3.9), we have

$$\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1.$$

By the property of β, we obtain that $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$. Thus $\lim_{n\to\infty} d(x_{n-1}, x_n) = r = 0$ for all $n \geq 1$ which is a contradiction.

Finally, we have that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0 \text{ for all } n \geq 1. \tag{3.10}$$

Now we are ready to show that $\{x_n\}$ is a Cauchy sequence. Suppose for a contradiction, then there exists $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$ with $m_k > n_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \text{ and } d(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{3.11}$$

Using (3.11), we have that

$$\begin{aligned} \varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + d(x_{m_k}, x_{n_k-1}). \end{aligned} \tag{3.12}$$

Taking $k \to \infty$ in (3.12) and by (3.10), it follows that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.13}$$

From (3.2), we have

$$d(x_{n_k+1}, Tx_{n_k}) = d(A, B) \text{ and } d(x_{m_k+1}, Tx_{m_k}) = d(A, B). \tag{3.14}$$

Using the P-property, it follows that $d(x_{n_k+1}, x_{m_k+1}) = d(Tx_{n_k}, Tx_{m_k})$.
Since $(x_{n_k}, x_{n_k+1}) \in E_G$ and G is transitive, $(x_{n_k}, x_{m_k}) \in E_G$.
Consequently, by the property of T,

$$\begin{aligned} d(x_{n_k+1}, x_{m_k+1}) &= d(Tx_{n_k}, Tx_{m_k}) \\ &\leq \beta(d(x_{n_k}, x_{m_k}))M(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) \end{aligned} \tag{3.15}$$

where $M(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) =$
$\max\left\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), \dfrac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\right\}.$

Next, let us consider all the possible cases of $M(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1})$ as follows.
If $M(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = d(x_{n_k}, x_{m_k})$, from (3.15), we have

$$\begin{aligned} d(x_{n_k+1}, x_{m_k+1}) &= d(Tx_{n_k}, Tx_{m_k}) \\ &\leq \beta(d(x_{n_k}, x_{m_k}))d(x_{n_k}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{m_k}). \end{aligned} \tag{3.16}$$

Then, by (3.16) and $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon > 0$,

$$1 \leq \lim_{n\to\infty} \beta(d(x_{n_k}, x_{m_k})) \leq 1.$$

Thus $\lim_{n\to\infty} \beta(d(x_{n_k}, x_{m_k})) = 1$. By the definition of β, we have

$$\lim_{n\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon = 0$$

which is a contradiction.

If $M(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = d(x_{n_k}, x_{n_k+1})$, from (3.15), we have

$$\begin{aligned} d(x_{n_k+1}, x_{m_k+1}) &= d(Tx_{n_k}, Tx_{m_k}) \\ &\leq \beta(d(x_{n_k}, x_{m_k}))d(x_{n_k}, x_{n_k+1}) \\ &\leq d(x_{n_k}, x_{n_k+1}). \end{aligned} \tag{3.17}$$

By taking $k \to \infty$ and using (3.10),

$$\lim_{n\to\infty} d(x_{n_k+1}, x_{m_k+1}) = \epsilon = 0$$

which is a contradiction.

If $M(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = d(x_{m_k}, x_{m_k+1})$, it is similar to the previous case.

If $M(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = \dfrac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}$, by the triangular inequality, we have

$$\begin{aligned} &\frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2} \\ &\leq \frac{d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})}{2}. \end{aligned} \tag{3.18}$$

Also, from (3.10),

$$\lim_{n\to\infty} \frac{d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})}{2} = \epsilon. \tag{3.19}$$

Using (3.18) and (3.19) in (3.15) and taking $k \to \infty$, we have that

$$1 \leq \lim_{n\to\infty} \beta(d(x_{n_k}, x_{m_k})) \leq 1.$$

By the property of β, $\lim_{n\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon = 0$ which is a contradiction.

Thus, $\{x_n\}$ is a Cauchy sequence in A which is a closed subset of X. Therefore there exists $x^* \in A$ such that $\lim_{n\to\infty} x_n = x^*$. By the G-continuity of T, $\lim_{n\to\infty} Tx_n = Tx^*$.

Therefore, by (3.2),

$$\lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(x^*, Tx^*).$$

By the uniqueness of limit, $d(x^*, Tx^*) = d(A, B)$. This implies that $x^* \in A$ is a best proximity point of T.

Suppose that there is another best proximity point for T, namely y^*, such that $(x^*, y^*) \in E_G$. Thus $d(x^*, y^*) > 0$ and $d(x^*, Tx^*) = d(y^*, Ty^*) = d(A, B)$. By the P-property, $d(x^*, y^*) = d(Tx^*, Ty^*) > 0$.

Since T is a G-proximal Geraghty mapping and
$M(x^*, y^*, x^*, y^*) = \max\{d(x^*, y^*), d(x^*, x^*), d(y^*, y^*)\}$, we obtain that

$$\begin{aligned} d(x^*, y^*) = d(Tx^*, Ty^*) &\leq \beta(d(x^*, y^*))M(x^*, y^*, x^*, y^*) \\ &= \beta(d(x^*, y^*))d(x^*, y^*) \\ &\leq d(x^*, y^*). \end{aligned}$$

Since $d(x^*, y^*) > 0$, we have $1 \leq \beta(d(x^*, y^*))$. Using the fact that $\beta(d(x^*, y^*)) \leq 1$, we have that $\beta(d(x^*, y^*)) = 1$. Finally by the property of β,

$$d(x^*, y^*) = 0.$$

The proof is now completed. ∎

Example 3.3. Let $X = \mathbb{R}^2$ be equipped with the metric d defined by

$$d((x, y), (u, v)) = \sqrt{(x-u)^2 + (y-v)^2}.$$

Let

$$\begin{aligned} A &= \{(x, 1) : 0 \leq x \leq 1\} \text{ and} \\ B &= \{(x, -1) : 0 \leq x \leq 1\} \cup \{(0, y) : -2 \leq y \leq -1\}. \end{aligned}$$

Then A and B are closed, $d(A, B) = 2, A_0 = A$ and $B_0 = \{(x, -1) : 0 \leq x \leq 1\}$. Also, (A, B) satisfies the P-property.

Define a directed graph $G = (V_G, E_G)$ by $V_G = X$ and

$$E_G = \{((x, y), (u, v)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x < u \text{ and } y \leq v\}.$$

We can see that G is transitive. Let $T : A \to B$ be a mapping defined by

$$T(x, 1) = (\ln(x+1), -1), \text{ for all } (x, 1) \in A.$$

Then T is G-continuous and $T(A_0) \subseteq B_0$.

Next, we will show that T is a G-proximal Geraghty mapping. Let $(x, 1)$, $(y, 1)$, $(u, 1)$, $(v, 1) \in A$ such that $((x, 1), (y, 1)) \in E_G$ and
$d((u, 1), T(x, 1)) = d(A, B) = d((v, 1), T(y, 1))$. Then

$$x \leq y \text{ and } d((u, 1), (\ln(x+1), -1)) = d(A, B) = d((v, 1), (\ln(y+1), -1)).$$

This implies that $u = \ln(x+1)$ and $v = \ln(y+1)$.

Since $x < y$ and $x, y \in [0, 1]$, $u < v$. Thus $((u, 1), (v, 1)) \in E_G$ and so T is G-proximal.

We also note that there is $\beta \in \mathcal{B}$ defined by $\beta(t) = \begin{cases} 1, & , t = 0; \\ \dfrac{\ln(1+t)}{t} & , t > 0. \end{cases}$

Now,

$$\begin{aligned}
d(T(x,1),T(y,1)) &= d((\ln(x+1),-1),(\ln(y+1),-1)) \\
&= |\ln(x+1)-\ln(y+1)| \\
&= \left|\ln\left(\frac{x+1}{y+1}\right)\right| \\
&= \left|\ln\left(\frac{y+1+x+1-y-1}{y+1}\right)\right| \\
&= \left|\ln\left(1+\frac{x-y}{y+1}\right)\right| \\
&\leq \ln(1+|x-y|) = \frac{\ln(1+|x-y|)}{|x-y|}|x-y| \\
&= \beta(d((x,1),(y,1)))d((x,1),(y,1)) \\
&\leq \beta(d((x,1),(y,1)))M((x,1),(y,1),(u,1),(v,1)).
\end{aligned}$$

Therefore by Theorem 3.2, T is a G-proximal Geraghty mapping and hence $(0,1)$ is a best proximity point of T.

4. Consequences

Several consequences of our main result are given in this section. Put $\beta(t)=k$, where $k\in[0,1)$ in Theorem 3.2, we obtain the next corollary.

Definition 4.1. A non-self mapping $T:A\to B$ is said to be a *G-proximal generalized contraction* if the following hold;

(i) T is G-proximal;
(ii) there exists $k\in[0,1)$ such that for all $x,y,u,v\in A$ if $d(u,Tx)=d(v,Ty)=d(A,B)$ and $(x,y)\in E_G$,

$$d(Tx,Ty)\leq kM(x,y,u,v)$$

where $M(x,y,u,v)=\max\left\{d(x,y),d(x,u),d(y,v),\frac{d(x,v)+d(y,u)}{2}\right\}$.

Corollary 4.2. *Let $T:A\to B$ be a G-proximal generalized contraction. Suppose that X is complete, G is transitive and A_0 is non-empty. If the following conditions hold:*

(i) T is G-continuous on A and $T(A_0)\subseteq B_0$;
(ii) there exist $x_0,x_1\in A$ such that $d(x_1,Tx_0)=d(A,B)$ and $(x_0,x_1)\in E_G$;
(iii) the pair (A,B) satisfies the P-property,

then T has a best proximity point.

Note that Corollary 4.2 is a generalization of the result in [25].
If $\beta(t)=e^{-kt}$, where $k>0$, we may have the next definition.

Definition 4.3. A non-self map $T:A\to B$ is said to be a *G-proximal exponential contraction* if the following hold:

(i) T is G-proximal;

(ii) there exists $k > 0$ such that for all $x, y, u, v \in A$ if $d(u, Tx) = d(v, Ty) = d(A, B)$ and $(x, y) \in E_G$,

$$d(Tx, Ty) \leq e^{-kd(x,y)} M(x, y, u, v)$$

when $M(x, y, u, v) = \max\left\{d(x, y), d(x, u), d(y, v), \dfrac{d(x, v) + d(y, u)}{2}\right\}$.

Corollary 4.4. *Let $T : A \to B$ be a G-proximal exponential contraction. Suppose that X is complete, G is transitive and A_0 is non-empty. If the following conditions hold:*

(i) T is G-continuous on A and $T(A_0) \subseteq B_0$;
(ii) there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $(x_0, x_1) \in E_G$;
(iii) the pair (A, B) satisfies the P-property,

then T has a best proximity point.

5. Applications

Let $\mathcal{R}$ be a binary relation over X. By applying our result, we obtain a best proximity point result for a map on a metric space endowed with $\mathcal{R}$. We first list some definitions.

Definition 5.1 ([24])**.** A mapping $T : A \to B$ is called *proximally comparative* if for all $x, y, u_1, u_2 \in A$,

$$x\mathcal{R}y \text{ and } d(u_1, Tx) = d(u_2, Ty) = d(A, B) \Longrightarrow u_1\mathcal{R}u_2.$$

Definition 5.2. Let $x \in X$. A map $T : A \to B$ is called *$\mathcal{R}$-continuous* at x if for each sequence $\{x_n\}$ in A,

$$x_n \to x \text{ and } x_n\mathcal{R}x_{n+1} \text{ for all } n \Longrightarrow Tx_n \to Tx.$$

Definition 5.3. A mapping $T : A \to B$ is said to be a *proximally comparative, Geraghty mapping* if the following hold:

(i) T is a proximally comparative mapping;
(ii) there exists $\beta \in \mathcal{B}$ such that for all $x, y, u, v \in A$ if $d(u, Tx) = d(v, Ty) = d(A, B)$ and $x\mathcal{R}y$,

$$d(Tx, Ty) \leq \beta(d(x, y))M(x, y, u, v)$$

where $M(x, y, u, v) = \max\left\{d(x, y), d(x, u), d(y, v), \dfrac{d(x, v) + d(y, u)}{2}\right\}$.

Corollary 5.4. *Let $T : A \to B$ be a proximally comparative, Geraghty mapping. Suppose that X is complete, A_0 is nonempty and closed, and $\mathcal{R}$ is symmetric and transitive. If the following conditions hold:*

(i) T is $\mathcal{R}$-continuous on A and $T(A_0) \subseteq B_0$;
(ii) there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0\mathcal{R}x_1$;
(iii) the pair (A, B) satisfies the P-property,

then $\mathrm{BP}(T) \neq \emptyset$. Moreover, if $(x, y) \in E_G$ for all $x, y \in \mathrm{BP}(T)$, then T has a unique best proximity point.

Proof. We define a directed graph $G = (V_G, E_G)$ by $V_G = X$ and $E_G = \{(x, y) \in X \times X : x\mathcal{R}y\}$. In order to apply Theorem 3.2, all the hypotheses must hold.

(1) The condition (i) implies that T is G-continuous on A.

(2) Let $x_1, x_2, u_1, u_2 \in A$ such that $(x_1, x_2) \in E_G$ and $d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B)$. By the definition of E_G, we have $x\mathcal{R}y$. Since T is a proximally comparative, Geraghty mapping, we have $u_1 R u_2$. Then $(u_1, u_2) \in E_G$. Therefore T is G-proximal.

(3) From (2) and T is a proximally comparative, Geraghty mapping, T is a G-proximal Geraghty mapping.

(4) The condition (ii) implies that there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $(x_0, x_1) \in E_G$.

Finally, by applying Theorem 3.2, we have that $\mathrm{BP}(T) \neq \emptyset$. ∎

Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgements

This research was partially supported by Chiang Mai University.

References

[1] S. Banach, Sur les oprations dans les ensembles abstraits et leurs applications aux quations intgrales, Fundam. Math. 3 (1) (1922) 133–181.

[2] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Am. Math. Soc. 136 (2008) 1359–1373.

[3] W. Atiponrat, S. Dangskul, A. Khemphet, Coincidence point theorems for KC-contraction mappings in JS-metric spaces endowed with a directed graph, Carpathian Journal of Mathematics 35 (3) (2019) 263–72 .

[4] H.A. Hammad, W. Cholamjiak, D. Yambangwai, A modified shrinking projection methods for numerical reckoning fixed points of G-nonexpansive mappings in Hilbert spaces with graphs, Miskolc Math. Notes 20 (2) (2019) 941–956.

[5] S. Suantai, M. Donganont, W. Cholamjiak, Hybrid methods for a countable family of G-nonexpansive mappings in Hilbert spaces endowed with graphs, Multidisciplinary Digital Publishing Institute 7 (10) (2019) pages 936.

[6] R. Suparatulatorn, W. Cholamjiak, S. Suantai, A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs, Numerical Algorithms 77 (2018) 479–490.

[7] P. Charoensawan, W. Atiponrat, Common fixed point and coupled coincidence point theorems for Geraghty's type contraction mapping with two metrics endowed with a directed graph, Journal of Mathematics 2017 (2017) Art. ID 5746704.

[8] R. Suparatulatorn, S. Suantai, W. Cholamjiak, Hybrid methods for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs, AKCE Int. J. Graphs Comb. 14 (2017) (2) 101–111.

[9] P. Cholamjiak, Fixed point theorems for Banach type contraction on Tvs-cone metric spaces endowed with a graph, J. Comput. Anal. Appl. 16 (2014) 338–345.

[10] K. Fan, Extensions of two fixed point theorems of F.E. Browder, Math. Z. 122 (1969) 234–240.

[11] S. Reich, Approximate selections, best approximations, fixed points and invariant sets, J. Math. Anal. Appl. 62 (1) (1978) 104–113.

[12] S.S. Basha, P. Veeramani, Best proximity pair theorems for multifunctions with open fibres, J. Approx. Theory 103 (1) (2000) 119–129.

[13] S.S. Basha, Extensions of Banach's contraction principle, Numer. Funct. Anal. Optim. 31 (4-6) (2010) 569–576.

[14] W. A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (7-8) (2003) 851–862.

[15] A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2006) 1001–1006.

[16] P. Kumam and C. Mongkolekeha, Common best proximity points for proximity commuting mapping with Geraghtys functions, Carpath. J. Math. 31 (2015) 359–364

[17] N. Bilgili, E. Karapinar, K. Sadarangani, A generalization for the best proximity point of Geraghty-contractions, J. Inequal. Appl. 2013 (2013) pages 286.

[18] N. Bunlue, S. Suantai, Best proximity point for proximal Berinde nonexpansive mappings on starshaped sets, Archivum Mathematicum 54 (2018) 165–176.

[19] N. Bunlue, S. Suantai, Hybrid algorithm for common best proximity points of some generalized nonself nonexpansive mappings, Math. Methods Appl. Sci. 41 (17) (2018) 7655–7666.

[20] P. Sarnmeta, S. Suantai, Existence and convergence theorems for best proximity points of proximal multi-valued nonexpansive mappings, Communications in Mathematics and Applications, 10 (3) 369–377.

[21] R. Suparatulatorn, S. Suantai, A new hybrid algorithm for global minimization of best proximity points in Hilbert spaces, Carpathian Journal of Mathematics 35 (1) (2019) 95–102.

[22] R. Suparatulatorn, W. Cholamjiak S. Suantai, Existence and Convergence Theorems for Global Minimization of Best Proximity Points in Hilbert Spaces, Acta Appl Math 165 (2020) 81–90.

[23] V. Sankar Raj, A best proximity point theorem for weakly contractive non-self-mappings. Nonlinear Anal. 74 (14) (2011) 4804–4808.

[24] M. Jleli, E. Karapinar, B. Samet, Best proximity points for generalized $\alpha - \psi$ proximal contractive type mapping, J. Appl. Math. 2013 (2013) Article ID 534127.

[25] C. Klanarong, S. Suantai, Best proximity point theorems for G-proximal generalized contraction in complete metric spaces endowed with graphs, Thai J. Math. 15 (1) (2017) 261–276.

[26] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc. 40 (1973) 604–608.

[27] M.I. Ayari, A best proximity point theorem for α-proximal Geraghty non-self mappings. Fixed Point Theory Appl. 2019 (10) (2019) doi:10.1186/s13663-019-0661-8.