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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Proximal Point Algorithm Involving Best Proximity Point of Nonself Nonexpansive Mappings in Real Hilbert Spaces

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Abstract In this paper, we present a new algorithm for approximating a solution of the convex minimization problems and best proximity point problems. The weak and strong convergence theorems are proved under some appropriate conditions. Moreover, we provide numerical experiment to illustrate the convergence behavior of the solution and to show the effectiveness of the proposed algorithm.

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1. Introduction

Best proximity point for some nonlinear mappings plays very important role in optimization theory. It can be applied to solve equilibrium, see [1–4]. It is well known that the concept of a best proximity point include that of a fixed point as a special case.

In 2018, Pirbavafa and Vaezpour [5] applied the best proximity point theory to obtain equilibrium existence in abstract economies with two constraint set-valued maps. Accordingly, the concept of best proximity point attracted the attention of many mathematicians, see ([1, 2, 5–9]).

In recent years, many researchers have tried to develop different method for solving a best proximity point for nonexpansive mappings, see [6, 7]. In 2019, Suparatulatorn et al. [7] introduced a general Mann iteration process $\{x_n\}$ which is given by

$$\begin{cases} x_1 \in C_0, \\ x_{n+1} = P_C\left((1-\alpha_n)P_D y_n + \alpha_n T y_n\right), \quad \forall n \geq 1, \end{cases} \tag{1.1}$$

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where C and D be two nonempty closed convex subsets of a real Hilbert space H, $C_0 = \{x \in C : \|x-y\| = D(C,D)$, for some $y \in D\}$ and $D_0 = \{y \in D : \|x-y\| = D(C,D)$, for some $x \in C\}$. Let $T : C \to D$ be a nonself nonexpansive mapping such that $T(C_0) \subseteq D_0$. By using the iteration process (1.1), they proved weak convergence and strong convergence theorems for best proximity points of nonself nonexpansive mappings in real Hilbert spaces.

Let $g : H \to (-\infty, \infty]$ be a proper and convex function. One of the major problems for optimization is to find a point $x \in H$ such that

$$g(x) = \min_{y \in H} g(y).$$

We denote the set of all minimizers of g on H by $\mathrm{argmin}_{y\in H} g(y)$.

The proximal point algorithm is an important tool in solving optimization problem which was initiated by Martinet [10] in 1970. Later, Rockafellar [11] studied the convergence of a proximal point algorithm for finding a solution of the unconstrained convex minimization problem in H as follows. Let g be a proper, convex and lower semi-continuous function on H. The proximal point algorithm is defined by $x_1 \in H$ and

$$x_{n+1} = \operatorname*{argmin}_{u \in H} \left[g(y) + \frac{1}{2\lambda_n} \|u - x_n\|^2 \right], \quad \forall n \geq 1, \tag{1.2}$$

where $\lambda_n > 0$ for all $n \geq 1$. It was shown that if g has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$ then the sequence $\{x_n\}$ converges weakly to a minimizer of g; see also [12]. However, the proximal point algorithm does not necessarily converges strongly in general; see [13, 14].

Recently, several authors proposed modifications of Rockafellar's proximal point algorithm to have strong convergence, for example [15–17].

In recent years, many convergence results by the proximal point algorithm for solving optimization problems have been extended in many directions, see [18–23]. The minimizers of the objective convex functionals in the spaces with nonlinearlity play an important role in the branch of analysis and geometry. Several applications in machine learning, computer vision, system balancing and robot manipulation can be considered as solving optimization problems, see [19, 20, 24].

In this work, motivated by the research described above, we propose a new proximal point algorithm which is a modification of the iterative schemes (1.1) and (1.2) for finding a common elements of the set of best proximity point of nonself nonexpansive mappings and the set of minimizers of convex and lower semi-continuous functions. We prove some convergence theorems of the proposed algorithm under some mild conditions. A numerical example to support our main result is also given. Our results are refinements and generalizations of many recent results from the current literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is known in [25] that a Hilbert space H satisfies *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Recall that a mapping $T : H \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote a fixed point set of T by $F(T)$, that is,

$$F(T) = \{x \in A : x = Tx\}.$$

Let A and B be two nonempty closed convex subsets of H. We define A_0 and B_0 by the following sets:

$$A_0 = \{x \in A : \|x - y\| = D(A, B), \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : \|x - y\| = D(A, B), \text{ for some } x \in A\}.$$

We recall some useful definitions and lemmas, which will be used in the later sections.

Let C be a nonempty closed convex subset of Hilbert space H. For any $x \in H$, its projection onto C is defined as

$$P_C(x) = \operatorname{argmin}\{\|y - x\| : y \in C\}$$

The mapping $P_C : H \to C$ is called a *projection operator*, which has the well-known properties in the following lemma.

Lemma 2.1 ([26]). *Let C be a nonempty closed convex subset of Hilbert space H. Then for all $x, y \in H$ and $z \in C$,*

(1) $\langle P_C x - x, z - P_C x\rangle \geq 0$;
(2) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$;
(3) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$.

Definition 2.2 ([6]). Let A and B be two nonempty subsets of a real Hilbert space H. A mapping $T : A \to B$ is said to be *nonself nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in A$.

Definition 2.3 ([8]). An element $s \in A$ is said to be a *best proximity point* of the nonself mapping $T : A \to B$ if it satisfies the condition that

$$\|x - Tx\| = d(A, B).$$

We denote a best proximity point set of T by $Best_A T$, that is,

$$Best_A T = \{x \in A : \|x - Tx\| = d(A, B)\}.$$

Definition 2.4 ([3]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d) and $T : A \to B$ a nonself mapping. Then a sequence $\{x_n\}$ in A is said to be *an approximate best proximity point sequence for T* if

$$\lim_{n\to\infty} d(x_n, Tx_n) = d(A, B).$$

Definition 2.5 ([27]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d) and $T : A \to B$ a nonself mapping. We say that T satisfies the proximal property if for each sequence $\{x_n\}$ in A such that $x_n \rightharpoonup x \in A$ and $\{x_n\}$ is an approximate best proximity point sequence for T, we have $\|x - Tx\| = d(A, B)$.

We note that, if $A = B$, the proximal property reduces to the demiclosedness property of $I - T$ at 0.

Definition 2.6 ([8]). Let A and B be nonempty closed subsets of a metric space (X,d). Then (A,B) is said to satisfy the *UC property* if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n,y_n) = D(A,B)$ and $\lim_{n\to\infty} d(z_n,y_n) = D(A,B)$, then $\lim_{n\to\infty} d(x_n,z_n) = 0$.

Example 2.7 ([8]). Let A and B be nonempty subsets of a uniformly convex Banach space. If A is convex, then the pair (A,B) has the property UC.

Lemma 2.8 ([6]). *Let A,B be two nonempty subsets of a uniformly convex Banach spaces X such that A is closed and convex. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$. Then $F(P_A T|_{A_0}) = Best_A(T)$.*

Definition 2.9 ([6]). Let A and B be two nonempty subsets of a real Hilbert space H and C a subset of A. A mapping $T : A \to B$ is said to be *C-nonexpansive* if

$$\|Tx - Tz\| \leq \|x - z\|$$

for all $x \in A$ and $z \in C$. If $C = Best_A T$, we say that T is a *best proximally nonexpansive mapping.*

Lemma 2.10 ([6]). *Let A,B be two nonempty subsets of a uniformly convex Banach spaces X such that A is closed and convex. Suppose that $T : A \to B$ is a best proximally nonexpansive mapping such that $T(A_0) \subseteq B_0$. Then $P_A T|A_0$ is quasi-nonexpansive mapping.*

Lemma 2.11 ([7]). *Let (A,B) be a pair of two nonempty closed subsets of an uniformly convex Banach space E such that A is convex. Let $T : A \to B$ be a nonself nonexpansive mapping. Then T satisfies the proximal property.*

Lemma 2.12 ([6]). *Let (A,B) be a pair of nonempty subsets of a normed space E such that B is closed and convex. Then $\|x - P_B x\| = d(A,B)$ for all $x \in A_0$.*

Lemma 2.13 ([6]). *Let (A,B) be a pair of nonempty subsets of a uniformly convex Banach space E such that A is closed and convex. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$. Then $F(P_A T|A_0) = Best_A(T)$.*

Lemma 2.14 ([7]). *Let (A,B) be a pair of nonempty subsets of an uniformly sonvex Banach space E such that B is closed and convex. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$. Then $P_B z = Tz$ for all $z \in Best_A(T)$.*

Lemma 2.15 ([28]). *Let E be an uinformly convex Banach space, and $\{\alpha_n\}$ a sequence such that $0 < a \leq \alpha_n \leq b < 1$ for some positive real number a,b and for all $n \in \mathbb{N}$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in E are such that $\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 2.16 ([7]). Let (A,B) be a pair of nonempty subsets of a metric space (X,d). A nonself mapping $T : A \to B$ is said to be *proximal semicompact* if for any sequence $\{x_n\}$ in A which is an approximate best proximity point sequence for T, then $\{x_n\}$ has a convergent subsequence.

Let $f : H \to (-\infty,\infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in a real Hilbert space H as follows:

$$J_\lambda x = \operatorname*{argmin}_{u\in H} \left[f(u) + \frac{1}{2\lambda}\|u - x\|^2 \right]$$

for all $x \in H$. It was shown in [14] that the set of fixed points of the resolvent associated with f coincides with the set of minimizers of g. Also, the resolvent J_λ of f is nonexpansive for all $\lambda > 0$; see [29].

Lemma 2.17 ([29]). *Let H be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x \in H$ and $\lambda > \mu > 0$, we have the following identity holds:*

$$J_\lambda x = J_\mu \left(\frac{\lambda - \mu}{\lambda} J_\lambda x + \frac{\mu}{\lambda} x \right).$$

Lemma 2.18 ([30]). *Let H be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{2\lambda}\|J_\lambda x - y\|^2 - \frac{1}{2\lambda}\|x - y\|^2 + \frac{1}{2\lambda}\|x - J_\lambda x\|^2 \leq f(y) - f(J_\lambda x). \tag{2.1}$$

Lemma 2.19 ([31]). *Let H be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ with $x_n - Tx_n \to 0$, then $x = Tx$.*

3. Main Results

In this section, we propose our main algorithm and prove the strong convergence theorem for finding a common solution of minimizers of proper convex and lower-semi continuous functions and best proximity points of of nonself nonexpansive mappings in real Hilbert space.

Let C and D be two nonempty closed convex subsets of a real Hilbert space H. Let $T : C \to D$ be a nonself nonexpansive mapping such that $T(C_0) \subseteq D_0$. Let $f : C \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function. We propose a new proximal point algorithm for nonself nonexpansive mappings in real Hilbert spaces as follows:

Algorithm

$$\begin{cases} x_1 \in C_0, \\ y_n = \underset{u \in C}{\operatorname{argmin}} \left[f(u) + \frac{1}{2\lambda_n}\|u - x_n\|^2 \right], \\ x_{n+1} = P_C\left((1-\alpha_n)P_D y_n + \alpha_n T y_n\right), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$.

Theorem 3.1. *Let C and D be two nonempty closed convex subsets of a real Hilbert space H. Let $T : C \to D$ be a nonself nonexpansive mapping and $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = Best_C T \bigcap \operatorname{argmin}_{u \in C} f(u)$ is nonempty. Let $\{x_n\}$ be a sequence generated by (3.1) where $\{\alpha_n\}, \{\lambda_n\}$ are sequences in $(0, 1)$ such that $0 \leq a \leq \alpha_n \leq b < 1$ and $\lambda_n \geq \lambda$ for some a, b and λ. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.*

Proof. Let $q \in \mathcal{F}$. By Lemma 2.13 and Lemma 2.14, we get $q = P_C Tq$ and $P_D q = Tq$. Moreover, we have $f(q) \leq f(u)$ for all $u \in C$. It follows that

$$f(q) + \frac{1}{2\lambda_n}\|q-q\|^2 \leq f(u) + \frac{1}{\lambda_n}\|u-q\|^2, \quad \forall u \in C,$$

and hence $q = J_{\lambda_n} q$ for all $n \geq 1$. Since $y_n = J_{\lambda_n} x_n$, it implies by the nonexpansiveness of J_{λ_n} that

$$\|y_n - q\| = \|J_{\lambda_n} x_n - J_{\lambda_n} q\| \leq \|x_n - q\|. \tag{3.2}$$

Consider

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P_C\left((1-\alpha_n)P_D y_n + \alpha_n T y_n\right) - q\| \\
&= \|P_C\left((1-\alpha_n)P_D y_n + \alpha_n T y_n\right) - P_C Tq\| \\
&\leq \|\left((1-\alpha_n)P_D y_n + \alpha_n T y_n\right) - Tq\| \\
&\leq (1-\alpha_n)\|P_D y_n - P_D q\| + \alpha_n \|T y_n - Tq\| \\
&\leq \|y_n - q\|.
\end{aligned} \tag{3.3}$$

This implies by (3.2) and (3.3) that $\{\|x_n - q\|\}$ is nonincreasing and bounded below, thus $\lim_{n\to\infty} \|x_n - q\|$ exists. So we can assume that

$$\lim_{n\to\infty} \|x_n - q\| = d \quad \text{for some } d. \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$\limsup_{n\to\infty} \|y_n - q\| = d. \tag{3.5}$$

By the sub-differential inequality (2.1), we obtain

$$\frac{1}{2\lambda}\|y_n - q\|^2 - \frac{1}{2\lambda}\|x_n - q\|^2 + \frac{1}{2\lambda}\|x_n - y_n\|^2 \leq f(q) - f(y_n). \tag{3.6}$$

Since $f(q) \leq f(y_n)$ for all $n \geq 1$, we get

$$\|x_n - y_n\|^2 \leq \|x_n - q\|^2 - \|y_n - q\|^2.$$

It implies by (3.4) and (3.5) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.7}$$

It follows from Lemma 2.17, the nonexpansiveness of J_λ, and $\lambda_n \geq \lambda > 0$ that

$$\begin{aligned}
\|x_n - J_\lambda x_n\| &\leq \|x_n - y_n\| + \|y_n - J_\lambda x_n\| \\
&= \|x_n - y_n\| + \|J_{\lambda_n} x_n - J_\lambda x_n\| \\
&= \|x_n - y_n\| + \left\| J_\lambda \left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n \right) - J_\lambda x_n \right\| \\
&\leq \|x_n - y_n\| + \left\| \left(\frac{\lambda_n - \lambda}{\lambda_n} \right) J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n - x_n \right\| \\
&= \|x_n - y_n\| + \left(1 - \frac{\lambda}{\lambda_n} \right) \|J_{\lambda_n} x_n - x_n\| \\
&= \|x_n - y_n\| + \left(1 - \frac{\lambda}{\lambda_n} \right) \|y_n - x_n\| \\
&= \left(2 - \frac{\lambda}{\lambda_n} \right) \|x_n - y_n\| .
\end{aligned}$$

This implies by (3.7) that

$$\lim_{n\to\infty} \|x_n - J_\lambda x_n\| = 0. \tag{3.8}$$

By the nonexpansiveness of P_D and T, and Lemma 2.14, we obtain

$$\limsup_{n\to\infty} \|P_D x_n - Tq\| = \limsup_{n\to\infty} \|P_D x_n - P_D q\| \leq \limsup_{n\to\infty} \|x_n - q\| = d,$$

and

$$\limsup_{n\to\infty} \|T x_n - Tq\| \leq \limsup_{n\to\infty} \|x_n - q\| = d.$$

It follows that

$$\lim_{n\to\infty} \|(1-\alpha_n)(P_D y_n - Tq) + \alpha_n (T y_n - Tq)\| = d.$$

Using Lemma 2.15, we get

$$\lim_{n\to\infty} \|P_D y_n - T y_n\| = 0. \tag{3.9}$$

It implies by (3.7) and (3.9) that

$$\begin{aligned}
\|P_D x_n - T x_n\| &\leq \|P_D x_n - P_D y_n\| + \|P_D y_n - T y_n\| + \|T y_n - T x_n\| \\
&\leq \|x_n - y_n\| + \|P_D y_n - T y_n\| + \|y_n - x_n\| \to 0.
\end{aligned}$$

Using Lemma 2.12, we obtain

$$\begin{aligned}
\|x_n - T x_n\| &\leq \|x_n - P_D x_n\| + \|P_D x_n - T x_n\| \\
&= d(C, D) + \|P_D x_n - T x_n\| \\
&\to d(C, D) \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T x_n\| = d(C, D). \tag{3.10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q_1 \in C$. It implies by (3.10) and Lemma 2.11 that $q_1 \in Best_C T$. Since J_λ is a nonexpansive mapping, by (3.8) and Lemma 2.19, we have $q_1 \in F(J_\lambda) = \mathrm{argmin}_{u \in C} f(u)$. Hence, we have $q_1 \in \mathcal{F}$. We will show that $x_n \rightharpoonup q_1$. To show this, suppose not. So, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q_2 \in C$ and $q_2 \neq q_1$. Again, as above, we can conclude that $q_2 \in \mathcal{F}$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in \mathcal{F}$, by Opial's condition, we have

$$\begin{aligned}
\limsup_{k\to\infty} \|x_{n_k} - q_1\| &< \limsup_{k\to\infty} \|x_{n_k} - q_2\| \\
&= \lim_{n\to\infty} \|x_n - q_2\| \\
&= \limsup_{j\to\infty} \|x_{n_j} - q_2\| \\
&< \limsup_{j\to\infty} \|x_{n_j} - q_1\| \\
&= \lim_{n\to\infty} \|x_n - q_1\| \\
&= \lim_{k\to\infty} \|x_{n_k} - q_1\|.
\end{aligned}$$

This is a contradiction. Therefore, $q_1 = q_2$ and so $\{x_n\}$ converges weakly to an element of $\mathcal{F}$. ∎

Next, we prove the strong convergence theorem.

Theorem 3.2. *Let $\{x_n\}$ be as in Theorem 3.1 with $Best_C T \neq \emptyset$. If T is proximal semicompact, then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.*

Proof. It follows that from Theorem 3.1 that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = d(C, D).$$

Since T is proximal semicompact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ for some $q \in C_0$. Then, by the nonexpansiveness of T, we have

$$\begin{aligned}
\|q - Tq\| &\leq \|q - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tq\| \\
&\leq 2\|q - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\| \\
&\to d(C, D) \text{ as } k \to \infty.
\end{aligned}$$

Then $q \in Best_C T$. Since (3.8) and demiclosedness of $J_\lambda x_n$, we get $q \in F(J_\lambda)$. Therefore $q \in \mathcal{F}$. We know that $\lim_{n\to\infty} \|x_n - q\|$ exists. Therefore q is the strong limit of the sequence $\{x_n\}$. ∎

The following results are obtained from Theorem 3.1 by putting $C = D$.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \bigcap \operatorname{argmin}_{u\in C} f(u)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}
x_1 \in C, \\
y_n = \operatorname*{argmin}_{u\in C} \left[f(u) + \dfrac{1}{2\lambda_n} \|u - x_n\|^2 \right], \\
x_{n+1} = (1 - \alpha_n) y_n + \alpha_n T y_n, \quad \forall n \geq 1,
\end{cases}$$

where $\{\alpha_n\}, \{\lambda_n\}$ are sequences in $(0, 1)$ such that $0 \leq a \leq \alpha_n \leq b < 1$ and $\lambda_n \geq \lambda$ for some a, b and λ. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.

Theorem 3.4. *Let C and D be two nonempty closed convex subsets of a real Hilbert space H. Let $T : C \to D$ be a nonself nonexpansive mapping. Suppose that $Best_C T$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C_0, \\ x_{n+1} = P_C\left((1-\alpha_n)P_D x_n + \alpha_n T x_n\right), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 \leq a \leq \alpha_n \leq b < 1$ for some a and b. Then the sequence $\{x_n\}$ converges weakly to an element of $Best_C T$.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that* $\operatorname{argmin}_{u\in C} f(u)$ *is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \operatorname*{argmin}_{u\in C}\left[f(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\right], \quad \forall n \geq 1, \end{cases}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ such that $\lambda_n \geq \lambda$ for some λ. Then the sequence $\{x_n\}$ converges weakly to an element of $\operatorname{argmin}_{u\in C} f(u)$.

4. Numerical Example

In this section, we give an numerical experiment for solving the best proximity point problem and the convex minimization problem by using Algorithm (3.1).

Example 4.1. Let $H = \mathbb{R}^2$ with usual norm, $C = (-\infty, -3] \times [1,4]$ and $D = [3,\infty) \times (-\infty,\infty)$. Define $T : C \to D$ by

$$T(x,y) = (-x, 2 - \cos(y-1)), \quad \text{for all } (x,y) \in C.$$

Then $C_0 = \{-3\} \times [1,4]$, $D_0 = \{3\} \times [1,4]$ and $d(C,D) = 6$. It is easy to see that T is nonself nonexpansive such that $T(C_0) \subseteq D_0$. Define $f : C \to (-\infty,\infty]$ by

$$f(x,y) = \frac{1}{2}\left[(x+3)^2 + (y+3)^2\right], \quad \text{for all } (x,y) \in C.$$

We see that f is proper convex and lower semi-continuous with $Best_C T \bigcap \operatorname{argmin}_{u\in C} f(u) = \{(-3,1)\}$. Using proximity operator [32], we know that

$$\operatorname*{argmin}_{(u,v)\in C}\left[f(u,v) + \frac{1}{2}\|(u,v) - (x,y)\|^2\right] = \operatorname{prox}_f(x,y) = \left(\frac{x-3}{2}, \frac{y+1}{2}\right).$$

Suppose the sequence $\{x_n\}$ generated by (3.1) and put $x_1 = (-3,4)$. In the experiment, we choose the stopping criterion $E_n := \|x_n - z\| < 10^{-10}$ where $z = (-3,1)$ or the maximum iteration exceeds $10,000$ iterations. Since the α_n effects to the rate of convergence, therefore α_n are proposed in different values i.e. $0.1, 0.2, \ldots, 0.9$ respectively. The proposed algorithm is coded in MATLAB2014b, and run on MacBook Air (1.4 GHz Intel Core i5 and 4 GB 1600 MHz DDR3). The number of iterations of each case are shown in Table 1.

α_n	number of iterations	α_n	number of iterations
0.1	235	0.6	29
0.2	112	0.7	23
0.3	71	0.8	17
0.4	50	0.9	13
0.5	37		

TABLE 1. The number of iterations for Algorithm (3.1) with difference α_n

The results in Table 1 show that the biggest size of parameter $\alpha_n(\alpha_n = 0.9)$ provides less iterations than other cases. When the size of parameter α_n get close to 0, the number of iteration is increased. While the higher size of parameters α_n make the number of iterations slightly increased.

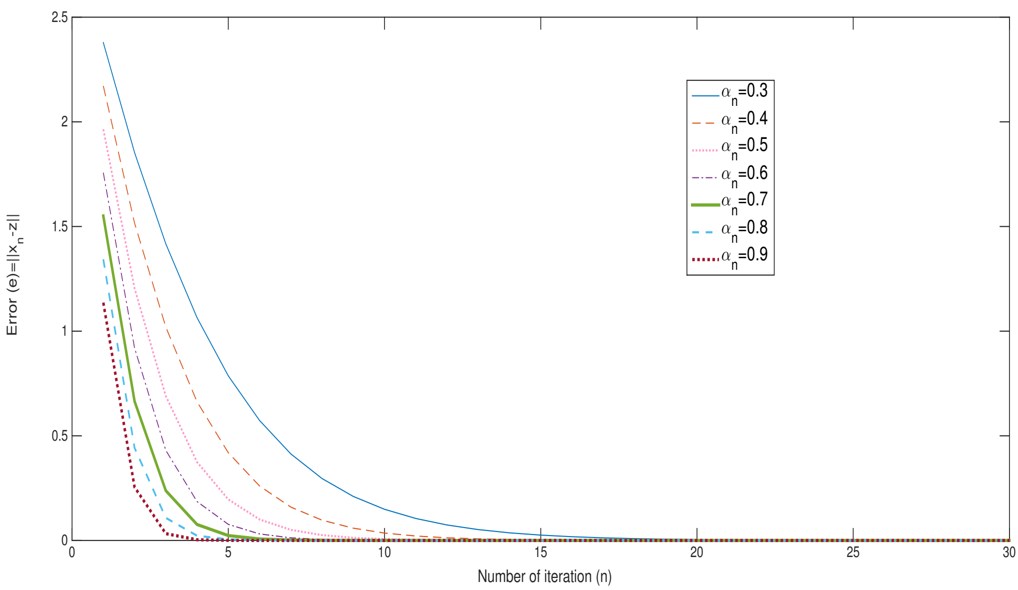

FIGURE 1. Error ($E_n := \|x_n - z\|$) of the sequence $\{x_n\}$ in different parameters α_n.

We choose the different parameters α_n which are 0.1, 0.5 and 0.9 to show the sequence of $\{x_n\}$ generated by Algorithm (3.1). The sequences of $\{x_n\}$ obtained by the proposed algorithm are shown in Table 2. In case $\alpha_n = 0.1$, the second term of sequence $\{x_n\}$ is slightly decreased. In case $\alpha_n = 0.5$, the second term of sequence $\{x_n\}$ is moderately decreased. While the sequence $\{x_n\}$ is rapidly decreased in the last case.

$\alpha_n = 0.1$		$\alpha_n = 0.5$		$\alpha_n = 0.9$	
n	x_n	n	x_n	n	x_n
1	$(-3, 4)$	1	$(-3, 4)$	1	$(-3, 4)$
2	$(-3, 3.792926279833230)$	2	$(-3, 2.96463139916615)$	2	$(-3, 2.13633651849907)$
3	$(-3, 3.59628850481420)$	3	$(-3, 2.20476676330093)$	3	$(-3, 1.03281082055393)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
234	$(-3, 1.00000000010482)$	36	$(-3, 1.00000000019188)$	12	$(-3, 1.00000000034167)$
235	$(-3, 1.00000000009434)$	37	$(-3, 1.00000000009594)$	13	$(-3, 1.00000000003417)$

TABLE 2. The sequence of $\{x_n\}$ generated by Algorithm (3.1) with different constant parameters α_n

In addition, the parameters α_n depend on the number of iteration are studied in 2 cases: decreasing α_n to 0 and increasing α_n to 1. In decreasing case, we defined the parameter $\alpha_n = 0.0001 + \dfrac{1}{n+1}$. In increasing case, the parameter is defined by $\alpha_n = 0.99 - \dfrac{1}{n+1}$. The comparision of the sequence $\{x_n\}$ generated by Algorithm (3.1) with vary parameters α_n and constant $\alpha_n = 0.5$ is shown in Table 3.

$\alpha_n = 0.0001 + \frac{1}{n+1}$		$\alpha_n = 0.5$		$\alpha_n = 0.99 - \frac{1}{n+1}$	
n	x_n	n	x_n	n	x_n
1	$(-3, 4)$	1	$(-3, 4)$	1	$(-3, 4)$
2	$(-3, 3.99979292627983)$	2	$(-3, 2.96463139916615)$	2	$(-3, 1.94997017034897)$
3	$(-3, 2.96426916040579)$	3	$(-3, 2.20476676330093)$	3	$(-3, 1.53872801098075)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$129,584$	$(-3, 1.00000000010001)$	36	$(-3, 1.00000000019188)$	14	$(-3, 1.00000000050447)$
$129,585$	$(-3, 1.00000000009434)$	37	$(-3, 1.00000000009594)$	15	$(-3, 1.00000000004108)$

TABLE 3. The sequence of $\{x_n\}$ generated by Algorithm (3.1) with different parameters α_n

From Table 3, we observe that the number of iterations extremely increase in the decreasing case of α_n ($\alpha_n = 0.0001 + \dfrac{1}{n+1}$). While the number of iterations in increasing case of α_n ($\alpha_n = 0.99 - \dfrac{1}{n+1}$) is slightly different from the number of iterations in constant α_n ($\alpha_n = 0.5$).

From the numerical experiment, the choosing large parameter α_n (close to 1) causes the proposed algorithm works faster. While tiny parameter (close to 0) causes Algorithm (3.1) work extremely slow.

5. Conclusion

In this paper, we have presented a proximal point algorithm for solving a common solution of the minimization problems and best proximity point problem. Under appropriate conditions, we proved that the sequence generated by the proposed algorithms converges to best proximity points and minimum point. Numerical results have been demonstrated the behaviour of algorithm's convergence and its effectiveness.

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