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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

A New Hybrid Iterative Method for Solving Mixed Equilibrium and Fixed Point Problems for Bregman Relatively Nonexpansive Mappings

Kanikar Muangchoo[1], Poom Kumam[1,*], Yeol Je Cho[2], Sakulbuth Ekvittayaniphon[3] and Wachirapong Jirakitpuwapat[1]

[1] *KMUTTFixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
e-mail : ni_003@hotmail.com (K. Muangchoo); poom.kumam@mail.kmutt.ac.th (P. Kumam); wachirapong.jira@hotmail.com (W. Jirakitpuwapat)

[2] *Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea, and School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China*
e-mail : yjchomath@gamil.com (Y. J. Cho)

[3] *Faculty of Science and Technology, Rajamangala University of Technology Phra Nakhon (RMUTP) 1381 Pracharat 1 Road, Wongsawang, Bang Sue, Bangkok 10800, Thailand*
e-mail : sakulbuth.e@rmutp.ac.th (S. Ekvittayaniphon)

Abstract In this paper, we introduce a new hybrid iterative method for finding a common solution of a mixed equilibrium problem and a fixed point problem for a Bregman relatively nonexpansive mapping in reflexive Banach spaces. We prove that a sequence generated by the hybrid iterative algorithm converges strongly to a common solution of these problems. Finally, we provide some consequences of the main result and give a numerical example to demonstrate the applicability of the iterative algorithm.

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1. Introductions

In 2008, the equilibrium problem was generalized by Ceng and Yao [1] to the mixed equilibrium problem: Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $\Theta : C \times C \to \mathbb{R}$ be

*Corresponding author.

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an equilibrium bifunction. The mixed equilibrium problem (for short, MEP) is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) \geq \varphi(x^*), \quad \forall y \in C. \tag{1.1}$$

The solution set of MEP (1.1) is denoted by Sol(MEP). In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (for short, EP), which is to find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C, \tag{1.2}$$

which is introduced and studied by Blum and Oettli [2]. The solution set of EP(1.2) is denoted by Sol(EP). It is known that the equilibrium problems have a great impact and influence in the development of several topics of science and engineering, so that they can be solved using the novel and unified framework. The problem of finding the set of solutions in (1.2) arises in various areas such as nonlinear analysis, optimization, economic, financial, and game theory. Some notable problems are mathematical programming problem, variational inequality problem, complementary problem, saddle point problem, Nash equilibrium problem in noncooperative games and fixed point problem, see [2–6]

The concept of nonexpansivity plays an important role in the study of Mann-type iteration for finding fixed points of a mapping. In 1953, Mann [7] introduced an iterative method for finding fixed points in a Banach space of a mapping $T : C \to C$, utilizing the concept of nonexpansivity. For an arbitrary starting point $x_0 \in C$, we can generate a sequence of points $\{x_n\}$ using the Mann-type iteration formula:

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n, \quad x_1 \in C, \tag{1.3}$$

given that $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence satisfying some appropriate conditions. The construction of fixed points of nonexpansive mappings via Mann's algorithm has been extensively investigated in the literature.

In 1979, Reich [8] proved that if we choose a sequence $\{\alpha_n\}$ such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then a sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Mann's algorithm converges weakly to a fixed point of T. Since a sequence generated by Mann's algorithm does not converge strongly in general, some attempts to modify the Mann iteration algorithm (1.3) to guarantee strong convergence have recently been made.

In 2008, Takahashi et al. [9] came up with an algorithm, called *The Shrinking Projection Method*, to find a common fixed point of an infinite family of nonexpansive mappings in Hilbert space. Let $T_i : C \to C$ for $i \in \mathbb{N}$ be mappings such that $F(T) := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Starting with $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$, they defined the iteration

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $P_{C_{n+1}}$ is the metric projection from C onto C_{n+1} and $\{\alpha_n\}$ is chosen so that $0 \leq \alpha_n \leq a < 1$ for some $a \in [0, 1)$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

In 2010, Reich and Sabach [10] proposed two algorithms for finding a common fixed point of finitely many Bergman strongly nonexpansive mappings $T_i : C \to C$ ($i =$

$1, 2, ..., N$) satisfying $\cap_{i=1}^{N} F(T_i) \neq \emptyset$ in a reflexive Banach space E as follows:

$$\begin{cases} x_0 = x \in E, \text{ chosen arbitrarily}, \\ y_n^i = T_i(x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \cap_{i=1}^{N} C_n^i, \\ Q_n^i = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = proj_{C_n \cap Q_n}^f (x_0), \ \forall n \geq 0, \end{cases} \tag{1.5}$$

and

$$\begin{cases} x_0 \in E, C_0^i = E, i = 1, 2, ..., N, \\ y_n^i = T_i(\nu_n + e_n^i), \\ C_{n+1}^i = \{z \in C_n^i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_{n+1} = \cap_{i=1}^{N} C_{n+1}^i, \\ x_{n+1} = proj_{C_{n+1}}^f (x_0), \ \forall n \geq 0, \end{cases} \tag{1.6}$$

where $proj_C^f$ is the Bregman projection with respect to f from E onto a closed and convex subset C of E. They proved that $\{x_n\}$ converges strongly to a common fixed point in $\cap_{i=1}^{N} F(T_i)$.

In 2011, Martín Marquez et al. [11] used the following Mann-Type iterative scheme for Bregman strongly nonexpansive mappings.

Theorem 1.1. *Let $T : C \to C$ be a Bregman strongly nonexpansive mapping with $\widehat{F}(T) \neq \emptyset$. Let $f : C \to \mathbb{R}$ be a Legendre function which is totally convex on bounded subsets of E. Suppose that ∇f is weakly sequentially continuous and ∇f^* is bounded on bounded subsets of int domf^*. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence generated by the iterative scheme*

$$x_{n+1} = \nabla f^*\big(\alpha_n \nabla f(z_n) + (1 - \alpha_n)\nabla f(Tx_n)\big), \tag{1.7}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0, 1]$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then, for each $x_0 \in C$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a point in $\widehat{F}(T)$.

In 2012, Suantai et al. [12] used the following Halpern's iterative scheme for Bregman strongly nonexpansive self mapping T on E. Suppose that $u \in E$ and define the sequence $\{x_n\}_{n\in\mathbb{N}}$ as follows: $x_1 \in E$ and

$$x_{n+1} = \nabla f^*\big(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)\big), \ \forall n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that above sequence converges strongly to $P_{F(T)}^f(u)$, where $P_{F(T)}^f(u)$ is the Bregman projection of E onto $F(T)$.

In 2009, Takahashi and Zembayashi [13] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1.2. *Let E be a uniformly smooth and strictly convex Banach space. Let $f : C \times C \to \mathbb{R}$ be a function satisfying*

(A1) *$f(x, x) = 0$ for all $x \in C$;*

(A2) *f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;*

(A3) *for each $y \in C$, the function $x \longmapsto f(x, y)$ is weakly upper semicontinuous;*

(A4) *for each $x \in C$, the function $y \longmapsto f(x, y)$ is convex and lower semicontinuous.*

Let $T : C \to C$ be a relatively nonexpansive (as defined by Matsushita and Takahashi [14]) such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_n = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{1.9}$$

where J is the normalized duality mapping on E and $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\}_{n\in\mathbb{N}} \subset [a,\infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)}x$ as $n \to \infty$.

In 2010, Plubtieng and Ungchiterakool [15] proved a strong convergence theorem by the hybrid method for finding a common fixed point of two relatively nonexpansive mappings and an element of the set of solutions of an equilibrium problem in a Banach space.

Theorem 1.3. *Let E be a uniformly smooth and strictly convex Banach space. Let $\hat{C}$ and C be nonempty, closed and convex subsets of E, and $f : C \times C \to \mathbb{R}$ be a function satisfying $(A1) - (A4)$ in Theorem 1.2 and $EP(f) \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ be sequences generated by*

$$\begin{cases} x_0 = x \in E, \\ u_n \in C = C_1 \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \ \forall y \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Ju_n), \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.10}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies either

(a) $0 \leq \alpha_n < 1$ *for all* $n \in \mathbb{N}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$, *or,*

(b) $0 \leq \alpha_n \leq 1$ *for all* $n \in \mathbb{N}$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$,

and where $\{r_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ and $\Sigma_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to $\Pi_{EP(f)}x_0$ as $n \to \infty$.

In 2018, Biranvand and Darvish [16] studied a new iterative method for a common fixed point of a finite family of Bregman strongly nonexpansive mappings in the frame work of reflexive real Banach spaces. They proved the strong convergence theorem for finding common fixed points with the solutions of a mixed equilibrium problem.

Last but not least, Kazmi et al. [17] developed an iterative method to find a common solution to a fixed point problem for a Bregman relatively nonexpansive mapping in reflexive Banach spaces and a generalized equilibrium problem (GEP):

$$G(x^*, y) + \langle \phi(x^*), y - x^*\rangle \geq 0, \forall y \in C, \tag{1.11}$$

where $\phi : C \to \mathbb{R}$ is a real-valued function. The solution set of GEP (1.11) is denoted by Sol(GEP(G, ϕ)).

$$\begin{cases} x_0, z_0 = x \in C, \\ u_n = \nabla f^*\big(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)\big), \\ z_{n+1} = res^f_{G,\phi} u_n, \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, x_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \geq 0\}, \\ x_{n+1} = proj^f_{C_n \cap Q_n} x_0, \ \ \forall n \geq 0, \end{cases} \tag{1.12}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then, a sequence $\{x_n\}$ generated by the iterations (1.12) converges strongly to $proj^f_{GEP(f)\cap F(T)} x_0$.

In this paper, motivated by the works given in [16, 17], we study a solution to the mixed equilibrium problem (MEP), denoted by Sol(MEP(Θ, φ)). Then, we propose the following modified iterative method for finding both a solution to (1.1) and a fixed point:

$$\begin{cases} x_0 \in C = C_0, \\ y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)], \\ u_n \in Res^f_{\Theta,\varphi}(y_n), \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n)\}, \\ x_{n+1} = proj^f_{C_{n+1}}(x_0), \ \ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.13}$$

where $T : C \to C$ be a Bregman relatively nonexpansive mapping and $MEP(f) \cap F(T) \neq \emptyset$. We will prove that a sequence $\{x_n\}$ generated by (1.13) converges strongly to the point $proj^f_{MEP(f)\cap F(T)}(x_0)$. In addition, we give some consequences of the main result, and provide a numerical example to demonstrate the applicability of the iterative algorithm.

2. Preliminaries

In this section, we introduce several definitions and results, which are used in the following sections.

Let E be a real Banach space with E^* as its dual space and the norm $\|\cdot\|$. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x, x^*\rangle$. If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in E, we denote the strong convergence and the weak convergence of $\{x_n\}_{n\in\mathbb{N}}$ to a point $x \in E$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let C be a nonempty, closed and convex subset of E and $T : C \to C$ be a mapping. Then a point $x \in C$ is called a *fixed point* of T if $Tx = x$ and the set of all fixed points of T is denoted by $F(T)$.

We call a map T

(i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall x, y \in C;$$

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \ \ \forall x \in C, \ \ y \in F(T);$$

(iii) *closed* if $x_n \to x$ and $Tx_n \to y$ implies $Tx = y$.

(iv) *relatively nonexpansive* [14], if the following conditions are satisfied:
   (1) $F(T)$ is nonempty;
   (2) $\phi(u, Tx) \leq \phi(u, x), \ \forall u \in F(T), \ x \in C$;
   (3) $\widehat{F}(T) = F(T)$.

A Banach space E is said to be *uniformly convex* if

$$\delta(\varepsilon) := \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\} > 0$$

for every $\varepsilon > 0$. We denote $\delta(\varepsilon)$ the modulus δ of convexity of E.

Let $S_E(r) = \{x \in E : \|x\| = r\}$ be a sphere of radius r centered at 0 of E and $S_E = S_E(1)$ be the unit sphere. E is said to be strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$ whenever $x, y \in S_E$, and $x \neq y$. The norm of E is said to be *Gâteaux differentiable* if, for each $x, y \in S_E$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. If the limit (2.1) is attained uniformly for all $x, y \in S_E$, then E is called uniformly smooth. It is well known (see [18, 19]) that E is uniformly convex if and only if E^* is uniformly smooth. Moreover, If E is reflexive, then E is strictly convex if and only if E^* is smooth.

For any $x \in \text{int}(\text{dom}f)$, the *right-hand derivative* of f at x in the direction $y \in E$ is defined by

$$f'(x, y) = \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{2.2}$$

A function f is called *Gâteaux differentiable at* x if $f'(x, y)$ is defined (i.e. the limit exists) for all $y \in E$. In this case, $f'(x, y)$ coincides with the value of the *gradient* (∇f) of f at x. Furthermore, if f is Gâteaux differentiable for any $x \in \text{int}(\text{dom}f)$, we say that f is *Gâteaux differentiable.*

A function f is called *Fréchet differentiable* at x if this limit is attained uniformly for $\|y\| = 1$. Moreover, f is *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for all $x \in C$ and $\|y\| = 1$.

It is well known that if a continuous convex function $f : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇f is norm-to-weak* continuous (see [20]). Also, it is known that if f is Fréchet differentiable, then ∇f is norm-to-norm continuous (see, [21]).

In the sequel we shall denote by $\Gamma(E)$ the class of proper lower semi-continuous convex functions on E, and $\Gamma^*(E^*)$ the class of proper weak* lower semi-continuous convex functions on E^*.

For $f \in \Gamma(E)$, the *subdifferential* ∂f of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y), \forall y \in E\}$$

for all $x \in E$. Rockafellar's theorem [22, 23] ensures that $\partial f \subset E \times E^*$ is maximal monotone. If $f \in \Gamma(E)$ and $g : E \to \mathbb{R}$ is a continuous convex function, then $\partial(f+g) = \partial f + \partial g$.

For $f \in \Gamma(E)$, *the (Fenchel) conjugate function* f^* of f is defined by

$$f^*(x^*) = \sup_{x \in E}\{\langle x, x^*\rangle - f(x)\}, \ \text{ for all } x^* \in E^*.$$

It is well known that

$$f(x) + f^*(x^*) \geq \langle x, x^*\rangle, \ \text{ for all } (x, x^*) \in E \times E^*,$$

and that $(x, x^*) \in \partial f$ is equivalent to

$$f(x) + f^*(x^*) = \langle x, x^*\rangle. \tag{2.3}$$

We also know that if $f \in \Gamma(E)$, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous convex function; see [24] for more details on convex analysis.

A function $f : E \to \mathbb{R}$ is said to be *strongly coercive* if

$$\lim_{\|x_n\| \to \infty} \frac{f(x_n)}{\|x_n\|} = \infty.$$

It is also said to be *bounded on bounded sets* if $f(S_E(r))$ is bounded for each $r > 0$.

A function $f : E \to \mathbb{R}$ is said to be *uniformly convex on bounded sets* [25] if the function $\rho_r : [0, +\infty) \to [0, +\infty]$, called the *uniform convexity* of f, defined by

$$\rho_r(t) = \inf_{\substack{x,y \in S_E(r),\\ \|x-y\|=t,\\ \alpha \in (0,1)}} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)}$$

is positive for all r and $t > 0$. It is known that $\rho_r(t)$ is a nondecreasing function. If $f : E \to \mathbb{R}$ is uniformly convex on bounded sets of E, then we have

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) - \alpha(1-\alpha)\rho_r(\|x - y\|) \tag{2.4}$$

for all x, y in $S_E(r)$ and $\alpha \in (0, 1)$.

A function f is said to be *locally uniformly smooth on bounded sets* [25] if the function $\sigma_r : [0, +\infty) \to [0, +\infty]$ defined by

$$\sigma_r(t) = \sup_{\substack{x \in S_E(r), y \in S_E,\\ \alpha \in (0,1)}} \frac{\alpha f(x + (1-\alpha) t y) + (1-\alpha) f(x - \alpha t y) - f(x)}{\alpha(1-\alpha)},$$

satisfies

$$\lim_{t \to 0^+} \frac{\sigma_r(t)}{t} = 0, \ \text{ for all } r > 0.$$

The Legendre function f is defined from a general Banach space $E \to (-\infty, +\infty]$, see [26]. Since E is reflexive, according to [26], the function f is Legendre if it satisfies the following conditions:

(1) The interior of the domain of f, $\operatorname{int}(\operatorname{dom} f)$, is nonempty; f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f)$.

(2) The interior of the domain of f^*, $\operatorname{int}(\operatorname{dom} f^*)$, is nonempty; f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$.

We note that for a Legendre function f, the following properties hold [26]:

(a) f is a Legendre function if and only if f^* is a Legendre function;

(b) $(\partial f)^{-1} = \partial f^*$;

(c) $\nabla f = \big(\nabla f^*\big)^{-1}$, $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\big(\operatorname{dom} f^*\big)$, $\operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\big(\operatorname{dom} f\big)$;

(d) The functions f and f^* are strictly convex on the interior of respective domains.

Definition 2.1 ([27]). Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, +\infty)$ defined by

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{2.5}$$

is called *Bregman distance* with respect to f.

In particular, it can be easily seen that

$$D_f(x, y) = -D_f(y, x) + \langle \nabla f(y) - \nabla f(x), y - x\rangle.$$

Using the above observation and the definition, we can derive the following properties.

- The *three point identity*: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$, we have

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{2.6}$$

- The *four point identity*: for any $y, w \in \operatorname{dom} f$ and $x, z \in \operatorname{int}(\operatorname{dom} f)$, we have

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w\rangle.$$

It is clear that $D_f(x, y) \geq 0$ for all $x, y \in E$. In the case when E is a smooth Banach space, we set $f(x) = \|x\|^2$ for all $x \in E$ to obtain that $\nabla f(x) = 2Jx$ for all for all $x \in E$ and hence $D_f(x, y) = \phi(x, y)\ \ \forall x, y \in E$ [28].

Furthermore, let E be a Banach space and C be a nonempty closed convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Then it follows from [29] that, for any $x \in E$ and $x_0 \in C$, we have

$$D_f(x_0, x) = \min_{y \in C} D_f(y, x).$$

The *Bregman projection* $proj_C^f$ from E onto C is defined by $proj_C^f(x) = x_0$ for all $x \in E$. It is well known that $x_0 = proj_C^f(x)$ if and only if

$$\langle y - x_0, \nabla f(x) - \nabla f(x_0)\rangle \leq 0,\ \ \forall y \in C. \tag{2.7}$$

It is also known that $proj_C^f$ from E onto C has the following property:

$$D_f(y, proj_C^f(x)) + D_f(proj_C^f(x), x) \leq D_f(y, x),\ \ \forall y \in C,\ x \in E. \tag{2.8}$$

For more details on Bregman projection $proj_C^f$, see [20].

Definition 2.2 ([20]). Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called:

(1) totally convex at $x \in \operatorname{int}(\operatorname{dom} f)$ if its modulus of total convexity at x, that is, the function $\nu_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x, t) = \inf\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\},$$

is positive whenever $t > 0$;

(2) totally convex if it is totally convex at every point $x \in \operatorname{int}(\operatorname{dom} f)$;

(3) totally convex on bounded sets if $\nu_f(B,t)$ is positive for any nonempty bounded subset B of E and $t>0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f:\mathrm{int}(\mathrm{dom} f)\times[0,+\infty)\to[0,+\infty)$ defined by

$$\nu_f(B,t)=\inf\{\nu_f(x,t):x\in B\cap \mathrm{dom}\, f\}.$$

Definition 2.3 ([10, 30])**.** Let $T:C\to\ \mathrm{int}(\mathrm{dom} f)$ be a mapping and let $F(T)$ denote the set of fixed points of T , i.e., $F(T)=\{x\in C:Tx=x\}$. We recall the necessary notations of the nonlinear mapping related to Bregman distance as follows:

(1) A point $p\in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$ [8];

(2) T is said to be *Bregman quasi-nonexpansive* if $F(T)\neq\emptyset$ and

$$D_f(p,Tx)\le D_f(p,x),\forall x\in C,p\in F(T);$$

(3) T is said to be *Bregman relatively nonexpansive* [10] if $\widehat{F}(T)=F(T)$ and

$$D_f(p,Tx)\le D_f(p,x),\forall x\in C,p\in F(T);$$

(4) T is said to be *Bregman firmly nonexpansive* if

$$\langle\nabla f(Tx)-\nabla f(Ty),Tx-Ty\rangle\le\langle\nabla f(x)-\nabla f(y),Tx-Ty\rangle,\ \ \forall x,y\in C,$$

or, equivalently

$$D_f(Tx,Ty)+D_f(Ty,Tx)+D_f(Tx,x)+D_f(Ty,y)\le D_f(Tx,y)+D_f(Ty,x),$$
$$\forall x,y\in C.$$

Note that the notions of an asymptotic fixed point of T and Bregman relatively nonexpansive operators were first introduced and studied by Reich [31] and Butnariu et al. [32] respectively.

Example 2.4 ([33])**.** Let E is a real Banach space, $A:E\to 2^{E^*}$ be a maximal monotone mapping. If $A^{-1}(0)\neq\emptyset$ and the Legendre function $f:E\to(-\infty,+\infty]$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the resolvent with respect to A,

$$Res_A^f(x)=\left(\nabla f+A\right)^{-1}\circ\nabla f(x)$$

is a single-valued, closed and Bregman relatively nonexpansive mapping from E onto $D(A)$ and $F\left(Res_A^f\right)=A^{-1}(0)$.

Let $f:E\to\mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. In addition, if $f:E\to(-\infty,+\infty]$ is a proper lower semicontinuous function, then $f^*:E^*\to(-\infty,+\infty]$ is a proper weak* lower semicontinuous and convex function (see [11, 24]). Hence V_f is convex in the second variable. Thus, for all $z\in E, wehaves$

$$D_f\left(z,\nabla f^*\left(\sum_{i=1}^N t_i\nabla f(x_i)\right)\right)\le\sum_{i=1}^N t_iD_f(z,x_i). \tag{2.9}$$

where $\{x_i\}_{i=1}^N\subset E$ and $\{t_i\}_{i=1}^N\subset(0,1)$ with $\sum_{i=1}^N t_i=1$.

The following definition is slightly different from that in [20]:

Definition 2.5 ([21]). Let E be a Banach space. Then a function $f : E \to \mathbb{R}$ is said to be a *Bregman function* if the following conditions are satisfied:

(1) f is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D_f(x, y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [20] and Zălinscu [25]:

Lemma 2.6. *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then we have the following:*

(1) $\nabla f : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;
(2) $\langle x - y, \nabla f(x) - \nabla(y)\rangle = 0$ if and only if $x = y$;
(3) $\{x \in E : D_f(x, y) \le r\}$ is bounded for all y in E and $r > 0$;
(4) dom $f^* = E^*$, f^* is Gâteaux differentiable function and $\nabla f^* = (\nabla f)^{-1}$.

In addition, we have the following Proposition; see [25].

Proposition 2.7 ([25]). *Let $f \in \Gamma(E)$ be convex. Consider the following statements:*

(1) f is bounded and uniformly smooth on bounded sets;

(2) f is Fréchet differentiable on $E =$ dom f and ∇f is uniformly continuous on bounded sets;

(3) f^* is strongly coercive and uniformly convex on bounded sets;

Then we have (1) $\Longleftrightarrow$ (2) $\Leftarrow$ (3). Moreover, if f is strongly coercive then (1) $\Rightarrow$ (3); in this case E^* is reflexive (also E is reflexive if E is a Banach space).

Proposition 2.8 ([25]). *Let $f \in \Gamma(E)$. Consider the following statements:*

(1) f is strongly coercive and uniformly convex on bounded sets;
(2) f^* is bounded and uniformly smooth on bounded sets;
(3) f^* is Fréchet differentiable on E^* dom f^* and ∇f^* is uniformly continuous on bounded sets;

Then (1) $\Rightarrow$ (2) $\Longleftrightarrow$ (3). Moreover, if f is bounded on bounded sets then (2) $\Rightarrow$ (1); in this case E^* is reflexive (also E is reflexive if E is a Banach space).

Lemma 2.9 ([21]). *Let E be a Banach space and let $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded sets. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following lemmas was first proved in Kohsaka and Takahashi [21].

Lemma 2.10 ([21]). *Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and V be the function defined by*

$$V(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \text{ for all } x \in E,\ x^* \in E^*.$$

The following assertions hold:

(1) $D_f\big(x, \nabla f^*(x^*)\big) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.
(2) $V(x, x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \le V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.

It also follows from the definition that V is convex in the second variable x^* and

$$V\big(x, \nabla f(y)\big) = D_f(x, y).$$

Lemma 2.11 ([34]). *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f^* is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.12 ([20]). *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.13 ([10]). *Let $f : E \to \mathbb{R}$ be a totally convex and Gâteaux differentiable function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.14 ([30]). *Let E be a reflexive Banach space and C be a nonempty closed convex subset of $int(domf)$ and $f : E \to \mathbb{R}$ be Legendre function. Let $T : C \to C$ be a Bregman quasi-nonexpansive mapping with respect to f, then $F(T)$ is closed and convex.*

Lemma 2.15 ([10]). *Let $f : E \to \mathbb{R}$ be a totally convex and Gâteaux differentiable function, $x_0 \in E$ and C be a nonempty closed convex subset of a reflexive Banach space E. Suppose that $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_0) \leq D_f(proj_C^f x_0, x_0)$ for any $n \in N$, then$\{x_n\}$ converges strongly to $proj_C^f x_0$.*

Lemma 2.16 ([35]). *Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers satisfying*

$$x_{n+1} \leq (1 - \gamma_n)x_n + \gamma_n\delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy the conditions:

(a) $\{\gamma_n\}_{n\in\mathbb{N}} \subset [0, 1]$ *and* $\Sigma_{n=1}^{\infty}\gamma_n = +\infty$ *or, equivalently,* $\Pi_{n=1}^{\infty}(1 - \gamma_n) = 0$;

(b) $\limsup_{n\to\infty} \delta_n < 0$ *or* $\Sigma_{n=1}^{\infty}\gamma_n\delta_n < \infty$.

Then, we have $\lim_{n\to\infty} x_n = 0$.

In order to solve the mixed equilibrium problem, let us recall the following assumptions, as stated in Theorem 1.2, for bifunction Θ on the set C :

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $y \in C$, the function $x \longmapsto \Theta(x, y)$ is weakly upper semicontinuous;

(A4) for each $x \in C$, the function $y \longmapsto \Theta(x, y)$ is convex and lower semicontinuous.

Definition 2.17 ([16]). Let C be a nonempty, closed and convex subsets of a real reflexive Banach space and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunctional satisfying $(A1) - (A4)$. The mixed resolvent of Θ is the operator $Res_{\Theta,\varphi}^f : E \to 2^C$

$$Res_{\Theta,\varphi}^f(x) = \{z \in C : \Theta(z, y) + \varphi(y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq \varphi(z), \quad \forall y \in C\}. \tag{2.10}$$

The following results can be deduced from Lemma 2.6 and Lemma 2.9 due to Reich and Sabach [10]. Their proof is provided for reader's convenience.

Lemma 2.18 ([16]). *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ is a coercive and Gâteaux differentiable function. Let C be a nonempty, closed and convex subset of E. Assume that $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional and the bifunctional $\Theta : C \times C \to \mathbb{R}$ satisfies conditions $(A1)-(A4)$, then $dom\big(Res^f_{\Theta,\varphi}\big) = E$.*

Proof. Since f is a coercive function, the function $h : E \times E \to \mathbb{R}$ defined by

$$h(x, y) = f(y) - f(x) - \langle x^*, y - x\rangle$$

satisfies the following for all $x^* \in E^*$ and $y \in C$;

$$\lim_{\|x-y\|\to+\infty} \frac{h(x, y)}{\|x - y\|} = +\infty.$$

Then from [2], there exists $\hat{x} \in C$ such that

$$\Theta(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) + f(y) - f(\hat{x}) - \langle x^*, y - \hat{x}\rangle \ge 0$$

for any $y \in C$. This implies that

$$\Theta(\hat{x}, y) + \varphi(y) + f(y) - f(\hat{x}) - \langle x^*, y - \hat{x}\rangle \ge \varphi(\hat{x}). \tag{2.11}$$

We know that (2.11) holds for $y = t\hat{x} + (1 - t)\hat{y}$ where $\hat{y} \in C$ and $t \in (0, 1)$. Therefore, we have

$$\begin{aligned}\Theta(\hat{x}, t\hat{x} + (1 - t)\hat{y}) + \varphi(t\hat{x} + (1 - t)\hat{y}) + f(t\hat{x} + (1 - t)\hat{y}) - f(\hat{x})&\\ - \langle x^*, t\hat{x} + (1 - t)\hat{y} - \hat{x}\rangle&\\ \ge \varphi(\hat{x})&\end{aligned} \tag{2.12}$$

for all $\hat{y} \in C$. By convexity of φ, we have

$$\begin{aligned}\Theta(\hat{x}, t\hat{x} + (1 - t)\hat{y}) + (1 - t)\varphi(\hat{y}) + f(t\hat{x} + (1 - t)\hat{y}) - f(\hat{x})&\\ - \langle x^*, t\hat{x} + (1 - t)\hat{y} - \hat{x}\rangle&\\ \ge (1 - t)\varphi(\hat{x}).&\end{aligned} \tag{2.13}$$

Since

$$f(t\hat{x} + (1 - t)\hat{y}) - f(\hat{x}) \le \langle \nabla f(t\hat{x} + (1 - t)\hat{y}), t\hat{x} + (1 - t)\hat{y} - \hat{x}\rangle,$$

we can conclude from (2.13) and $(A4)$ that

$$\begin{aligned}t\Theta(\hat{x}, \hat{x}) + (1 - t)\Theta(\hat{x}, \hat{y}) + (1 - t)\varphi(\hat{y}) + \langle \nabla f(t\hat{x} + (1 - t)\hat{y}), t\hat{x} + (1 - t)\hat{y} - \hat{x}\rangle&\\ - \langle x^*, t\hat{x} + (1 - t)\hat{y} - \hat{x}\rangle&\\ \ge (1 - t)\varphi(\hat{x})&\end{aligned} \tag{2.14}$$

for all $\hat{y} \in C$. From $(A1)$ we have

$$\begin{aligned}(1 - t)\Theta(\hat{x}, \hat{y}) + (1 - t)\varphi(\hat{y}) + \langle \nabla f(t\hat{x} + (1 - t)\hat{y}), (1 - t)(\hat{y} - \hat{x})\rangle&\\ - \langle x^*, (1 - t)(\hat{y} - \hat{x})\rangle&\\ \ge (1 - t)\varphi(\hat{x}),&\end{aligned} \tag{2.15}$$

or, equivalently,

$$(1 - t)\big[\Theta(\hat{x}, \hat{y}) + \varphi(\hat{y}) + \langle \nabla f(t\hat{x} + (1 - t)\hat{y}), \hat{y} - \hat{x}\rangle - \langle x^*, \hat{y} - \hat{x}\rangle\big] \ge (1 - t)\varphi(\hat{x}).$$

Thus, we have

$$\Theta(\hat{x}, \hat{y}) + \varphi(\hat{y}) + \langle \nabla f(t\hat{x} + (1 - t)\hat{y}), \hat{y} - \hat{x}\rangle - \langle x^*, \hat{y} - \hat{x}\rangle \ge \varphi(\hat{x}),$$

for all $\hat{y} \in C$. Since f is Gâteaux differentiable function, it follows that ∇f is norm-to-weak* continuous (see [24]). Hence, letting $t \to 1^-$ we then get

$$\Theta(\hat{x}, \hat{y}) + \varphi(\hat{y}) + \langle \nabla f(\hat{x}), \hat{y} - \hat{x} \rangle - \langle x^*, \hat{y} - \hat{x} \rangle \geq \varphi(\hat{x}).$$

By taking $x^* = \nabla f(x)$ we obtain $\hat{x} \in C$ such that

$$\Theta(\hat{x}, \hat{y}) + \varphi(\hat{y}) + \langle \nabla f(\hat{x}) - \nabla f(x), \hat{y} - \hat{x} \rangle \geq \varphi(\hat{x}),$$

for all $\hat{y} \in C$, i.e., $\hat{x} \in Res^f_{\Theta,\varphi}(x)$. Hence, we conclude that $\mathrm{dom}\big(Res^f_{\Theta,\varphi}\big) = E$. ■

Lemma 2.19 ([16])**.** *Let $f : E \to \mathbb{R}$ be a Legendre function. Let C be a closed and convex subset of E. If the bifunctional $\Theta : C \times C \to \mathbb{R}$ satisfies conditions $(A1)-(A4)$, then the following hold:*

(1) $Res^f_{\Theta,\varphi}$ is single-valued;

(2) $Res^f_{\Theta,\varphi}$ is a Bregman firmly nonexpansive mapping [10], i.e., for all $x, y \in E$,

$$\langle T_r x - T_r y, \nabla f(T_r x) - \nabla f(T_r y) \rangle \leq \langle T_r x - T_r y, \nabla f(x) - \nabla f(y) \rangle,$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x);$$

(3) $F\big(Res^f_{\Theta,\varphi}\big) = Sol(MEP)$ is closed and convex;

(4) $D_f(q, Res^f_{\Theta,\varphi}(x)) + D_f(Res^f_{\Theta,\varphi}(x), x) \leq D_f(q, x), \ \forall q \in F(Res^f_{\Theta,\varphi})$, $x \in E$;

(5) $Res^f_{\Theta,\varphi}$ is a Bregman quasi-nonexpansive mapping.

3. Main Results

In this section, we prove the following strong convergence theorem for finding a common solution of a mixed equilibrium problem and a fixed point problem for Bregman relatively nonexpansive mappings in reflexive Banach space.

Theorem 3.1. *Assume the following conditions:*

- *E is a Banach space and C is a nonempty, closed and convex subset of a reflexive Banach space of E with dual E^* such that $C \subset int(domf)$ and $F(T) = \widehat{F}(T)$.*
- *$f : E \to \mathbb{R}$ is a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E.*
- *$\Theta : C \times C \to \mathbb{R}$ is a bifunction satisfying conditions $(A1)-(A4)$.*
- *$\varphi : C \to \mathbb{R}$ is a real-valued function.*
- *$T : C \to C$ is a Bregman relatively nonexpansive mapping.*
- *$\Omega = Sol(MEP(\Theta, \varphi)) \cap F(T)$ such that $\Omega \neq \emptyset$.*
- *$\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.*

Let $\{x_n\}$ be a sequence generated by the iterative method:

$$\begin{cases} x_0 \in C = C_0, \\ y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)], \\ u_n \in Res^f_{\Theta,\varphi}(y_n), \\ C_{n+1} = \{z \in C_n : D_f(z,u_n) \le D_f(z,x_n)\}, \\ x_{n+1} = proj^f_{C_{n+1}}(x_0), \ \ n \in \mathbb{N}\cup\{0\}. \end{cases} \tag{3.1}$$

Then, $\{x_n\}$ converges strongly to $proj^f_\Omega(x_0)$ where $proj^f_\Omega(x_0)$ is the Bregman projection of C onto Ω.

Proof. We divide the proof into eight steps.

Step 1: We show that Ω is closed and convex. It follows from Lemma 2.14 and from (3) of Lemma 2.19 that Ω is a closed and convex set and hence $proj^f_\Omega(x_0)$ is well defined.

Step 2: We show that C_n is closed and convex for each $n \in \mathbb{N}\cup\{0\}$ by mathematical induction. The base case when $n = 0$ holds since we already assume that $C_0 = C$ is closed and convex. Then, we suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For each $z \in C_k$, we see that $D_f(z,u_k) \le D_f(z,x_k)$ is equivalent to

$$\langle \nabla f(x_k) - \nabla f(u_k), z\rangle \le f(u_k) - f(x_k) + \langle \nabla f(x_k), x_k\rangle - \langle \nabla f(u_k), u_k\rangle. \tag{3.2}$$

By the construction of the set C_{k+1}, we see that

$$C_{k+1} = \{z \in C_k : D_f(z,u_k) \le D_f(z,x_k)\}. \tag{3.3}$$

Hence C_{k+1} is also closed and convex. Therefore $\{x_n\}$ is well defined.

Step 3: We show that $\Omega \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$ by mathematical induction. Note that $\Omega \subset C_0 = C$. Suppose $\Omega \subset C_k$ for some $k \in \mathbb{N}$ and let $p \in \Omega \subset C_n$. In view of Lemma 2.10 and (3.1), we obtain

$$\begin{aligned} D_f(p,u_k) &= D_f\big(p, Res^f_{\Theta,\varphi}(y_k)\big) \\ &\le D_f\big(p,y_k\big) \\ &= D_f\big(p, \nabla f^*[\alpha_k\nabla f(x_k) + (1-\alpha_k)\nabla f(Tx_k)]\big) \\ &= V\big(p, \alpha_k\nabla f(x_k) + (1-\alpha_k)\nabla f(Tx_k)\big) \\ &\le \alpha_k V\big(p,\nabla f(x_k)\big) + (1-\alpha_k)V\big(p,\nabla f(Tx_k)\big) \\ &= \alpha_k D_f\big(p,x_k\big) + (1-\alpha_k)D_f\big(p,Tx_k\big) \\ &\le \alpha_k D_f\big(p,x_k\big) + (1-\alpha_k)D_f\big(p,x_k\big) \\ &= D_f\big(p,x_k\big). \end{aligned} \tag{3.4}$$

This shows that $p \in C_{k+1}$, which implies $\Omega \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$.

Step 4: We show that the sequences $\{x_n\}$, $\{Tx_n\}$ and $\{u_n\}$ are bounded in C. In view of (2.8), we conclude that

$$\begin{aligned} D_f(x_n,x_0) &= D_f\big(proj^f_{C_n}(x_0), x_0\big) \\ &\le D_f\big(p,x_0\big) - D_f\big(p, proj^f_{C_n}(x_0)\big) \\ &\le D_f\big(p,x_0\big), \ \forall p \in \Omega \subset C_n, \ \ n \in \mathbb{N}\cup\{0\}. \end{aligned} \tag{3.5}$$

This implies that the sequence $\{D_f(x_n, x_0)\}$ is bounded, and hence it follows from Lemma 2.13 that the sequence $\{x_n\}$ is also bounded. Further, the inequality

$$\begin{aligned} D_f(q, x_n) &= D_f\big(q, \, proj^f_{C_{n+1}}(x_0)\big) \\ &\le D_f\big(q, x_0\big) - D_f\big(x_n, x_0\big) \end{aligned}$$

implies that $\{D_f(q, x_n)\}$ is bounded. Now it follows from the fact $D_f(q, Tx_n) \le D_f\big(q, x_n\big), \ \forall q \in \Omega$ that $\{Tx_n\}$ is bounded.

Now, we put $M = \max\{D_f(q, u_0), \sup D_f(q, x_n)\}$ so that $D_f(q, u_0) \le M$. Then, it follows from (3.1) that $D_f(q, u_n) \le M$. Thus, $\{D_f(q, u_n)\}$ is bounded, which implies that $\{u_n\}$ is bounded.

Step 5: We show that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. Since $x_n = proj^f_{C_n}(x_0)$ and $x_{n+1} = proj^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have

$$D_f(x_n, x_0) \le D_f(x_{n+1}, x_0), \ \ n \in \mathbb{N} \cup \{0\}. \tag{3.6}$$

This shows that $\{D_f(x_n, x_0)\}$ is nondecreasing. From (2.8), we obtain

$$\begin{aligned} D_f(x_{n+1}, x_n) &= D_f\big(x_{n+1}, proj^f_{C_n}(x_0)\big) \\ &\le D_f\big(x_{n+1}, x_0\big) - D_f\big(proj^f_{C_n}(x_0), x_0\big) \\ &\le D_f\big(x_{n+1}, x_0\big) - D_f\big(x_n, x_0\big) \end{aligned} \tag{3.7}$$

which implies that $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$.

From (3.6) and (3.7), we get that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

Step 6: We show that $\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since $x_m = proj^f_{C_m}(x_0) \in C_m \subset C_n$ for any positive integer $m > n$, we obtain from (2.8) that

$$\begin{aligned} D_f(x_m, x_n) &= D_f\big(x_m, proj^f_{C_n}(x_0)\big) \\ &\le D_f\big(x_m, x_0\big) - D_f\big(proj^f_{C_n}(x_0), x_0\big) \\ &= D_f\big(x_m, x_0\big) - D_f\big(x_n, x_0\big). \end{aligned} \tag{3.8}$$

Therefore, $D_f(x_m, x_n) \to 0$ as $m, n \to \infty$, and since f is totally convex on bounded subsets of E, f is sequentially consistent by Lemma 2.12. Therefore it follows that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence. The inequality (3.8) implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.9}$$

Since f is totally convex on bounded sets, it follows from Lemma 2.11 that f is sequentially consistent and so we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

From the three point identity of the Bregman distance (2.6), we have

$$D_f(q, x_{n+1}) + D_f(x_{n+1}, u_n) - D_f(q, u_n) = \langle \nabla f(u_n) - \nabla f(x_{n+1}), q - x_{n+1} \rangle.$$

Since f is bounded on bounded subsets of E, then ∇f is bounded on bounded subsets of E^* and hence it follows from boundedness of $\{x_n\}, \{Tx_n\}$ and $\{u_n\}$ that the sequences $\{\nabla f(x_n)\}$, $\{\nabla f(Tx_n)\}$ and $\{\nabla f(u_n)\}$ are bounded in E^*, which implies that $\{D_f(x_{n+1}, u_n)\}$ is bounded.

Since $x_{n+1} = proj^f_{C_{n+1}}(x_0) \in C_{n+1}$, we conclude that

$$D_f\big(x_{n+1}, u_n\big) \le D_f\big(x_{n+1}, x_n\big) \to 0, \;\; n \to \infty.$$

This, together with (3.9), implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0. \tag{3.11}$$

Therefore, by Lemma 2.10, we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.12}$$

Taking into account

$$\|x_{n+1} - u_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|,$$

using (3.10) and (3.12), we get

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.13}$$

By Lemma 2.11, ∇f is uniformly continuous on bounded subsets of E, and so we have

$$\lim_{n\to\infty} |\nabla f(x_n) - \nabla f(u_n)| = 0 \tag{3.14}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0. \tag{3.15}$$

Next, we estimate the following difference.

$$\begin{aligned} D_f(p, x_n) - D_f(p, u_n) &= f(p) - f(x_n) - \langle \nabla f(x_n), p - x_n\rangle - f(p) + f(u_n) \\ &\quad + \langle \nabla f(u_n), p - u_n\rangle \\ &= f(u_n) - f(x_n) + \langle \nabla f(u_n), p - u_n\rangle - \langle \nabla f(x_n), p - x_n\rangle \\ &= f(u_n) - f(x_n) + \langle \nabla f(u_n), x_n - u_n\rangle \\ &\quad + \langle \nabla f(x_n) - \nabla f(x_n), p - x_n\rangle. \end{aligned} \tag{3.16}$$

Since $\{u_n\}$ and $\{\nabla f(u_n)\}$ are bounded, it follows from (3.13), (3.14), (3.15) and (3.16) that

$$\lim_{n\to\infty} |D_f(p, x_n) - D_f(p, u_n)| = 0. \tag{3.17}$$

Further, it follows from property (5) in Lemma 2.19 that

$$\begin{aligned} D_f(u_n, y_n) &\le D_f(p, y_n) - D_f(p, u_n) \\ &\le D_f\big(p, \nabla f^*\big(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)\big)\big) - D_f(p, u_n) \\ &\le \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, Tx_n) - D_f(p, u_n) \\ &\le \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, x_n) - D_f(p, u_n) \\ &\le D_f(p, x_n) - D_f(p, u_n). \end{aligned} \tag{3.18}$$

Since $\{D_f(p, x_n)\}$ is bounded, it follows from (3.17) and (3.18) that

$$\lim_{n\to\infty} D_f(u_n, y_n) = 0$$

and hence

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.19}$$

Further, it follows from (3.13), (3.19) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.20}$$

Again, since ∇f is uniformly continuous on bounded subsets of E, it follows from Lemma 2.11, (3.19) and (3.20) we have that

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(y_n)\| = 0 \tag{3.21}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{3.22}$$

Now,

$$\begin{aligned}\|\nabla f(x_n) - \nabla f(y_n)\| &= \|\nabla f(x_n) - \nabla f\big(\nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)]\big)\| \\ &= \|\nabla f(x_n) - \alpha_n \nabla f(x_n) - (1-\alpha_n)\nabla f(Tx_n)\| \\ &= \|(1-\alpha_n)\nabla f(x_n) - (1-\alpha_n)\nabla f(Tx_n)\| \\ &= \|(1-\alpha_n)\big(\nabla f(x_n) - \nabla f(Tx_n)\big)\| \\ &= (1-\alpha_n)\|\nabla f(x_n) - \nabla f(Tx_n)\|.\end{aligned} \tag{3.23}$$

Since $\{\nabla f(x_n)\}$ and $\{\nabla f(u_n)\}$ are bounded, it follows from (3.22), (3.23) and the condition $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(Tx_n)\| = 0. \tag{3.24}$$

Moreover, we have from (3.24) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.25}$$

Hence, we obtain $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ as desired.

Step 7: We show that $w \in \Omega = Sol(MEP(\Theta, \varphi)) \cap F(T)$. First, we show that $w \in F(T)$. Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$ as $k \to \infty$. It follows from (3.13), (3.19) that $\{x_n\}, \{y_n\}$ and $\{u_n\}$ all have the same asymptotic behaviour and hence there exist subsequences $\{y_{n_k}\}$ of $\{y_n\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that $y_{n_k} \rightharpoonup w$ and $u_{n_k} \rightharpoonup w$ as $k \to \infty$. It follows from the fact $x_{n_k} \rightharpoonup w$ and (3.25) that

$$\lim_{n\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{3.26}$$

Since T is Bregman relatively nonexpansive, it follows from (3.26) that $w \in \widehat{F}(T) = F(T)$.

Next, we show that $w \in Sol(MEP(\Theta, \varphi))$. Since $u_n = Res^f_{\Theta,\varphi}(y_n)$, we have

$$\Theta(u_n, z) + \varphi(z) + \langle \nabla f(u_n) - \nabla f(x_n), z - u_n \rangle \geq \varphi(u_n), \ \ \forall z \in C.$$

From $(A2)$, we have

$$\Theta(z, u_n) \leq -\Theta(u_n, z) \leq \varphi(z) - \varphi(u_n) + \langle \nabla f(u_n) - \nabla f(x_n), z - u_n \rangle, \ \ \forall z \in C.$$

Hence, we obtain

$$\Theta(z, u_{n_k}) \leq \varphi(z) - \varphi(u_{n_k}) + \langle \nabla f(u_{n_k}) - \nabla f(x_{n_k}), z - u_{n_k} \rangle, \ \ \forall z \in C.$$

Since $u_{n_k} \rightharpoonup w$ and Θ is lower semi-continuous in the second argument, φ is continuous. Then, using (3.15) and taking $k \to \infty$ in the above inequality,
we have

$$\Theta(z, w) + \varphi(w) - \varphi(z) \leq 0, \ \ \forall z \in C.$$

We define $z_t = tz + (1-t)w$ for $t \in [0,1]$. Since $z \in C$ and $w \in C$ we have $z_t \in C$ and hence

$$\Theta(z_t, w) + \varphi(w) - \varphi(z_t) \leq 0.$$

Now, we have

$$\begin{aligned} 0 &= \Theta(z_t, z_t) + \varphi(z_t) - \varphi(z_t) \\ &\leq t\Theta(z_t, z) + (1-t)\Theta(z_t, w) + t\varphi(z) + (1-t)\varphi(w) - \varphi(z_t) \\ &\leq t[\Theta(z_t, z) + \varphi(z) - \varphi(z_t)]. \end{aligned}$$

Since, $\Theta(z_t, z) + \varphi(z) - \varphi(z_t) \geq 0$. Then, we have

$$\Theta(w, z) + \varphi(z) - \varphi(w) \geq 0, \ \ \forall z \in C.$$

Therefore, $w \in Sol(MEP(\Theta, \varphi))$. Thus, we prove that $w \in \Omega$.

Step 8: We show that $x_n \to w = proj^f_{\Omega}(x_0)$.
Let $\bar{x} = proj^f_{\Omega}(x_0)$. Since $\{x_n\}$ is a weakly convergent sequence, $x_{n+1} = proj^f_{\Omega}(x_0)$ and $proj^f_{\Omega}(x_0) \in \Omega \subset C_{n+1}$. It follows from (3.5) that

$$D_f(x_{n+1}, x_0) \leq D_f\big(proj^f_{\Omega}(x_0), x_0\big). \tag{3.27}$$

Now, by Lemma 2.15, $\{x_n\}$ converges strongly to $\bar{x} = proj^f_{\Omega}(x_0)$. Therefore, by the uniqueness of the limit, we have that the sequence $\{x_n\}$ converges strongly to $w = proj^f_{\Omega}(x_0)$. This completes the proof. ■

The next two corollaries follow immediately from Theorem 3.1 that we have just proved.

Corollary 3.2. *Let E be a Banach space and C be a nonempty, closed and convex subset of a reflexive Banach space E with dual E^* such that $C \subset int(domf)$ and $F(T) = \widehat{F}(T)$. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping. Assume $\Omega = F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the iterative schemes:*

$$\begin{cases} x_0 \in C = C_0, \\ y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)], \\ C_{n+1} = \{z \in C_n : D_f(z, y_n) \leq D_f(z, x_n)\}, \\ x_{n+1} = proj^f_{C_{n+1}}(x_0), \ \ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.28}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ Then, $\{x_n\}$ converges strongly to $proj^f_{\Omega}(x_0)$.

Corollary 3.3. *Let E be a Banach space and C be a nonempty, closed and convex subset of a reflexive Banach space E with dual E^* such that $C \subset int(domf)$. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions $(A1) - (A4)$ and let φ be a real-valued function. Assume $\Omega = Sol(MEP(\Theta, \varphi)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the iterative schemes:*

$$\begin{cases} x_0 \in C = C_0, \\ u_n \in Res^f_{\Theta,\varphi}(x_n), \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \le D_f(z, x_n)\}, \\ x_{n+1} = proj^f_{C_{n+1}}(x_0), \;\; n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.29}$$

Then, $\{x_n\}$ converges strongly to $proj^f_\Omega(x_0)$.

4. Numerical Examples

In this section, we employ the methods obtained in this paper to solve a particular problem. Given a mixed equilibrium problem and a fixed point problem for a Bregman relatively nonexpansive mapping, we show that our algorithm generates a sequence that converges to a desired common solution. Finally, we compare our method with (1.12) by Kazmi et al. [17] on performance.

Example 4.1. We set the following parameters for the implementation of Theorem 3.1

Let $E = \mathbb{R}, C = [0, 10]$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{4}{5}x^2$. Let $\Theta : C \times C \to \mathbb{R}$ be defined by $\Theta(x, y) = x(y - x), \;\forall x, y \in C$ and $\varphi : C \to \mathbb{R}$ be defined by $\varphi(x) = x, \;\forall x \in C$. Let $T : C \to C$ be defined by $Tx = \frac{1}{5}x$, and let $\{\alpha_n\}_{n\in\mathbb{N}} = \left\{\frac{1}{n}\right\}$ Let be a sequences $\{x_n\}$ generated by the iterative schemes:

$$\begin{cases} x_0 \in C = C_0, \\ y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(Tx_n)], \\ u_n \in Res^f_{\Theta,\varphi}(y_n), \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \le D_f(z, x_n)\}, \\ x_{n+1} = proj^f_{C_{n+1}}(x_0), \;\; n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{4.1}$$

and

$$Res^f_{\Theta,\varphi}(y_n) = \{z \in C : \Theta(z, y) + \varphi(y) + \langle \nabla f(z) - \nabla f(y_n), y - z\rangle \ge \varphi(z), \;\; \forall y \in C\}. \tag{4.2}$$

It is easy to observe that $f : \mathbb{R} \to \mathbb{R}$ is a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $\mathbb{R}$ and $\nabla f(x) = \frac{8}{5}x$. Since $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{R}\}$, then $f^*(z) = \frac{5}{16}z^2$ and $\nabla f^*(z) = \frac{5}{8}z$.

Further, it is easy to observe that T is a Bregman relatively nonexpansive mapping with $F(T) = \{0\}$. We also observe that Θ satisfy conditions $(A1)-(A4)$, φ is a real-valued function, and Sol(MEP $(\Theta, \varphi)) = \{0\} \neq \emptyset$. Therefore, $\Omega = \text{Sol(MEP}(\Theta, \varphi)) \cap F(T) \neq \emptyset$.

After simplification, the iterative scheme (4.1) is reduced to the following:

$$\begin{aligned}
y_n &= \left(\frac{4}{n}+1\right)\frac{x_n}{5};\\
u_n &= \frac{8}{65}x_n\left(\frac{4}{n}+1\right)-\frac{5}{13};\\
C_{n+1} &= \left[0, \frac{x_n+u_n}{2}\right];\\
x_{n+1} &= proj^f_{C_{n+1}}(x_0), \quad n\in\mathbb{N}\cup\{0\}.
\end{aligned}$$

We set the initial value $x_0 = 5$. Using the software Matlab 2017b, we produce the following Figure 1 and Figure 2. By Theorem 3.1, $\{x_n\}$ converges to $0\in\Omega$.

Example 4.2. We set the following parameters for the implementation of the methods by Kazmi et al.[17].

Let $E=\mathbb{R}, C=[0,10]$ and let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\frac{4}{5}x^2$. Let $G:C\times C\to\mathbb{R}$ be defined by $G(x,y)=x(y-x)$, $\forall x,y\in C$ and $\phi:C\times C\to\mathbb{R}$ be defined by $\phi(x,y)=xy$, $\forall x,y\in C$. Let $T:C\to C$ be defined by $Tx=\frac{1}{5}x$, and $\{\alpha_n\}_{n\in\mathbb{N}}=\left\{\frac{1}{n}\right\}$. Let $\{x_n\}$ be a sequence generated by the iterative schemes:

$$\begin{cases}
x_0, z_0 = x\in C,\\
u_n = \nabla f^*\big(\alpha_n\nabla f(z_n)+(1-\alpha_n)\nabla f(Tx_n)\big),\\
z_{n+1} = res^f_{G,\phi}u_n,\\
C_n=\{z\in C: D_f(z,z_{n+1})\le \alpha_n D_f(z,z_n)+(1-\alpha_n)D_f(z,x_n)\},\\
Q_n=\{z\in C:\langle\nabla f(x_0)-\nabla f(x_n), z-x_n\rangle\le 0\},\\
x_{n+1}=proj^f_{C_n\cap Q_n}x_0, \quad \forall n\ge 0,
\end{cases} \tag{4.3}$$

where

$$\begin{aligned}
Res^f_{G,\phi}(u_n)=\{z\in C:&G(z,y)+\langle\nabla f(z)-\nabla f(u_n), y-z\rangle\\
&+\phi(z,y)-\phi(z,z)\ge 0, \ \forall y\in C\}.
\end{aligned} \tag{4.4}$$

All the conditions for f, T, Θ, φ, and Ω are met as explained in the proof of Example 2. After simplification, the iterative scheme (4.3) is reduced to the following:

$$\begin{aligned}
u_n &= \frac{z_n}{n}+\left(1-\frac{1}{n}\right)\frac{x_n}{5};\\
z_n &= \frac{4}{9}\left(\frac{z_n}{n}+\left(1-\frac{1}{n}\right)\frac{x_n^2}{25}\right);\\
C_n &= \begin{cases}\left[0, \ \frac{\frac{1}{2}\left[\frac{1}{n}\left(z_n^2-x_n^2\right)-z_{n+1}+x_n^2\right]}{\frac{1}{n}(z_n-x_n)-z_{n+1}+x_n}\right] & ; \frac{1}{n}(z_n-x_n)-z_{n+1}+x_n\neq 0,\\ \{0\} & ; otherwise;\end{cases}\\
Q_n &= [0, \ x_n];\\
x_{n+1} &= proj^f_{C_n\cap Q_n}(x_0), \quad n\in\mathbb{N}\cup\{0\}.
\end{aligned}$$

We set the initial values $x_0 = 5$ and $z_0 = 8$. Using the software Matlab 2017b, we produce the following Figure 1 and Figure 2. Then, by Kazmi et al. [17], $\{x_n\}$ converges to $0\in\Omega$.

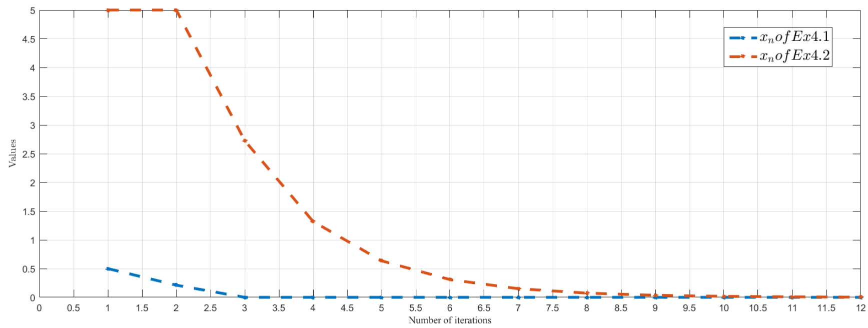

FIGURE 1. Plotting of values of x_n for $n = 1$ to 12

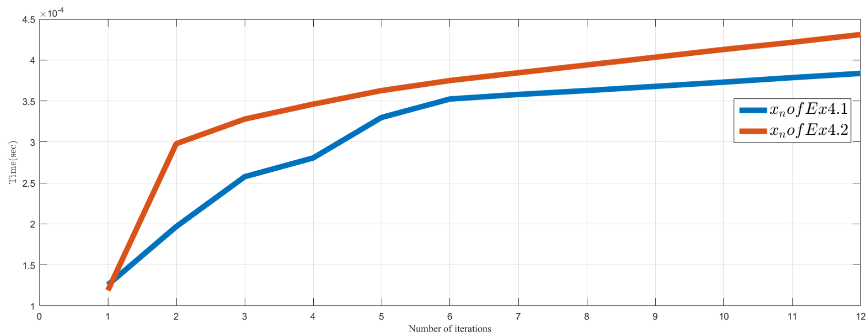

FIGURE 2. Plotting of run times to compute x_n for $n = 1$ to 12

Comparing the results of Example 2 and Example 3 from the figures, we see that the result from Example 2 seems to converge faster than the result from Example 3. This suggests that our new iteration scheme performs better than the iteration scheme given by Kazmi, yielding a result that converges faster but slightly longer run time.

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