Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 899–911

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Fixed Point Theorems for Multivalued Gerghaty Type Contractions via Generalized Simulation Functions

Azhar Hussain[1], **Ahsan Ali**[1], **Nuttapol Pakkaranang**[2] and **Nopparat Wairojjana**[3,*]

[1] *Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan*
e-mail : hafiziqbal30@yahoo.com (A. Hussain); ali_ahsan45@yahoo.com (A. Ali)

[2] *KMUTTFixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
e-mail : nuttapol.pakka@gmail.com (N. Pakkaranang)

[3] *Applied Mathematics Program, Faculty of Science and Technology, Valaya Alongkorn Rajabhat University under the Royal Patronage (VRU), 1 Moo 20 Phaholyothin Road, Klong Neung, Klong Luang, Pathumthani, 13180, Thailand*
e-mail : nopparat@vru.ac.th (N. Wairojjana)

Abstract The aim of this paper is to introduce the notion of a multivalued Gerghaty type contractive mapping via simulation functions along with $\mathcal{C}$-class functions and prove some fixed point results. As consequences, we derive some fixed point results endowed with graph. An example is given to show the validity of our results given herein.

MSC: 54H25; 47H10

Keywords: simulation functions; $\mathcal{C}$-class function; gerghaty contraction

Submission date: 15.12.2019 / Acceptance date: 02.05.2020

1. Introduction and Preliminaries

The theory of fixed point is an essential tool to tackle with many problems in mathematics. In this regards, Banach contraction principle [1] plays an important role in the development of metric fixed point theory. Many authors have investigated this principle due to its applicability (see e.g. [2–11]). In particular, Geraghty [12] obtained a generalization of the Banach contraction principle by considering an auxiliary function. Nadler [13] generalized Banach contraction principle to multivalued mappings which opens new doors in metric fixed point theory. Later on, this concept of Banach contraction is extended by using multivalued mappings and concept of the measure of non-compactness (see e.g. [14–21]).

*Corresponding author.

Published by The Mathematical Association of Thailand.

Khojasteh *et al.* [22] introduced the notion of $\mathcal{Z}$-contraction using a class of control functions called simulation functions and presented a generalized version of Banach contraction principle. Olgun *et al.* [23] obtained some fixed point results for generalized $\mathcal{Z}$-contractions. De-Hierro *et al.* [24] enlarged the class of simulation functions for a pair of mappings and obtained some coincidence point theorems. Later on Chandok *et al.* [25] combined the idea of simulation functions with $\mathcal{C}$-class functions and proved the existence and uniqueness of point of coincidence which generalized the results in [22, 23].

On the other hand, Samet *et al.* [4] introduced the idea of α-admissibility and generalized the Banach contraction principle. Karapinar [26] introduced the notion of α-admissible $\mathcal{Z}$-contraction and generalized the results of Samet *et al.* [4] and Khojasteh *et al.* [22]. Very recently, Patel [27] proved some fixed point theorems for multivalued contractions via generalized simulation functions in α-complete metric spaces.

In 2015, Khojasteh *et al.* [22] introduced a class Ξ of functions $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$, satisfying the following assertions:

(ζ_1): $\zeta(0,0) = 0$;
(ζ_2): $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ_3): If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\lim_{n\to\infty} sup\, \zeta(t_n, s_n) < 0$$

called the simulation functions and defined the notion of $\mathcal{Z}$-contraction with respect to the function ζ to generalized Banach contraction principle [1] and unify several known contractions involving the combination of $d(Tx,Ty)$ and $d(x,y)$. Following Theorem is due to Khojasteh *et al.* [22]:

Theorem 1.1 ([22]). *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \ for\ all\ x, y \in X. \tag{1.1}$$

where $\zeta \in \Xi$. Then T has a unique fixed point $u \in X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$ where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to this fixed point of T.

Example 1.2 ([22]). Let $\zeta_i : [0,\infty)\times[0,\infty) \to \mathbb{R}$, $i = 1,2,3$ be defined by

(i): $\zeta_1(t,s) = \lambda s - t$, where $\lambda \in (0,1)$;
(ii): $\zeta_2(t,s) = s\varphi(s) - t$, where $\varphi : [0,\infty) \to [0,1)$ is a mapping such that $\lim_{t\to r^+} sup\, \psi(t) < 1$ for all $r > 0$;
(iii): $\zeta_3 = s - \psi(s) - t$, where $\psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi(t) = 0$ if and only if $t = 0$.

Then ζ_i for $i = 1,2,3$ are simulation functions.

Karapinar [26] presented some fixed point results in the setting of a complete metric spaces by defining a new contractive condition via admissible mapping imbedded in simulation function. Hakan [28] *et al.* introduced the generalized simulation function on a quasi metric space and presented a fixed point theorem. Roldán-López-de-Hierro *et al.* [24] modified the notion of a simulation function by replacing (ζ_3) by (ζ_3'),

(ζ_3'): : if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then

$$\lim_{n\to\infty} sup \zeta(t_n, s_n) < 0.$$

The class of functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying $(\zeta_1 - \zeta 2)$ and (ζ_3') is called simulation function in the sense of Roldán-López-de-Hierro and is denoted by Ω.

Definition 1.3 ([2])**.** A mapping $\mathcal{G} : [0,+\infty)^2 \to \mathbb{R}$ is called a $\mathcal{C}$-class function if it is continuous and satisfies the following conditions:

(1): $\mathcal{G}(s,t) \leq s$;
(2): $\mathcal{G}(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,+\infty)$.

Definition 1.4 ([7])**.** A mapping $\mathcal{G} : [0,+\infty)^2 \to \mathbb{R}$ has the property $\mathcal{C}_{\mathcal{G}}$, if there exists $\mathcal{C}_{\mathcal{G}} \geq 0$ such that

($\mathcal{G}_1$): $\mathcal{G}(s,t) > \mathcal{C}_{\mathcal{G}}$ implies $s > t$;
($\mathcal{G}_2$): $\mathcal{G}(s,t) \leq \mathcal{C}_{\mathcal{G}}$, for all $t \in [0,+\infty)$.

Some examples of $\mathcal{C}$-class functions that have property $\mathcal{C}_{\mathcal{G}}$ are as follows:

(a): $\mathcal{G}(s,t) = s - t,\ \mathcal{C}_{\mathcal{G}} = r, r \in [0,+\infty)$;
(b): $\mathcal{G}(s,t) = s - \dfrac{(2+t)t}{(1+t)}, \mathcal{C}_{\mathcal{G}} = 0$;
(c): $\mathcal{G}(s,t) = \dfrac{s}{1+kt}, k \geq 1, \mathcal{C}_{\mathcal{G}} = \dfrac{r}{1+k}, r \geq 2$.

For more examples of $\mathcal{C}$-class functions that have property $\mathcal{C}_{\mathcal{G}}$ see [2, 7].

Definition 1.5 ([7])**.** A $\mathcal{C}_{\mathcal{G}}$ simulation function is a mapping $\mathcal{G} : [0,+\infty)^2 \to \mathcal{R}$ satisfying the following conditions:

(1): $\zeta(t,s) < \mathcal{G}(s,t)$ for all $t,s > 0$, where $\mathcal{G} : [0,+\infty)^2 \to \mathbb{R}$ is a $\mathcal{C}$-class function;
(2): if $\{t_n\}, \{s_n\}$ are sequences in $(0,+\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < \mathcal{C}_{\mathcal{G}}$.

Some examples of simulation functions and $\mathcal{C}_{\mathcal{G}}$-simulation functions are:

(1): $\zeta(t,s) = \dfrac{s}{s+1} - t$ for all $t,s > 0$.
(2): $\zeta(t,s) = s - \phi(s) - t$ for all $t,s > 0$, where $\phi : [0,+\infty) \to [0,+\infty)$ is a lower semi continuous function and $\phi(t) = 0$ if and only if $t = 0$.

For more examples of simulation functions and $\mathcal{C}_{\mathcal{G}}$-simulation functions see [2, 7, 9, 22, 24, 29].

We denote $\mathcal{F}$ by the class of all functions $\beta : [0,\infty) \to [0,1)$ satisfying $\beta(t_n) \to 1$, implies $t_n \to 0$ as $n \to \infty$.

Definition 1.6 ([12])**.** Let (X,d) be a metric space. A map $T : X \to X$ is called Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y).$$

By using such maps, Geraghty *et al.* [12] proved the following fixed point result:

Theorem 1.7. *Let (X,d) be a complete metric space. Mapping $T : X \to X$ is Geraghty contraction. Then T has a fixed point $x \in X$, and $\{T^n x_1\}$ converges to x.*

Definition 1.8 ([4])**.** Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-admissible if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.

Definition 1.9 ([6])**.** An α-admissible map T is said to be triangular α-admissible if $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ implies $\alpha(x,y) \geq 1$

Cho *et al.* [30] generalized the concept of Geraghty contraction to α-Geraghty contraction and prove the fixed point theorem for such contraction.

Definition 1.10 ([30])**.** Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called α-Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Theorem 1.11 ([30])**.** *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $T : X \to X$ satisfying the following conditions:*

(1) T is continuous α-Geraghty contraction;
(2) T be a triangular α-admissible;
(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

Then T has a fixed point $x \in X$, and $\{T^n x_1\}$ converges to x.

For a non-empty set X, let $\mathcal{P}(X)$ denotes the power set of X. If (X, d) is a metric space, then let

$$\begin{aligned}
\mathcal{N}(X) &= \mathcal{P}(X) - \{\emptyset\}, \\
\mathcal{CB}(X) &= \{A \in \mathcal{N}(X) : A \text{ is closed and bounded}\}, \\
\mathcal{K}(X) &= \{A \in \mathcal{N}(X) : A \text{ is compact}\}, \\
d(A, B) &= \inf\{d(a, b) : a \in A \text{ and } b \in B\}, \\
d(a, B) &= \inf\{d(a, b), a \in X \text{ and } b \in B\}, \\
\mathcal{H}(A, B) &= \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.
\end{aligned}$$

Mohammadi *et al.* [31] introduced the concept of α-admissibility for multivalued mappings as follows:

Definition 1.12 ([31])**.** Let X be a non empty set, $T : X \to \mathcal{N}(X)$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. Then T is said to be an α-admissible whenever for each $x \in X$ and $y \in Tx$

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(y, z) \geq 1, \text{ for all } z \in Ty.$$

Definition 1.13 ([32])**.** Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$. The metric space (X, d) is said to be α-complete if and only if every Cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Definition 1.14 ([33])**.** Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to \mathcal{K}(X)$ mappings. Then T is said to be an α-continuous multivalued mapping on $(\mathcal{K}(X), \mathcal{H})$, if for all sequences $\{x_n\}$ with $x_n \to x \in X$ as $n \to \infty$, and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $Tx_n \to Tx$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mathcal{H}(Tx_n, Tx) = 0.$$

Definition 1.15 ([27])**.** Let X be a nonempty set, $T : X \to \mathcal{N}(X)$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. Then T is said to be triangular α-admissible if T is α-admissible and

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1 \Rightarrow \ \alpha(x, z) \geq 1, \ \forall \ z \in Ty.$$

Lemma 1.16 ([27])**.** *Let $T : X \to \mathcal{N}(X)$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then for a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Lemma 1.17 ([34]). *Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.2}$$

If $\{x_n\}$ is not a Cauchy sequence in X, then there exists $\varepsilon > 0$ and two sequences $x_{m(k)}$ and $x_{n(k)}$ of positive integers such that $x_{n(k)} > x_{m(k)} > k$ and the following sequences tend to ε^+ when $k \to \infty$:

$$d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{n(k)+1}), d(x_{m(k)-1}, x_{n(k)}),$$
$$d(x_{m(k)-1}, x_{n(k)+1}), d(x_{m(k)+1}, x_{n(k)+1}).$$

The purpose of this paper is to prove some fixed point results for multivalued Gerghaty contraction via simulation functions with $\mathcal{C}$-class functions. Also, we give an example to show the validity of our results. Moreover, as consequences we present some related results endowed with graph.

2. Main Results

We begin with the following definition:

Definition 2.1. Let (X,d) be a metric space and $T : X \to \mathcal{K}(X)$ and $\alpha : X \times X \to [0,\infty)$ be a function. We say T is $\mathcal{Z}_{(\alpha,\mathcal{G})}$ Geraghty multivalued contraction with respect to ζ such that

$$\zeta(\alpha(x,y)\mathcal{H}(Tx,Ty), \beta(M(x,y))M(x,y)) \geq \mathcal{C}_{\mathcal{G}}, \tag{2.1}$$

for all $x, y \in X$ with $x \neq y$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Theorem 2.2. *Let (X,d) be a metric space and $T : X \to \mathcal{K}(X)$ be $\mathcal{Z}_{(\alpha,\mathcal{G})}$ Geraghty multivalued contraction satisfying:*

- **(1):** *(X,d) is an α-complete metric space;*
- **(2):** *there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;*
- **(3):** *T is triangular α-admissible;*
- **(4):** *T is an α-continuous multivalued mapping.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T and proof is complete. Assume that $x_1 \notin Tx_1$. Since T is $\mathcal{Z}_{(\alpha,\mathcal{G})}$ Geraghty multivalued contraction therefore taking $x = x_0$ and $y = x_1$ in (2.1), we get

$$\zeta(\alpha(x_0,x_1)\mathcal{H}(Tx_0,Tx_1), \beta(M(x_0,x_1))M(x_0,x_1)) \geq \mathcal{C}_{\mathcal{G}}.$$

Also we get that there exists $x_2 \in Tx_1$, $x_2 \neq x_1$ such that

$$\zeta(\alpha(x_1,x_2)\mathcal{H}(Tx_1,Tx_2), \beta(M(x_1,x_2))M(x_1,x_2)) \geq \mathcal{C}_{\mathcal{G}}$$

and α-admissibility of T gives $\alpha(x_1,x_2) \geq 1$. Repeating this process, we find that there exists a sequence $\{x_n\}$ with initial point x_0 such that $x_{n+1} \in Tx_n$, $x_n \neq x_{n+1}$ $\forall\, n \geq 0$, we derive

$$\alpha(x_n, x_{n+1}) \geq 1 \textit{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

By taking $x = x_n$ and $y = x_{n+1}$ in (2.1), we get that

$$\zeta(\alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1}), \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1})) \geq \mathcal{C}_{\mathcal{G}}.$$

Since T is $\mathcal{Z}_{(\alpha,\mathcal{G})}$ Geraghty multivalued contraction, we have

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leq \zeta(\alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1}), \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1})) \\ &< \mathcal{G}(\beta(M(x_n, x_{n+1}))M(x_n, x_{n+1}), \alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1})).\end{aligned}$$

Using $(\mathcal{G}_1)$, we get that

$$\alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1}) < \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1}). \tag{2.3}$$

Since T is compact, therefore

$$d(x_{n+1}, x_{n+2}) \leq \mathcal{H}(Tx_n, Tx_{n+1}). \tag{2.4}$$

Thus, from inequalities (2.3) and (2.4) we have

$$\begin{aligned}d(x_{n+1}, x_{n+2}) &\leq \alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1}) \\ &< \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1}) \\ &< M(x_n, x_{n+1}),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1})\}.\end{aligned}$$

If $M(x_n, x_{n+1}) = d(x_{n+1}, Tx_{n+1})$, inequality (2.5) gives

$$d(x_{n+1}, x_{n+2}) < d(x_{n+1}, Tx_{n+1}),$$

a contradiction. Hence $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$, and consequently from (2.5), we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}).$$

Hance for all $n \in \mathbb{N} \cup \{0\}$, we have $d(x_n, x_{n+1}) > d(x_{n+1}, x_{n+2})$. Therefore, $d(x_n, x_{n+1})$ is a decreasing sequence of non-negative real numbers, hence there exists $L \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} M(x_n, x_{n+1}) = L.$$

Assume that $L > 0$. So by inequality (2.5) we obtain,

$$\lim_{n\to\infty} \alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1}) = L \tag{2.6}$$

and

$$\lim_{n\to\infty} \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) = L. \tag{2.7}$$

Using (2.1) and (2) of Definition 1.5, we get

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leq \lim_{n\to\infty} \sup \zeta(\alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1}), \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1})) \\ &= \lim_{n\to\infty} \sup \zeta(\alpha(x_n, x_{n+1})\mathcal{H}(Tx_n, Tx_{n+1}), \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1})) \\ &< \mathcal{C}_{\mathcal{G}},\end{aligned}$$

which is a contradiction and hence $L = 0$. Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose on contrary that it is not, then by Lemma 1.17, we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon \tag{2.8}$$

and consequently,

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.9}$$

Let $x = x_{m(k)}, y = x_{n(k)}$. Since T is triangular α-admissible, so by Lemma 1.16, we have $\alpha(x_{m(k)}, x_{n(k)}) \geq 1$. Then by (2.1),

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\leq \zeta(\alpha(x_{m(k)}, x_{n(k)})\mathcal{H}(Tx_{m(k)}, Tx_{n(k)}), \beta(M(x_{m(k)}, x_{n(k)}))M(x_{m(k)}, x_{n(k)})) \\ &< \mathcal{G}(\beta(M(x_{m(k)}, x_{n(k)}))M(x_{m(k)}, x_{n(k)}), \alpha(x_{m(k)}, x_{n(k)})\mathcal{H}(Tx_{m(k)}, Tx_{n(k)})). \end{aligned}$$

Here $M(x_{m(k)}, x_{n(k)}) = d(x_{m(k)}, x_{n(k)})$, by $(\mathcal{G}_1)$, we get

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) &\leq \alpha(x_{m(k)}, x_{n(k)})\mathcal{H}(Tx_{m(k)}, Tx_{n(k)}) \\ &< \beta(M(x_{m(k)}, x_{n(k)}))M(x_{m(k)}, x_{n(k)}) \\ &< M(x_{m(k)}, x_{n(k)}) \\ &= d(x_{m(k)}, x_{n(k)}). \end{aligned} \tag{2.10}$$

Using (2.8) and (2.9) in (2.10), we get

$$\lim_{k\to\infty} \alpha(x_{m(k)}, x_{n(k)})\mathcal{H}(Tx_{m(k)}, Tx_{n(k)}) = \varepsilon,$$

and

$$\lim_{k\to\infty} \beta(M(x_{m(k)}, x_{n(k)}))M(x_{m(k)}, x_{n(k)}) = \varepsilon,$$

Therefore using (2.1) and (2) of Definition 1.5, we get

$$\mathcal{C}_{\mathcal{G}} \leq \zeta(\alpha(x_{m(k)}, x_{n(k)})\mathcal{H}(Tx_{m(k)}, Tx_{n(k)}), \beta(M(x_{m(k)}, x_{n(k)}))M(x_{m(k)}, x_{n(k)})) < \mathcal{C}_{\mathcal{G}},$$

which is a contradiction. Hence x_n is a Cauchy sequence. From (2.2) and the α-completeness of (X, d), there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. By α-continuity of a multivalued mapping T, we get

$$\lim_{n\to\infty} \alpha(x_n, u)\mathcal{H}(Tx_n, Tu) = 0. \tag{2.11}$$

Thus we obtain

$$d(u, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) \leq \lim_{n\to\infty} \alpha(x_n, u)\mathcal{H}(Tx_n, Tu) = 0.$$

Therefore, $u \in Tu$ and hence T has a fixed point. ■

Example 2.3. Let $X = [-10, 10]$ with the metric $d(x, y) = |x - y|$ and $T : X \to \mathcal{K}(X)$ be defined as:

$$Tx = \begin{cases} \{0\} & if\ x \in (-10, 0), \\ [0, \dfrac{1+x}{7}] & if\ x \in [0, 2], \\ [\dfrac{2x-5}{3}, \dfrac{2x-4}{3}] & if\ x \in (2, 5], \\ \{5\} & if\ x \in (5, 10), \end{cases}$$

and a function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if\ x \in [0, 2], \\ 0 & otherwise. \end{cases}$$

Then the space (X, d) is α-complete and T is not continuous but it is α-continuous. Also T is an triangular α-admissible multivalued mapping, since if $\alpha(x, y) \geq 1$, then we have $x, y \in [0,2]$, and so $Tx \in [0, \frac{1+x}{7}]$ and $Ty \in [0, \frac{1+y}{7}]$, which implies $\alpha(p,q) \geq 1$ for all $p \in Tx$ and $q \in Ty$. Thus, T is α-admissible. Further, if $\alpha(x,y) \geq 1$, then $x, y \in [0,2]$. So $x \in [0,2]$ and $Ty \in [0, \frac{1+y}{7}]$. Let $z \in Ty$. Then we have $\alpha(y,z) \geq 1$. Finally, $x \in [0,2]$ and $z \in [0, \frac{1+z}{7}]$ gives $\alpha(x,z) \geq 1$. Hence T is an triangular α-admissible. If we choose $x_0 = 2$ then condition (2) of Theorem 2.2 holds. Consider $\zeta \in \Omega$, $\mathcal{G}(s,t) = s - t$ and $\beta(t) = \frac{1}{1+t}$ for all $t \geq 0$, then it is clear that $\beta \in \mathcal{F}$. Then for $x, y \in [0,2]$, $x \neq y$, $\alpha(x,y) = 1$ and we will evaluate the values of $H(Tx,Ty)$ and $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}$. So

$$\zeta(\alpha(x,y)\mathcal{H}(Tx,Ty), \beta(M(x,y)M(x,y))) = \frac{8}{9}\left(\frac{M(x,y)}{1+M(x,y)}\right) - H(Tx,Ty) \quad (2.12)$$

and

$$\mathcal{G}(\beta(M(x,y)M(x,y)), \alpha(x,y)\mathcal{H}(Tx,Ty)) = \left(\frac{M(x,y)}{1+M(x,y)}\right) - H(Tx,Ty). \quad (2.13)$$

From (2.12) and (2.13), we obtain

$$\begin{aligned} 0 &< \zeta(\alpha(x,y)\mathcal{H}(Tx,Ty), \beta(M(x,y)M(x,y))) \\ &< \mathcal{G}(\beta(M(x,y)M(x,y)), \alpha(x,y)\mathcal{H}(Tx,Ty)). \end{aligned} \quad (2.14)$$

Hence, from (2.14) it is clear that T is an $\mathcal{Z}_{(\alpha,\mathcal{G})}$ Geraghty multivalued contraction with $\mathcal{C}_{\mathcal{G}} = 0$. Thus all the conditions of Theorem 2.2 are satisfied. Consequently T has fixed points in X.

The next results shows that the α-continuity of the mapping T can be relaxed by assuming the condition (4) as follows.

Theorem 2.4. *Let (X,d) be a metric space, and $T : X \to \mathcal{K}(X)$ be generalized $\mathcal{Z}_{(\alpha,\mathcal{G})}$ Geraghty multivalued contraction. Suppose the following conditions hold:*

(1): *(X,d) is an α-complete metric space;*
(2): *there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;*
(3): *T is triangular α-admissible;*
(4): *if x_n is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. Following the proof of Theorem 2.2, we know that x_n is a Cauchy sequence in X such that $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. From condition (4), we get $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$. By using (2.1), we have

$$\zeta(\alpha(x_n,x)\mathcal{H}(Tx_n,Tx), \beta(M(x_n,x))M(x_n,x)) \geq \mathcal{C}_{\mathcal{G}} \quad (2.15)$$

where

$$M(x_n,x) = \max\{d(x_n,x), d(x_n,Tx_n), d(x,Tx)\}$$

for all $n \in \mathbb{N}$. Suppose that $d(x, Tx) > 0$. Let $\varepsilon = \frac{d(x, Tx)}{2}$. Since $x_n \to x \in X$ as $n \to \infty$, we can find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{d(x, Tx)}{2} \tag{2.16}$$

for all $n \geq n_1$. Also, as x_n is a Cauchy sequence, then there exists $n_2 \in \mathbb{N}$ such that

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) < \frac{d(x, Tx)}{2} \tag{2.17}$$

for all $n \geq n_2$. Thus, we get

$$M(x_n, x) = d(x, Tx) \tag{2.18}$$

for all $n \geq n_0 = \max\{n_1, n_2\}$. Hence from (2.15), we have

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\leq \zeta(\alpha(x_n, x)\mathcal{H}(Tx_n, Tx), \beta(d(x, Tx))d(x, Tx)) \\ &< \mathcal{G}(\beta(d(x, Tx))d(x, Tx), \alpha(x_n, x)\mathcal{H}(Tx_n, Tx)). \end{aligned}$$

Using $\mathcal{G}_1$, we get

$$\alpha(x_n, x)\mathcal{H}(Tx_n, Tx) < \beta(d(x, Tx))d(x, Tx) < d(x, Tx).$$

Since we know that $d(x_{n+1}, Tx) \leq \mathcal{H}(Tx_n, Tx)$ and $\alpha(x_n, x) \geq 1$, therefore

$$d(x_{n+1}, Tx) \leq \alpha(x_n, x)\mathcal{H}(Tx_n, Tx).$$

So,

$$d(x_{n+1}, Tx) \leq \alpha(x_n, x)\mathcal{H}(Tx_n, Tx) < \beta(d(x, Tx))d(x, Tx) < d(x, Tx). \tag{2.19}$$

Letting $n \to \infty$, we get $d(x, Tx) < d(x, Tx)$, which is a contradiction. Therefore, $d(x, Tx) = 0$, that is, $x \in Tx$. This completes the proof. ∎

3. Consequences

In 2008, Jachymski [35] introduced the concept of $\mathcal{G}$-contraction on a metric space endowed with a graph and proved a fixed point theorem which extends the results of Ran and Reurings [36]. Later on, results of Jachymski [35] have been extended to multivalued mappings in [37, 38]. We present some consequences of our main result endowed with graph. For this purpose, following notions are essential. Let (X, d) be metric space and $\triangle = \{(x, x) : x \in X\}$. Consider a graph G a set $V(G)$ of its vertices equal to X and the set $E(G)$ of its edges, that is $(x, y), (y, x) \in E(G)$ implies that $x = y$. Also, G is directed if the edges have a direction associated with them. Now, we can identify the graph $\mathcal{G}$ with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 3.1 ([27])**.** Let X be a non empty set endowed with a graph G and $T : X \to \mathcal{N}(X)$ be a multivalued mapping. Then T is said to be triangular edge preserving if for each $x \in X$ and $y \in Tx$ with $(x, y), (y, z) \in E(G)$, we have $(x, z) \in E(G)$ for all $z \in Ty$.

Definition 3.2 ([27])**.** Let (X, d) be a metric space endowed with a graph G. The metric space X is said to be $E(G)$-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, converges in X.

Definition 3.3 ([27]). Let (X,d) be a metric space endowed with a graph G. We say that $T: X \to \mathcal{K}(X)$ is an $E(G)$-continuous mapping to $(\mathcal{K}(X), \mathcal{H})$, if for given $x \in X$ and sequence $\{x_n\}$ with

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } (x_n, x_{n+1}) \in E(G) \ \forall\, n \in \mathbb{N} \ \Rightarrow \ \lim_{n\to\infty} \mathcal{H}(Tx_n, Tx) = 0.$$

Definition 3.4. Let (X,d) be a metric space endowed with a graph G. A mapping $T: X \to \mathcal{K}(X)$ is said to be $E(G) - \mathcal{Z}_G$ Geraghty multivalued contraction, if there exist $\zeta \in \mathcal{Z}_G$ and $\alpha : X \times X \to [0, \infty\}$ such that

$$x, y \in X, (x,y) \in E(G) \Rightarrow \zeta(\alpha(x,y)\mathcal{H}(Tx,Ty), \beta(d(x,y))d(x,y)) \geq \mathcal{C}_G.$$

Similarly, by taking $M(x,y)$ instead of $d(x,y)$.

Theorem 3.5. *Let (X,d) be a metric space endowed with a graph G, and $T: X \to \mathcal{K}(X)$ be $\mathcal{Z}_G$ Geraghty multivalued contraction. Suppose the following conditions hold:*

(1): *(X,d) is an $E(G)$-complete metric space;*
(2): *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \in E(G)$;*
(3): *T is triangular edge preserving;*
(4): *T is an $E(G)$-continuous multivalued mapping.*

Then T has a fixed point.

Proof. This result can be obtained from Theorem 2.2 by defining a mapping $\alpha : X \times X \to [0, \infty\}$, such that

$$\alpha(x,y) = \begin{cases} 1 & if \ (x,y) \in E(G) \\ 0 & otherwise. \end{cases}$$

Hence all the conditions of Theorem 2.2 are satisfied, therefore proof is completed. Thus T has a fixed point in X. ∎

By Theorem 2.4, we get the following result.

Theorem 3.6. *Let (X,d) be a metric space endowed with a graph G, and $T: X \to \mathcal{K}(X)$ be $\mathcal{Z}_G$ Geraghty multivalued mapping. Suppose the following conditions hold:*

(1): *(X,d) is an $E(G)$-complete metric space;*
(2): *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \in E(G)$;*
(3): *T is triangular edge preserving;*
(4): *if x_n is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

4. Conclusions

In this paper, we have presented some fixed point theorems for a class of multivalued mappings via simulation functions with $\mathcal{C}$-class functions. As consequences of obtained results we established some fixed point results for multivalued mapping endowed with a graph. Our results extended many existing results in the literature.

Acknowledgements

I would like to thank the referee(s) for comments and suggestions on the manuscript. Nopparat Wairojjana was partially supported by Valaya Alongkorn Rajabhat University under the Royal Patronage, Thailand.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux equations intégrales, Fund. Math. 3 (1922) 133–181.

[2] A.H. Ansari, H. Isik, S. Radenović, Coupled fixed point theorems for contractive mappings involving new function classes and applications, Filomat. 31 (2017) 1893–1907.

[3] L. Ćirić, A generalization of Banachs contraction principle, Proc. Am. Math. Soc. 45 (1974) 267–273.

[4] M. Jleli, B. Samet, A new generalization of the Banach contraction principle, J. Ineq. Appl. 2014 (2014) 1–8.

[5] P. Kumam, D. Gopal, L. Budha, A new fixed point theorem under Suzki type $\mathcal{Z}$-contraction mappings, J. Math. Anal. 8 (2017) 113–119.

[6] E. Karapinar, P. Kumam, P. Salimi, On α-ψ-Meir-Keeler contractive mappings, Fixed Point Theo. and Appl. 94 (2013).

[7] X. Liu, A.H. Ansari, S. Chandok, S. Radenovic, On some results in metric spaces using auxiliary simulation functions via new functions, J. Comput. Anal. Appl. 24 (2018) 1103–1114.

[8] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal. Theo. and Meth. Appl. 71 (2009) 5313–5317.

[9] S. Wang, A.H. Ansari, S. Chandok, Some fixed point results for non-decreasing and mixed monotone mappings with auxiliary functions, Fixed Point Theo. Appl. 209 (2015).

[10] X. Li, A. Hussain, M. Adeel, E. Savas, Fixed point theorems for $\mathcal{Z}_\theta$-contraction and applications to nonlinear integral equations, IEEE Access. 7 (2019) 120023–120029.

[11] A. Al-Mazrooei, A. Hussain, M. Ishfaq, J. Ahmad, Simulative Suzuki-Gerghaty type contraction with $\mathcal{C}$-Class functions and applications, IEEE Access. 7 (2019) 75284–75291.

[12] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc. 40 (1973) 604–608.

[13] SB. Nadler, Jr, Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475–488.

[14] S. Suantai, W. Chloamjiak, P. Cholamjiak, An implicit Iteration process for a finite family of multi valued mappings in Banach spaces, Appl. Math. Lett. 25 (2012) 1656–1660.

[15] W. Cholamjiak, P. Cholamjiak, S. Suantai, Convergence of iterative schemes for solving fixed point of multi valued nonself mappings and equilibrium problems, J. Nonlinear Sci. Appl. 8 (2015) 1245–1256.

[16] S. Suantai, W. Cholamjiak, P. Cholamjiak, Weak and strong convergence to common fixed points of a countable family of multi valued mappings in Banach spaces, Thai J. Math. 9 (2011) 505–520.

[17] S. Suantai, W. Cholamjiak, P. Cholamjiak, Strong convergence theorems of a finite family of quasi- nonexpansive and Lipschitz multivalued mappings, Afrika Mat. 26 (2015) 345–355.

[18] A. Mukheimer, J. Vujaković, A. Hussain, H. Aydi, S. Radenović, S. Yaqoob, A new approach to multivalued nonlinear weakly Picard operators, J. Ineq. App. 288 (2019).

[19] H. ur Rehman, D. Gopal, P. Kumam, Generalizations of Darbo's fixed point theorem for new condensing operators with application to a functional integral equation, Demonstratio Mathematica. 52 (2019) 166–182.

[20] H. ur Rehman, P. Kumam, S. Dhompongsa, Existence of tripled fixed points and solution of functional integral equations through a measure of noncompactness, Carpathian J. Math. 35 (2019) 193–208.

[21] N. Wairojjana, H. ur Rehman, M. Sirajo, N. Pakkaranang, Fixed point theorems for Meir-Keeler condensing operators in partially ordered Banach spaces, Thai J. Math. 18 (2019) 77–93.

[22] F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theorems via simulation functions, Filomat. 26 (2015) 1189–1194.

[23] M. Olgun, O. Bicer, T. Alyildiz, A new aspect to Picard operators with simulation functions, Turk J. Math. 40 (2016) 832–837.

[24] A. Rold, E. Karapinar, C. Rold, J. Martinez, Coincidence point theorems on metric spaces via simulation function, J. Comput. Appl. Math. 275 (2015) 345–355.

[25] S. Chandok, A. Chanda, LK. Dey, M. Pavlović, S. Radenović, Simulation Functions and Geraghty Type Results, Bol. Soc. Paran. Mat. In press.

[26] E. Karapinar, Fixed points results via simulation functions, Filomat 30 (2016) 2343–2350.

[27] DK. Patel, Fixed points of multivalued contractions via generalized class of simulation functions, Bol. Soc. Paran. Mat. 38 (2020) 161–179.

[28] H. Simsek, M. Yalcin, Generalized $\mathcal{Z}$-contraction on quasi metric spaces and a fixed point result, J. Nonlinear Sci. Appl. 10 (2017) 3397–3403.

[29] A. Nastasi, P. Vetro, Fixed point results on metric and partial metric spaces via simulation functions, J. Nonlinear Sci. Appl. 8 (2015) 1059–1069.

[30] SH. Cho, JS. Bae, E. Karapinar, Fixed point theorems for α-Geraghty contraction typemaps inmetric spaces, Fixed Point Theo. and Appl. 329 (2013).

[31] B. Mohammadi, S. Rezapour, N. Shahzad, Some results on fixed points of α-ψ-Ciric generalized multifunctions, Fixed Point Theo. Appl. 24 (2013) https://doi.org/10.1186/1687-1812-2012-212.

[32] N. Hussain, MA. Kutbi, P. Salimi, Fixed point Theory in α-complete metric spaces with applications, Abstract and Applied Analysis 2014 (2014) Article ID. 280817.

[33] MA. Kutbi, W. Sintunavarat, On new fixed point results for (α, ψ, ζ)-contractive multivalued mappings on α-complete metric spaces and their consequences, Fixed Point Theo. and Appl. 2015 (2015) https://doi.org/10.1186/1687-1812-2015-2.

[34] S. Radenović, S. Chandok, Simulation type functions and coincidence points, Filomat 32 (2018) 141–147.

[35] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Am. Math. Soc. 136 (2008) 1359–1373.

[36] A. Ran, M. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.

[37] I. Beg, A. Butt, Fixed point of set-valued graph contractive mappings, J. Ineq. Appl. 252 (2013).

[38] T. Dinevari, M. Frigon, Fixed point results for multivalued contractions on a metric space with a graph, J. Math. Anal. Appl. 405 (2013) 507–517.