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Additive s-Functional Inequalities

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Abstract In this paper, we solve the following additive s-functional inequality

$$\|f((k+1)x-y)-f(kx-y)-f(x)\| \leq \|s(f(x+y)-f(x)-f(y))\| \tag{0.1}$$

and

$$\|f(x+y)-f(x)-f(y)\| \leq \|s(f((k+1)x-y)-f(kx-y)-f(x))\| \tag{0.2}$$

where k is an integer greater than 1 and s is acomplex number with $|s|<1$. Furthermore, we prove the Hyers-Ulam stability of the additive s-functional inequalities (0.1) and (0.2) in complex Banach spaces.

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1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [6] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

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then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

Fechner [7] and Gilányi [8] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park [9, 10] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [11–18]).

This paper is organized as follows: In Section 2, we solve the additive s-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive s-functional inequality (0.1) in complex Banach spaces. In Section 3, we solve the additive s-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive s-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, let G be a 2-divisible abelian group. Assume that X is a real or complex normed space with norm $\|\cdot\|$ and that Y is a complex Banach space with norm $\|\cdot\|$.

2. Additive s-Functional Inequality (0.1)

Throughout this section, assume that s is a fixed complex number with $|s| < 1$.

In this section, we solve and investigate the additive s-functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. *If $k \in \mathbb{N}$ and a mapping $f : G \to Y$ satisfies*

$$\|f((k+1)x - y) - f(kx - y) - f(x)\| \leq \|s(f(x+y) - f(x) - f(y))\| \tag{2.1}$$

for all $x, y \in G$, then $f : G \to Y$ is additive.

Proof. Assume that $f : G \to Y$ satisfies (2.1).

Letting $x = 0$ and $y = 0$ in (2.1), we get $\|f(0)\| \leq \|s(f(0))\|$ and so $f(0) = 0$, since $|s| < 1$.

Letting $x = p$ and $y = kp - q$ in (2.1), we get

$$\|f(p+q) - f(q) - f(p)\| \leq \|s\left(f((k+1)p - q) - f(p) - f(kp - q)\right)\| \tag{2.2}$$

for all $p, q \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) - f(y) - f(x)\| &\leq \|s(f((k+1)x - y) - f(kx - y) - f(x))\| \\ &\leq \|s^2(f(x+y) - f(y) - f(x))\| \end{aligned}$$

and so $f(x + y) = f(y) + f(x)$ for all $x, y \in G$. ∎

We prove the Hyers-Ulam stability of the additive s-functional inequality (2.1) in complex Banach spaces.

Theorem 2.2. *Let $r > 1$ and θ be nonnegative real numbers, k be an integer greater than 1 and $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f((k+1)x - y) - f(kx - y) - f(x)\| \\ &\leq \|s(f(x+y) - f(x) - f(y))\| + \theta\left(\|x\|^r + \|y\|^r\right) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{\theta \sum_{t=1}^{k-1}(t^r + 1)}{(1 - |s|)(k^r - k)} \|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.3), we get $\|f(0)\| \le \|sf(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (2.3), we get

$$\|f((k+1)x) - f(kx) - f(x)\| \le \theta \|x\|^r \tag{2.5}$$

for all $x \in X$.

For $t \in \mathbb{N}$, letting $y = tx$ in (2.3), we get

$$\begin{aligned} &\|f((k-t+1)x) - f((k-t)x) - f(x)\| \\ &\le \|s(f((t+1)x) - f(tx) - f(x))\| + \theta((t^r + 1)\|x\|^r) \end{aligned} \tag{2.6}$$

for all $x, y \in X$.

From (2.5) and (2.6), we get

$$\begin{aligned} &\sum_{t=1}^{k-1} \|f((k-t+1)x) - f((k-t)x) - f(x)\| \\ &\le \sum_{t=1}^{k-1} \|s(f((t+1)x) - f(tx) - f(x))\| + \theta(\sum_{t=1}^{k-1}(t^r + 1)\|x\|^r) \end{aligned} \tag{2.7}$$

for all $x \in X$. By (2.6) and (2.7) and the triangle inequality, we get

$$\begin{aligned} &(1 - |s|)\|f(kx) - kf(x)\| \\ &= (1 - |s|)\left\| \sum_{t=1}^{k-1}(f((t+1)x) - f(tx) - f(x)) \right\| \\ &\le \sum_{t=1}^{k-1}(1 - |s|)\|(f((t+1)x) - f(tx) - f(x))\| \\ &\le \sum_{t=1}^{k-1}\|(f((t+1)x) - f(tx) - f(x))\| - \sum_{t=1}^{k}\|s(f((t+1)x) - f(tx) - f(x))\| \\ &\le \theta(\sum_{t=1}^{k-1}(t^r + 1)\|x\|^r) \end{aligned}$$

for all $x \in X$, since

$$\sum_{t=1}^{k-1}\|f((k-t+1)x) - f((k-t)x) - f(x)\| = \sum_{t=1}^{k-1}\|f((t+1)x) - f(tx) - f(x)\|.$$

Since $|s| < 1$, the mapping f satisfies the inequality

$$\|f(kx) - kf(x)\| \le \frac{\theta(\sum_{t=1}^{k-1}(t^r + 1)\|x\|^r)}{1 - |s|}$$

for all $x \in X$. So

$$\left\| f(x) - kf\left(\frac{x}{k}\right) \right\| \leq \frac{\sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k^r}\theta\|x\|^r \tag{2.8}$$

for all $x \in X$. Thus

$$\begin{aligned} \left\| k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right) \right\| \\ &\leq \frac{\sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k^r} \sum_{j=l}^{m-1} \frac{k^j}{k^{rj}}\theta\|x\|^r \end{aligned} \tag{2.9}$$

for all nonnegative integers m, l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ converges. So it is possible to define the mapping $h : X \to Y$ as

$$h(x) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in X$. Also, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.9), we get (2.4). It follows from (2.3) that

$$\begin{aligned} &\|h((k+1)x+y) - h(kx-y) - h(x)\| \\ &= \lim_{n\to\infty} k^n \left\| f\left(\frac{(k+1)x+y}{k^n}\right) - f\left(\frac{kx+y}{k^n}\right) - f\left(\frac{x}{k^n}\right) \right\| \\ &\leq \lim_{n\to\infty} k^n \left\| s\left\{ f\left(\frac{x+y}{k^n}\right) - f\left(\frac{x}{k^n}\right) - f\left(\frac{y}{k^n}\right) \right\} \right\| \\ &+ \lim_{n\to\infty} \frac{k^n}{k^{nr}}\theta(\|x\|^r + \|y\|^r) = |s|\|h(x+y) - h(x) - h(y)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h((k+1)x+y) - h(kx-y) - h(x)\| \leq |s|\|h(x+y) - h(x) - h(y)\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is additive.

Now, let $u : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$\begin{aligned} \|h(x) - u(x)\| &= k^n \left\| h\left(\frac{x}{k^n}\right) - u\left(\frac{x}{k^n}\right) \right\| \\ &\leq k^n \left(\left\| h\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right) \right\| + \left\| u\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right) \right\| \right) \\ &\leq \frac{2k^n \cdot \theta \sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k^{nr}(k^r-k)}\|x\|^r \end{aligned}$$

which tends to 0 when $n \to \infty$ for all $x \in X$. So it means that $h(x) = u(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.4). ■

Theorem 2.3. *Let $r < 1$ and θ be nonnegative real numbers, k be an integer greater than 1 and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{\theta \sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)(k-k^r)}\|x\|^r \tag{2.10}$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\| \frac{f(kx)}{k} - f(x) \right\| \leq \frac{\sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k} \theta \|x\|^r$$

for all $x \in X$. So

$$\begin{aligned} \left\| \frac{1}{k^l} f(k^l x) - \frac{1}{k^m} f(k^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{k^j} f(k^j x) - \frac{1}{k^{j+1}} f(k^{j+1} x) \right\| \\ &\leq \frac{\sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k} \sum_{j=l}^{m-1} \frac{k^{jr}}{k^j} \theta \|x\|^r \end{aligned} \tag{2.11}$$

for all nonnegative integers m, l with $m > l$ and all $x \in X$. It follows from (2.11) that the sequence $\left\{\frac{1}{k^n} f(k^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{k^n} f(k^n x)\right\}$ converges. So it is possible to define the mapping $h : X \to Y$ as

$$h(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$$

for all $x \in X$. Also, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.11), we get (2.10).

The rest of the proof is similar to the proof of Theorem 2.2. ∎

Remark 2.4. If s is a real number such that $-1 < s < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive s-Functional Inequality (0.2)

Throughout this section, assume that s is a fixed complex number with $|s| < 1$.

In this section, we solve and investigate the additive s-functional inequality (0.2) in complex Banach spaces.

Lemma 3.1. *If $k \in \mathbb{N}$ and a mapping $f : G \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \|s(f((k+1)x - y) - f(kx - y) - f(x))\| \tag{3.1}$$

for all $x, y \in G$, then $f : G \to Y$ is additive.

Proof. Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = 0$ and $y = 0$ in (3.1), we get $\|f(0)\| \leq \|s(f(0))\|$ and so $f(0) = 0$, since $|s| < 1$.

Letting $x = p$ and $y = kp - q$ in (3.1), we get

$$\|f((k+1)p - q) - f(kp - q) - f(p)\| \leq \|s(f(p+q) - f(p) - f(q))\| \tag{3.2}$$

for all $p, q \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|f(p+q) - f(p) - f(q)\| &\leq \|s(f((k+1)p - q) - f(kp - q) - f(p))\| \\ &\leq \|s^2(f(p+q) - f(p) - f(q))\| \end{aligned}$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in G$. Thus $f : G \to Y$ is additive. ∎

Theorem 3.2. *Let $r > 1$ and θ be nonnegative real numbers, k be an integer greater than 1 and $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\leq \|s(f((k+1)x - y) - f(kx - y) - f(x))\| + \theta\left(\|x\|^r + \|y\|^r\right) \end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{\theta \sum_{t=1}^{k-1} (t^r + 1)}{(1-|s|)(k^r - k)} \|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$, in (3.3), we get $\|f(0)\| \leq \|sf(0)\|$. So $f(0) = 0$.

Letting $y = kx$ in (3.3), we get

$$\|f((k+1)x) - f(kx) - f(x)\| \leq \theta |k|^r \|x\|^r \tag{3.5}$$

for all $x \in X$.

For $t \in \mathbb{N}$, letting $y = tx$ in (3.3), we get

$$\begin{aligned} &\|f((t+1)x) - f(tx) - f(x)\| \\ &\leq \|s(f((k-t+1)x) - f((k-t)x) - f(x))\| + \theta((t^r + 1)\|x\|^r) \end{aligned} \tag{3.6}$$

for all $x, y \in X$.

From (3.5) and (3.6), we get

$$\begin{aligned} &\sum_{t=1}^{k-1} \|f((t+1)x) - f(tx) - f(x)\| \\ &\leq \sum_{t=1}^{k-1} \|s(f((k-t+1)x) - f((k-t)x) - f(x))\| + \theta(\sum_{t=1}^{k-1} (t^r + 1)\|x\|^r) \end{aligned} \tag{3.7}$$

for all $x \in X$. By (3.6) and (3.7) and the triangle inequality, (Summation Order) and the triangle inequality of norm $\|\cdot\|$, we get

$$\begin{aligned} &(1-|s|)\|f(kx) - kf(x)\| \\ &= (1-|s|)\left\| \sum_{t=1}^{k-1} (f((t+1)x) - f(tx) - f(x)) \right\| \\ &\leq \sum_{t=1}^{k-1} (1-|s|)\|(f((t+1)x) - f(tx) - f(x))\| \\ &\leq \sum_{t=1}^{k-1} \|(f((t+1)x) - f(tx) - f(x))\| - \sum_{t=1}^{k-1} \|s(f((t+1)x) - f(tx) - f(x))\| \\ &\leq \theta(\sum_{t=1}^{k-1} (t^r + 1)\|x\|^r) \end{aligned}$$

for all $x \in X$, since

$$\sum_{t=1}^{k-1} \|s(f((k-t+1)x) - f((k-t)x) - f(x))\| = \sum_{t=1}^{k-1} \|s(f((t+1)x) - f(tx) - f(x))\|.$$

Since $|s| < 1$, the mapping f satisfies the inequality

$$\|f(kx) - kf(x)\| \le \frac{\theta(\sum_{t=1}^{k-1}(t^r+1)\|x\|^r)}{1-|s|}$$

for all $x \in X$. So

$$\left\| f(x) - kf\left(\frac{x}{k}\right)\right\| \le \frac{\sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k^r}\theta\|x\|^r$$

for all $x \in X$. Thus

$$\begin{aligned} \left\| k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\| k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right)\right\| \\ &\le \frac{\sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k^r} \sum_{j=l}^{m-1} \frac{k^j}{k^{rj}}\theta\|x\|^r \end{aligned} \tag{3.8}$$

for all nonnegative integers m, l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{k^n f(\frac{x}{k^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{k^n f(\frac{x}{k^n})\}$ converges. So it is possible to define the mapping $h : X \to Y$ as

$$h(x) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in X$. Also, letting $l = 0$ and passing to the limit $m \to \infty$ in (3.8), we get (3.4). It follows from (3.3) that

$$\begin{aligned} &\|h(x+y) - h(x) - h(y)\| \\ &= \lim_{n\to\infty} k^n \left\| \left\{ f\left(\frac{x+y}{k^n}\right) - f\left(\frac{x}{k^n}\right) - f\left(\frac{y}{k^n}\right)\right\}\right\| + \lim_{n\to\infty}\frac{k^n}{k^{nr}}\theta(\|x\|^r + \|y\|^r) \\ &\le \lim_{n\to\infty} k^n|s| \left\| f\left(\frac{(k+1)x+y}{k^n}\right) - f\left(\frac{kx+y}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\| \\ &= |s|\|h((k+1)x+y) - h(kx-y) - h(x)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x+y) - h(x) - h(y)\| \le |s|\|h((k+1)x+y) - h(kx-y) - h(x)\|$$

for all $x, y \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is additive.

Now, let $u : X \to Y$ be another additive mapping satisfying (3.4). Then we have

$$\begin{aligned} \|h(x) - u(x)\| &= k^n \left\| h\left(\frac{x}{k^n}\right) - u\left(\frac{x}{k^n}\right)\right\| \\ &\le k^n \left(\left\| h\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\| + \left\| u\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\|\right) \\ &\le \frac{2k^n \cdot \theta\sum_{t=1}^{k-1}(t^r+1)}{(1-|s|)k^{nr}(k^r-k)}\|x\|^r \end{aligned}$$

which tends to 0 when $n \to \infty$ for all $x \in X$. So it means that $h(x) = u(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.4). ∎

Remark 3.3. If s is a real number such that $-1 < s < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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