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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Common Attractive Points Theorems of Widely More Generalized Hybrid Mappings in Hilbert Spaces

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Abstract We prove existence and weak convergence theorems for common attractive points of two widely more generalized hybrid mappings in Hilbert spaces. Consequently, we obtain a weak convergence theorem for common fixed points of two such mappings.

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1. Introduction

Throughout this article, let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and real numbers, respectively. The notion of attractive points for nonlinear mappings was introduced by Takahashi and Takeuchi [1] in Hilbert spaces. Let H be a real Hilbert space and C be a nonempty subset of H. For a mappings T from C into H, we denoted by $F(T)$ and $A(T)$ the set of all fixed points and attractive points, respectively, i.e.,

(1) $F(T) = \{x \in C : Tx = x\}$;
(2) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \forall x \in C\}$.

The authors also proved an existence theorem for attractive points without convexity in Hilbert spaces. Since then, attracive point theorems have been studied widely by many authors; see [2–5] for instance and references therein. A mapping $T : C \to H$ is called generalized hybrid [6] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|Ty - x\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

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for all $x, y \in C$. In 2013, Kawasaki and Takahashi [7] defined a class of widely more generalized hybrid mappings. A mapping $T : C \to H$ is called widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \epsilon\|x - Tx\|^2 \\ + \xi\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0, \quad \forall x, y \in C. \end{aligned} \tag{1.1}$$

We call such mapping an $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mapping. See [8, 9] for more results for the class of generalized hybrid mappings.

In the same time, Guu and Takahashi [10] proved the following existence theorem.

Theorem 1.1. *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mapping satisfying either the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \gamma > 0,\ \epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \beta > 0,\ \epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$.

Then $A(T) \neq \emptyset$ if and only if there exists $z \in C$ such that $\{T^n z, n = 0, 1, 2, \ldots\}$ is bounded.

A mapping $T : C \to H$ is called quasi-nonexpansive if $\|Tx - z\| \leq \|x - z\|$ for all $x \in C$ and $z \in F(T)$. In 2014, Kawasaki and Kobayashi [11] proved that if C is a nonempty closed and convex subset of a real Hilbert space H and $T : C \to H$ is an $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mapping such that $F(T) \neq \emptyset$ and suppose that one of the following holds:

(1) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \gamma > 0,\ \epsilon + \eta \geq 0$ and $\xi + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \beta > 0,\ \epsilon + \eta \geq 0$ and $\xi + \eta \geq 0$,

then T is a quasi-nonexpansive mapping.

In 2018, the concept of further generalized hybrid mappings was presented by Khan [12]. A mapping $T : C \to H$ is called further generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \epsilon\|x - Tx\|^2 \leq 0, \quad \forall x, y \in C.$$

Obviously, this is a spacial case of widely more generalized hybrid mappings when $\xi = \eta = 0$ in (1.1). Moreover, the author introduced the notion of common attractive point for two nonlinear mappings. For two mappings $S, T : C \to H$, denoted by $CAP(S, T)$ the set of common attractive points of S and T, i.e.,

$$CAP(S, T) = \{z \in H : \max(\|Sx - z\|, \|Tx - z\|) \leq \|x - z\|, \quad \forall x \in C\}.$$

The author also proved the following theorem.

Theorem 1.2 ([12])**.** *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $S, T : C \to C$ be two further generalized hybrid mappings which satisfy $\alpha+\beta+\gamma+\delta \geq 0, \epsilon \geq 0$ and either $\alpha + \beta > 0$ or $\alpha + \gamma > 0$. Then $CAP(S, T) \neq \emptyset$ if and only if there exists $z \in C$ such that both $\{S^n z : n = 0, 1, 2, \ldots\}$ and $\{T^n z : n = 0, 1, 2, \ldots\}$ are bounded.*

To study convergence theorems, the well-known iterative process was defined in 1974 by Ishikawa [13] to prove fixed point theorems for a continuous self mapping T, called

Ishikawa iterative process:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0,1)$. We can study more details about iteration algorithms in [14–16] and references therein.

Motivated by [10] and [12], in this paper, we use the Ishikawa iteration to prove existence and weak convergence theorems for common attractive points of two widely more generalized hybrid mappings without assuming the closedness of the domain. Moreover, our main result can be apply to a common fixed point theorem for such two mappings.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty subset of H. Then there exists a smallest convex set in H containing C, called a convex hull of C, denoted by $co(C) = \{\sum_{i=1}^{k} \lambda_i x_i : x_i \in C, \sum_i \lambda_i = 1, \lambda_i \geq 0\}$. Let $\{x_n\}$ be a sequence in H, we denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a Hilbert space H, it is known that

$$\|\alpha x + (1-\alpha)\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$. Furthermore, we have

$$2\langle x-y, z-w \rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.1}$$

for all $x, y, z, w \in H$. For every closed and convex subset C of H and $x \in H$, we know that there exists a unique nearest point $y \in C$ such that $\|x-y\| \leq \|x-z\|$ for all $z \in C$. We denote such a correspondence by $P_C x = y$. The mapping $P_C : H \to C$ is called metric projection of H onto C. It is known that

$$\langle x - P_C x, P_C x - z \rangle \geq 0$$

for all $x \in H$ and $z \in C$. For proving our main theorem, we need the following lemma due to Takahashi and Toyoda [17]

Lemma 2.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $P_C : H \to C$ be a metric projection. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in C$ and $n \in \mathbb{N}$, then $\{P_C x_n\}$ converges strongly to some $u_0 \in C$.*

The next result is useful for proving our main theorem.

Theorem 2.2 ([18]). *Let H be a Hilbert space and $\{x_n\}$ be a bounded sequence of H. Then $\{x_n\}$ is weakly convergent if and only if each weakly convergent subsequence of $\{x_n\}$ has the same weak limit, that is, for $x \in H$,*

$$x_n \rightharpoonup x \Leftrightarrow (x_{n_i} \rightharpoonup y \Rightarrow x = y).$$

We also know the following lemmas from Khan [12].

Lemma 2.3. *Let C be a nonempty closed and convex subset of H and let S, T be two mappings from C into itself. If $CAP(S,T) \neq \emptyset$, then $F(S) \cap F(T) \neq \emptyset$. In particular, if $z \in CAP(S,T)$, then $P_C z \in F(S) \cap F(T)$.*

Lemma 2.4. *Let C be a nonempty subset of H and let S,T be two mappings from C into H. Then $CAP(S,T)$ is a closed and convex subset of H.*

To prove our main result, we also need the following result of Guu and Takahashi [10].

Lemma 2.5. *Let H be a Hilbert space and let C be a nonempty subset of H. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mapping satisfying either of the conditions (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \gamma > 0,\ \epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \beta > 0,\ \epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$.

If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in A(T)$.

Furthermore, we also know the following lemma from Takahashi et al. [3] for quasi-nonexpansive mappings.

Lemma 2.6. *Let C be a nonempty subset of H and let T be a quasi-nonexpansive mapping from C into H. Then $A(T) \cap C = F(T)$.*

3. Main Results

We first prove an existence theorem of common attractive points for two widely more generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $S,T : C \to C$ be two $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mappings satisfying either of the conditions (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \gamma > 0,\ \epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \beta > 0,\ \epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$.

Then $CAP(S,T) \neq \emptyset$ if and only if there exists $z \in C$ such that both $\{S^n z, n = 0,1,2,\dots\}$ and $\{T^n z, n = 0,1,2,\dots\}$ are bounded.

Proof. Let $z \in CAP(S,T)$. Then $\max(\|Sx - z\|, \|Tx - z\|) \leq \|x - z\|$ for all $x \in C$. It follows that

$$\|S^{n+1}x - z\| \leq \|S^n x - z\| \quad \text{and} \quad \|T^{n+1}x - z\| \leq \|T^n x - z\|$$

for all $x \in C$ and $n = 0,1,2,\dots$. This implies that $\{S^n x, n = 0,1,2,\dots\}$ and $\{T^n x, n = 0,1,2,\dots\}$ are bounded.

In the other hand, suppose that there exists $z \in C$ such that both $\{S^n z, n = 0,1,2,\dots\}$ and $\{T^n z, n = 0,1,2,\dots\}$ are bounded. Suppose that

$$\max(\|Sx - z\|, \|Tx - z\|) = \|Tx - z\|.$$

As in the proof of Theorems 1.1 and T is a self-mapping, we have a unique point $p \in \bar{co}\{T^n z : n = 0,1,2,\dots\} \subseteq C \subseteq H$ such that $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$, that is, $p \in A(T)$. Using supposition on maximum, we get

$$\|Sx - p\| \leq \|x - p\|.$$

This means that $p \in A(S)$. Hence $p \in CAP(S,T)$. In the case of $\max(\|Sx - z\|, \|Tx - z\|) = \|Sx - z\|$, by interchanging the roles of S and T and use the same argument to conclude that $CAP(S,T) \neq \emptyset$. ■

As the direct consequnce of Theorem 3.1, we have the following results.

Theorem 3.2 ([10]). *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mapping satisfying either of the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$.

Then $A(T) \neq \emptyset$ if and only if there exists $z \in C$ such that $\{T^n z, n = 0, 1, 2, \ldots\}$ is bounded.

Proof. Let $S = T$ in Theorem 3.1, we obtain the desired result. ■

Since a class of $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mappings contains further generalized hybrid mappings, Theorem 3.1 extends the following theorem.

Theorem 3.3 ([12]). *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $S, T : C \to C$ be two further generalized hybrid mappings satisfying either the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$.

Then $CAP(S, T) \neq \emptyset$ if and only if there exists $z \in C$ such that both $\{S^n z, n = 0, 1, 2, \ldots\}$ and $\{T^n z, n = 0, 1, 2, \ldots\}$ are bounded.

Next, we prove a weak convergence theorem of a common attractive point for two widely more generalized hybrid mappings in a Hilbert space without assuming the closedness of the domain of such mappings.

Theorem 3.4. *Let H be a real Hilbert space, and let C be a nonempty and convex subset of H. Let $S, T : C \to C$ be two $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mappings satisfying either the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$

with $CAP(S, T) \neq \emptyset$ and $\{x_n\}$ be defined by

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - \beta_n) x_n + \beta_n S y_n \\ y_n = (1 - \alpha_n) x_n + \alpha_n T x_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0, 1)$. If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$, then $\{x_n\}$ converges weakly to $z \in CAP(S, T)$. Moreover, $z = \lim_{n\to\infty} P x_n$, where P is a projection of H onto $CAP(S, T)$.

Proof. Let $z \in CAP(S, T)$. Then by (3.1) and H is a Hilbert space, we have that

$$\begin{aligned} \|y_n - z\|^2 &= |(1 - \alpha_n)(x_n - z) + \alpha_n (T x_n - z)\|^2 \\ &= (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n \|T x_n - z\|^2 - \alpha_n (1 - \alpha_n)\|T x_n - x_n\|^2 \\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n \|x_n - z\|^2 - \alpha_n (1 - \alpha_n)\|T x_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n (1 - \alpha_n)\|T x_n - x_n\|^2 \qquad (3.2) \\ &= \|x_n - z\|^2 \end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|(1-\beta_n)(x_n-z)+\beta_n(Sy_n-z)\|^2\\ &= (1-\beta_n)\|x_n-z\|^2+\beta_n\|Sy_n-z\|^2-\beta_n(1-\beta_n)\|Sy_n-x_n\|^2\\ &\leq (1-\beta_n)\|x_n-z\|^2+\beta_n\|y_n-z\|^2-\beta_n(1-\beta_n)\|Sy_n-x_n\|^2 \quad (3.3)\\ &\leq \|x_n-z\|^2-\beta_n(1-\beta_n)\|Sy_n-x_n\|^2 \quad (3.4)\\ &= \|x_n-z\|^2. \quad (3.5)\end{aligned}$$

Thus, $\{\|x_n-z\|\}$ is nonincreasing which implies that $\lim_{n\to\infty}\|x_n-z\|$ exists. So $\{\|x_n-z\|\}$ is also bounded. That is, there exists $M>0$ such that

$$\|x_n-z\|\leq M \qquad \forall n\in\mathbb{N}.$$

This implies that

$$\|x_n\|\leq\|x_n-z\|+\|z\|\leq M+\|z\| \qquad \forall n\in\mathbb{N}$$

and

$$\|y_n\|\leq\|y_n-z\|+\|z\|\leq\|x_n-z\|+\|z\|\leq M+\|z\| \qquad \forall n\in\mathbb{N}.$$

This means that $\{x_n\}$ and $\{y_n\}$ are bounded. From (3.4), $\lim_{n\to\infty}\|x_n-z\|$ exists and $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, we obtain that

$$\lim_{n\to\infty}\|Sy_n-x_n\|=0. \quad (3.6)$$

From (3.3), we have that

$$\|x_{n+1}-z\|^2\leq(1-\beta_n)\|x_n-z\|^2+\beta_n\|y_n-z\|^2$$

and hence

$$\beta_n(\|x_n-z\|^2-\|y_n-z\|^2)\leq\|x_n-z\|^2-\|x_{n+1}-z\|^2.$$

Since $\beta_n(1-\beta_n)<\beta_n$ for all $\beta_n\in(0,1)$, we get that $\liminf_{n\to\infty}\beta_n>0$. This together with the existence of $\lim_{n\to\infty}\|x_n-z\|$, we have

$$\lim_{n\to\infty}(\|x_n-z\|^2-\|y_n-z\|^2)=0. \quad (3.7)$$

By $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, (3.2), and (3.7) we obtain that

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0. \quad (3.8)$$

Since $\|x_n-y_n\|\leq\alpha_n\|x_n-Tx_n\|$ and (3.8), $\lim_{n\to\infty}\|x_n-y_n\|=0$. Notice that

$$\|Sy_n-y_n\|\leq\|Sy_n-x_n\|+\|x_n-y_n\|$$

and let $n\to\infty$, then

$$\lim_{n\to\infty}\|Sy_n-y_n\|=0. \quad (3.9)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup z_0$. By Lemma 2.5 and (3.8), we have $z_0\in A(T)$. Using $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $x_{n_k}\rightharpoonup z_0$, by passing through a subsequence, if necessary, we can assume that there exists a weakly convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k}\rightharpoonup z_0$. By Lemma 2.5 and (3.9), $z_0\in A(S)$. Next, we prove that $x_n\rightharpoonup z_0$ by supposing a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup z_1$. It follows from what we have just proved that $z_1\in CAP(S,T)$, and from the initial step of this proof we can put $p = \lim_{n\to\infty}(\|x_n - z_0\|^2 - \|x_n - z_1\|^2)$. Furthermore, using (2.1) to get that

$$2\langle x_n, z_1 - z_0\rangle = \|x_n - z_0\|^2 + \|z_1\|^2 - \|x_n - z_1\|^2 - \|z_0\|^2.$$

This yields

$$\|x_n - z_0\|^2 - \|x_n - z_1\|^2 = 2\langle x_n, z_1 - z_0\rangle + \|z_0\|^2 - \|z_1\|^2.$$

It follows that

$$\|x_{n_j} - z_0\|^2 - \|x_{n_j} - z_1\|^2 = 2\langle x_{n_j}, z_1 - z_0\rangle + \|z_0\|^2 - \|z_1\|^2 \tag{3.10}$$

and

$$\|x_{n_k} - z_0\|^2 - \|x_{n_k} - z_1\|^2 = 2\langle x_{n_k}, z_1 - z_0\rangle + \|z_0\|^2 - \|z_1\|^2. \tag{3.11}$$

Taking $j \to \infty$ in (3.10) and $k \to \infty$ in (3.11), we get

$$p = 2\langle z_0, z_1 - z_0\rangle + \|z_0\|^2 - \|z_1\|^2,$$
$$p = 2\langle z_1, z_1 - z_0\rangle + \|z_0\|^2 - \|z_1\|^2.$$

This implies that $2\langle z_1 - z_0, z_1 - z_0\rangle = 0$ and hence $z_0 = z_1$. From Theorem 2.2, we can conclude that $x_n \rightharpoonup z_0$. Finally, to show that $z_0 = \lim_{n\to\infty} Px_n$, where P is a projection mapping from H onto $CAP(S,T)$. From (3.5), we have $\|x_{n+1} - z\| \le \|x_n - z\|$ for all $z \in CAP(S,T)$ and $n \in \mathbb{N}$. Using Lemma 2.4, we have $CAP(S,T)$ is closed and convex, applying Lemma 2.1, we have

$$\lim_{n\to\infty} Px_n = q$$

for some $q \in CAP(S,T)$. Since P is a projection, for each $z \in CAP(S,T)$ and $n \in \mathbb{N}$ we have

$$\langle x_n - Px_n, Px_n - z\rangle \ge 0.$$

Let $n \to \infty$, then for any $z \in CAP(S,T)$, we obtain that

$$\langle z_0 - q, q - z\rangle \ge 0.$$

Since $z_0 \in CAP(S,T)$,

$$\langle z_0 - q, q - z_0\rangle \ge 0$$

and hence

$$\langle z_0 - q, z_0 - q\rangle \le 0.$$

This means that

$$\|z_0 - q\|^2 \le 0.$$

Therefore, $z_0 = \lim_{n\to\infty} Px_n$ and the proof is complete. ∎

We obtain the following result when $\beta_n = 1$ and S is an identity mapping.

Corollary 3.5. *Let H be a real Hilbert space, and let C be a nonempty and convex subset of H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$-widely more generalized hybrid mapping satisfying either the conditions (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma > 0$, $\epsilon + \eta \ge 0$ *and* $\xi + \eta \ge 0$;
(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$, $\epsilon + \eta \ge 0$ *and* $\xi + \eta \ge 0$

with $A(T) \neq \emptyset$ *and* $\{x_n\}$ *be defined by* $x_1 \in C$ *and*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequence in* $(0, 1)$. *If* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, *then* $\{x_n\}$ *converges weakly to* $z \in A(T)$. *Moreover,* $z = \lim_{n\to\infty} Px_n$, *where* P *is a projection of* H *onto* $A(T)$.

If the domain C in Theorem 3.4 is closed, then we have the following result.

Theorem 3.6. *Let* H *be a real Hilbert space, and let* C *be a nonempty closed and convex subset of* H. *Let* $S, T : C \to C$ *be two* $(\alpha, \beta, \gamma, \delta, \epsilon, \xi, \eta)$*-widely more generalized hybrid mappings satisfying either of the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\epsilon + \eta \geq 0$ *and* $\xi + \eta \geq 0$

with $CAP(S, T) \neq \emptyset$ *and* $\{x_n\}$ *be defined by* (3.1). *If* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n(1 - \beta) > 0$, *then* $\{x_n\}$ *converges weakly to* $z \in F(S) \cap F(T)$.

Proof. By Theorem 3.4, we have $x_n \rightharpoonup z \in CAP(S, T)$. Since C is closed and convex, $z \in C$. By the quasi-nonexpansiveness of S and T and Lemma 2.6, we get that $A(T) \cap C = F(T)$ and $A(S) \cap C = F(S)$. It follows that

$$CAP(S, T) \cap C = [A(S) \cap A(T)] \cap C = F(S) \cap F(T).$$

Therefore, $z \in F(S) \cap F(T)$. ∎

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