Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 851–859

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Some Property of Common Fixed Point in Complex Valued b-Metric Spaces

Warinsinee Chantakun*, **Chonthicha Khun-inpho and Issara Inchan**

Department of Mathematics, Uttaradit Rajabhat University, Uttaradit, Thailand
e-mail : warinsinee@hotmail.com (W. Chantakun)

Abstract The purpose of this paper, is to introdue sufficient conditions for prove existence and uniqueness of common fixed point in complete complex valued b-metric spaces. Our main results extend and improve results of Dubey et. al [A K. Dubey, R. Shukla, R.P. Dubey, Some fixed point theorems in complex valued b-metric spaces, Hindawi Publishing Corporation Journal of Complex Systems 2015 (2015) 1–7].

MSC: 49K35; 47H10; 20M12

Keywords: common fixed point; complex valued metric space; complex valued b-metric space

Submission date: 09.03.2020 / Acceptance date: 20.04.2020

1. Introduction

The concept of a metric space was introduced by Frechet in 1906 [1]. Mathematicians studied the existence and uniqueness of fixed points by using Banach contraction principle. Banach contraction principle was proved in every new generalized metric spaces [2].

Fixed point theorems in metric spaces have been studied extensively by many researchers as in [3, 4] and [5]. In 1989, Bakhtin [6] introduced the notion of b-metric spaces. After, researchers extended fixed point theorems from metric space to b-metric spaces, for example in [7, 8].

In 2011, A. Azam, B. Fisher and M. Khan [8] introduced the notion of complex valued metric spaces and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. Complex valued metric space is a generalization of classical metric space. In 2011, S. Bhatt, S. Chaukiyal and R. C. Dimri [9] have proved a common fixed point theorems for weakly compatible mappings in a complex valued metric space. Recently, R.K. Verma and H.K. Pathak [10] introduced the concept of the property (E.A) in complex valued metric spaces for prove a common fixed point theorem for two pairs of weakly compatible mappings with property (E.A)

*Corresponding author.

Published by The Mathematical Association of Thailand.

and a common fixed point theorem using (CLRg) property which was introduced by Sintunavarat and Kumam [11].

In 2015, Dubey et. al [12] prove some fixed point results for the mapping satisfying rational expressions in complex valued b-metric spaces.

The aim of this paper is to introdue some contraction of two mappings and to prove some common fixed point theorems in complex valued b-metric spaces. Our results is generalized than the results of Dubey et. al [12].

2. Preliminaries

In this section, we present some definitions and lemmas for using in Section 3, and define the definition of b-metric space in the complex plane.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called metric if for $x, y, z \in X$ the following conditions are satisfied.

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called metric space, and d is called a metric on X.

Next, we suppose the definition of b-metric space which is generalized than metric spaces.

Definition 2.2 ([6])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called b-metric if for all $x, y, z \in X$ the following conditions are satisfied.

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

The following is some example of b-metric spaces.

Example 2.3 ([6])**.** Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegative property and triangle inequality of metric to prove the property (iii).

There is a completeness property in real number but on order relation is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex valued b-metric spaces, we define partial order in complex numbers (see [13]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order relation $\precsim$ on $\mathbb{C}$ as follows;

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \precsim z_2$ if and only if one of the following conditions holds:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
If one of the conditions $(ii), (iii)$, and (iv) holds, then we write $z_1 \prec z_2$. From the above partial order relation we have the following remark.

Remark 2.4. We can easily check the following:
(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \preccurlyeq z_2$ then $az_1 \preccurlyeq bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) If $0 \preccurlyeq z_1 \prec z_2$ then $|z_1| < |z_2|$.
(iii) If $z_1 \preccurlyeq z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$.
(iv) If $z \in \mathbb{C}$, for $a, b \in \mathbb{R}$ and $a \leq b$, then $az \preccurlyeq bz$.

A b-metric on a b-metric space is a function having real value. Based on the definition of partial order on complex number, real valued b-metric can be generalized into complex valued b-metric as follows.

Definition 2.5 ([8]). Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$ for all $x, y \in \mathbb{C}$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \preccurlyeq d(x, y) + d(y, z)$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Next, we give the definition of complex valued b-metric space.

Definition 2.6 ([5]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in X$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \preccurlyeq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space is defined in Definition 2.5, we can suppose some example of complex valued b-metric space.

Example 2.7 ([5]). Let $X = \mathbb{C}$. Define the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is complex valued b-metriic space with $s = 2$.

Definition 2.8 ([14]). Let (X, d) be a complex valued b-metric space.
(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in Y : d(x, y) \prec r\} \subseteq A.$$

(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

Definition 2.9 ([14]). Let (X, d) be complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
(i) The sequence $\{x_n\}$ is converges to $x \in X$ if for every $0 \prec r \in \mathbb{C}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus x is the limit of (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.

(iii) If for every Cauchy sequence in$X$ is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 2.10 ([14]). *Let (X, d) be a complex valued b-metric space ad let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.11 ([14]). . *Let (X, d) be a complex valued b-metric space ad let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

In this section, we give some contractive condition and prove the existence and uniqueness of common fixed point of a pair of mappings in a complete complex valued b-metric space.

Theorem 3.1. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be two self mappings satisfying the following condition:*

$$d(Sx, Ty) \preccurlyeq \lambda \frac{d^2(x, y)}{1 + d(x, y)} + \mu d(y, Ty) + \rho d(x, Sx), \tag{3.1}$$

for all $x, y \in X$, where λ, μ, ρ are nonnegative reals with $s(\lambda + \rho) + \mu < 1$. Then S and T has a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. We define

$$\begin{aligned} x_{2n+1} &= Sx_{2n} \text{ and} \\ x_{2n+2} &= Tx_{2n+1}, n = 0, 1, 2, 3, ... \end{aligned}$$

Now, we show that the sequence $\{x_n\}$ is a Cauchay sequence. Consider

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\preccurlyeq \lambda \frac{d^2(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \mu d(x_{2n+1}, Tx_{2n+1}) + \rho d(x_{2n}, Sx_{2n}) \\ &= \lambda \frac{d^2(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \mu d(x_{2n+1}, x_{2n+2}) + \rho d(x_{2n}, x_{2n+1}), \end{aligned}$$

which implies that

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leqslant \lambda \frac{|d(x_{2n}, x_{2n+1})|}{|1 + d(x_{2n}, x_{2n+1})|} d(x_{2n}, x_{2n+1}) \\ &\quad + \mu |d(x_{2n+1}, x_{2n+2})| + \rho |d(x_{2n}, x_{2n+1})|. \end{aligned}$$

We get,

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leqslant \lambda |d(x_{2n}, x_{2n+1})| + \mu |d(x_{2n+1}, x_{2n+2})| \\ &\quad + \rho |d(x_{2n}, x_{2n+1})|. \end{aligned}$$

This implies that,

$$\begin{aligned}
|d(x_{2n+1}, x_{2n+2})| - \mu|d(x_{2n+1}, x_{2n+2})| &\leqslant \lambda|d(x_{2n}, x_{2n+1})| + \rho|d(x_{2n}, x_{2n+1})| \\
(1-\mu)|d(x_{2n+1}, x_{2n+2})| &\leqslant (\lambda+\rho)|d(x_{2n}, x_{2n+1})| \\
|d(x_{2n+1}, x_{2n+2})| &\leqslant \frac{(\lambda+\rho)}{(1-\mu)}|d(x_{2n}, x_{2n+1})|.
\end{aligned}$$

Since $s(\lambda+\rho)+\mu < 1$ and $s \geq 1$, we have $\lambda+\mu < 1$. It follows that $\delta = \dfrac{(\lambda+\rho)}{(1-\mu)} < 1$ for all $n \geq 0$. Consequenly, we have

$$\begin{aligned}
|d(x_{2n+1}, x_{2n+2})| &\leqslant \delta|d(x_{2n}, x_{2n+1})| \\
&\leqslant \delta^2|d(x_{2n-1}, x_{2n})| \\
&\vdots \\
&\leqslant \delta^{2n+1}|d(x_0, x_1)| \\
|d(x_{n+1}, x_{n+2})| &\leqslant \delta|d(x_n, x_{n+1})| \\
&\leqslant \delta^2|d(x_{n-1}, x_n)| \\
&\vdots \\
&\leqslant \delta^{n+1}|d(x_0, x_1)|. \qquad (3.2)
\end{aligned}$$

For any $m, n \in \mathbb{N}$ with $m > n$ and since $s\delta = s(\lambda+\rho) < 1$, we get

$$\begin{aligned}
|d(x_n, x_{n+m})| &\leqslant s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\
&\leqslant s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_{n+m})| \\
&\leqslant s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| \\
&\qquad + s^3|d(x_{n+3}, x_{n+m})| \\
&\vdots \\
|d(x_n, x_{n+m})| &\leqslant s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| \\
&\quad + s^3|d(x_{n+3}, x_{n+m})| + \cdots + s^{n+m-1}|d(x_{n+m-2}, x_{n+m-1})| \\
&\quad + s^m|d(x_{n+m-1}, x_{n+m})|. \qquad (3.3)
\end{aligned}$$

By using (3.2) and (3.3), we get

$$\begin{aligned}
|d(x_n, x_{n+m})| &\leqslant s\delta^n|d(x_0, x_1)| + s^2\delta^{n+1}|d(x_0, x_1)| + s^3\delta^{n+2}|d(x_0, x_1)| \\
&\quad + s^3|d(x_{n+3}, x_m)| + \cdots + s^{m-n-1}\delta^{m-2}|d(x_0, x_1)| \\
&\quad + s^m\delta^{m-1}|d(x_0, x_1)| \\
&= (s\delta^n + s^2\delta^{n+1} + s^2\delta^{n+2} + \cdots + s^{n+m-1}\delta^{n+m-2} \\
&\quad + s^m\delta^{n+m-1})|d(x_0, x_1) \\
&\leqslant [(s\delta)^n + (s\delta)^{n+1} + (s\delta)^{n+2} + \cdots + (s\delta)^{n+m-2} + (s\delta)^{n+m-1} \\
&\quad + \delta^{n+m-1}]|d(x_0, x_1)| \\
&= \Sigma_{i=n}^{n+m-1}(s\delta)^i|d(x_0, x_1) \\
&\leqslant \Sigma_{i=n}^{\infty}(s\delta)^i|d(x_0, x_1)|.
\end{aligned}$$

Since $s(\lambda+\rho) < 1$ for $s \geqslant 1$ then $\delta < 1$ and $s\delta < 1$, it follows that

$$|d(x_n, x_{n+m})| \leqslant \frac{(s\delta)^n}{1-s\delta}|d(x_0, x_1)|. \qquad (3.4)$$

Taking $n \to \infty$ in (3.4), we have $(s\delta)^n \to 0$. This implies $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$. By Lemma 2.11, the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete complex valued b-metric space then $\{x_n\}$ is a convergent sequence. Let u be the limit of $\{x_n\}$. We show that u is a common fixed point of S and T. Consider,

$$\begin{aligned} d(Su, x_{2n+2}) &= d(Su, Tx_{2n+1}) \\ &\precsim \lambda \frac{d^2(u, Tx_{2n+1})}{1 + d(u, x_{2n+1})} + \mu d(x_{2n+1}, Tx_{2n+1}) + \rho d(u, Su) \\ &\precsim \lambda d(u, Tx_{2n+1}) + \mu d(x_{2n+1}, Tx_{2n+1}) + \rho d(u, Su). \end{aligned} \tag{3.5}$$

Since $\{x_n\}$ converges to u, (3.5), Remark 2.4 (ii) and Lemma 2.10, we have

$$\begin{aligned} |d(Su, u)| &= \lim_{n\to\infty} |d(Su, x_{2n+2})| \\ &\leqslant \lambda|d(u, u)| + \mu|d(u, u)| + \rho|d(u, Su)| = \rho|d(u, Su)|. \end{aligned} \tag{3.6}$$

From (3.6), we have $(1 - \rho)|d(Su, u)| \leqslant 0$. Since $0 < 1 - \rho < 1$, it follows that, $u = Su$. Hence u is a fixed point of S. Next, we show that u is a fixed point of T. Consider,

$$\begin{aligned} d(x_{2n+1}, Tu) &= d(Sx_{2n}, Tu) \\ &\precsim \lambda \frac{d^2(x_{2n}, u)}{1 + d(x_{2n}, u)} + \mu d(u, Tu) + \rho d(x_{2n}, Sx_{2n}) \\ &\precsim \lambda d(x_{2n}, u) + \mu d(u, Tu) + \rho d(x_{2n}, Sx_{2n}). \end{aligned} \tag{3.7}$$

Since $\{x_n\}$ converges to u, (3.7), Remark 2.4(ii) and Lemma 2.10, we have

$$\begin{aligned} |d(u, Tu)| &= \lim_{n\to\infty} |d(x_{2n+1}, Tu)| \\ &\leqslant \lambda|d(u, u)| + \mu|d(u, Tu)| + \rho|d(u, u)| = \mu|d(u, Tu)|. \end{aligned} \tag{3.8}$$

From (3.8), we have $(1 - \mu)|d(u, Tu)| \leqslant 0$. Since $0 < 1 - \mu < 1$, it follows that, $u = Tu$. Hence u is a fixed point of T. Hence, u is a common fixed point of S and T. Finally, we prove the uniqueness of common fixed point of S and T. Suppose v is a common fixed point of S and T. So, $Sv = v = Tv$. Now, we show that $u = v$, that consider

$$\begin{aligned} d(u, v) &= d(Su, Tv) \\ &\precsim \lambda \frac{d^2(u, v)}{1 + d(u, v)} + \mu d(v, Tv) + \rho d(u, Su) \\ &\precsim \lambda d(u, v) + \mu d(v, Tv) + \rho d(u, Su) \\ &= \lambda d(u, v) + \mu d(v, v) + \rho d(u, u) \\ &\precsim \lambda d(u, v). \end{aligned}$$

From Remark 2.4 (ii), taking absolute value of both side, we have

$$(1 - \lambda)|d(u, v)| \leqslant 0.$$

Since $0 < 1 - \lambda < 1$, it follows that $|d(u, v)| = 0$. By Definition 2.6 (ii), we have $u = v$. Therefore, S and T has a unique common fixed point in X. ∎

From Theorem 3.1, we can reduced to the result of Dubey et. al [12] as follows.

Corollary 3.2. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mappings satisfying the following condition*

$$d(Tx, Ty) \precsim \lambda \frac{d^2(x, y)}{1 + d(x, y)} + \mu d(y, Ty), \tag{3.9}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then T has a unique fixed point in X.

Proof. Let $\rho = 0$ and $S = T$ in the equation (3.1). From Theorem 3.1, T has a unique fixed point. ∎

From Theorem 3.1, we can applied to the theorem as follows.

Theorem 3.3. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be two self mappings satisfying the following condition:*

$$d(S^n x, T^n y) \precsim \lambda \frac{d^2(x, y)}{1 + d(x, y)} + \mu d(y, T^n y) + \rho d(x, S^n x), \tag{3.10}$$

for all $x, y \in X, n > 1$ and λ, μ, ρ are nonnegative reals with $s(\lambda + \rho) + \mu < 1$. Then S and T has a unique common fixed point in X.

Proof. Suppose $A = S^n$ and $B = T^n$, by Theorem 3.1, there exists a common fixed point u of A and B, such that

$$Au = u = Bu.$$

Thus $S^n u = u$ and $T^n u = u$. We claim that $Su = u$. Assume that $Su \neq u$, we have

$$\begin{aligned} d(Su, u) &= d(S(S^n u), T^n u) \\ &= d(S^n(Su), T^n u) \\ &\precsim \lambda \frac{d^2(Su, u)}{1 + d(Su, u)} + \mu d(u, T^n u) + \rho d(Su, S^n(Su)) \\ &= \lambda \frac{d^2(Su, u)}{1 + d(Su, u)} + \mu d(u, T^n u) + \rho d(Su, S(S^n u)) \\ &= \lambda \frac{d^2(Su, u)}{1 + d(Su, u)} + \mu d(u, T^n u) + \rho d(Su, Su) \\ &= \lambda \frac{d^2(Su, u)}{1 + d(Su, u)} \\ \therefore d(Su, u) &\precsim (\lambda - 1) d^2(Su, u). \end{aligned}$$

A contradiction, because $\lambda < 1$. Hence, $Su = u$. Next, we claim that $Tu = u$. Assume that $Tu \neq u$, we have

$$\begin{aligned} d(u, Tu) &= d(S^n u, T(T^n u)) \\ &= d(S^n u, T^n(Tu)) \\ &\precsim \lambda \frac{d^2(u, Tu)}{1 + d(u, Tu)} + \mu d(Tu, T^n(Tu)) + \rho d(u, S^n u) \\ &= \lambda \frac{d^2(u, Tu)}{1 + d(u, Tu)} + \mu d(Tu, T(T^n u)) + \rho d(u, S^n u) \\ &= \lambda \frac{d^2(u, Tu)}{1 + d(u, Tu)} + \mu d(Tu, Tu) + \rho d(u, S^n u) \\ &= \lambda \frac{d^2(u, Tu)}{1 + d(u, Tu)} \\ \therefore d(u, Tu) &\precsim (\lambda - 1) d^2(u, Tu). \end{aligned}$$

A contradiction, because $\lambda < 1$. Hence, $Tu = u$. Hence u is a common fixed point of S and T.

Finally, we show that u is a unique fixed point of S and T. Let v be a common fixed point of S and T, thus $S^n v = v = T^n v$. We must show that $u = v$. Assume that $u \neq v$, we have

$$\begin{aligned} d(u,v) &= d(S^n u, T^n v) \\ &\precsim \lambda \frac{d^2(u,v)}{1+d(u,v)} + \mu d(v, T^n v) + \rho d(u, S^n u) \\ &= \lambda \frac{d^2(u,v)}{1+d(u,v)} \\ \therefore d(u,v) &\precsim (\lambda - 1) d^2(u,v). \end{aligned}$$

A contradiction, because $\lambda < 1$. Hence, $u = v$. Therefore, u is a unique common fixed point of S and T. ■

Example 3.4. Let $X = \mathbb{C}$. Define a function $d : X \times X \to \mathbb{C}$ such that

$$d(z_1, z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2,$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$.

From Example 18 in [13], it implied that (X, d) is a complete complex valued b-metric space with $s = 2$. Now, we define two self-mappings $S, T : X \to X$ as follows:

$$Sz = \begin{cases} 0 & \text{if } a, b \in \mathbb{Q}, \\ 2 & \text{if } a \in \mathbb{Q}^C, b \in \mathbb{Q} \\ 2i & \text{if } a, b \in \mathbb{Q}^C \\ 2+2i & \text{if } a \in \mathbb{Q}, b \in \mathbb{Q}^C \end{cases} \quad \text{and } Tz = \begin{cases} 0 & \text{if } a, b \in \mathbb{Q}, \\ 1 & \text{if } a \in \mathbb{Q}^C, b \in \mathbb{Q} \\ i & \text{if } a, b \in \mathbb{Q}^C \\ 1+i & \text{if } a \in \mathbb{Q}, b \in \mathbb{Q}^C \end{cases}$$

where $z = a + bi \in X$. We see that $S^n z = 0 = T^n z$ for $n > 1$, so

$$d(S^n x, T^n y) = 0 \precsim \lambda \frac{d^2(x,y)}{1+d(x,y)} + \mu d(y, T^n y) + \rho d(x, S^n x),$$

for all $x, y \in X$ and $\lambda, \mu, \rho \geq 0$ with $2(\lambda + \rho) + \mu < 1$. So all conditions of Theorem 3.3 are satisfied to get a unique common fixed point 0 of S and T.

Acknowledgements

The author would like to thank Department of Mathematics, faculty of Science and Technology. Moreover, we would like to thank Uttaradit Rajabhat University for financial support.

References

[1] M. Frechet, Sur quelques points du calcul functional, Rendicontide Circolo Mathematico diPalermo. 22 (1) (1906) 1–72.

[2] O. Ege, I. Karaca, Banach fixed point theorem for digital images, J. Nonlinear Sci. Appl. 8 (2015) 237–245.

[3] T. Zamfirescu, Fix point theorems in metric spaces, Archiv der Mathematik 2 (1) (1972) 292–298.

[4] G. Emmanuele, Fixed point theorems in complete metric spaces, Nonlinear Analysis: Theory, Methods and Aplications 5 (3) (1981) 287–292.

[5] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Analysis: Theory, Methods and Applications 71 (11) (2009) 5313–5317.

[6] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Functional Analysis 30 (1989) 26–37.

[7] S. Ali, A common fixed point result in complex valued b-metric spaces under contractive condition, Global Journal of Pure and Applied Mathematics 13 (9) (2017) 4869–4876.

[8] A. Azam, F. Brain, M. Khan, Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim. 32 (3) (2011) 243–253.

[9] S. Bhatt, S. Chaukiyal, R.C. Dimri, A common fixed point theorem for weakly compatible maps in complex valued metric space, Int. J. Math. Sci. Appl. 1 (3) (2011) 1385–1389.

[10] R.K. Verma, H.K. Pathak, Common fixed point theorems using property (E.A) in complex valued metric spaces, Thai Journal of Mathematics 11 (2) (2013) 275–283.

[11] W. Sintunavarat, P. Kumam, Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. 2011 (2011) 637–958.

[12] A.K. Dubey, R. Shukla, R.P. Dubey, Some fixed point theorems in complex valued b-metric spaces, Hindawi Publishing Corporation Journal of Complex Systems 2015 (2015) 1–7.

[13] A.A. Mukheimer, Some common fixed point theorems in complex valued b-metric spaces, The Scientific World Journal 2014 (1014) 1–6.

[14] K. Rao, P. Swamy, J. Prasad, A common fixed point theorem in complex valued b-metric spaces, Bulletin of Mathmatics and Statistics Research 1 (1) 2013.