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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Hybrid Proximal Point Algorithm for Solution of Convex Multiobjective Optimization Problem over Fixed Point Constraint

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Abstract The purpose of this paper is to consider the convex constraint multiobjective optimization problem, as the constrained set is a fixed point set of nonexpansive mapping. By owing the concepts of proximal method and Mann algorithm, we introduce the algorithm and aim to establish the convergence results of the such proposed iterative algorithm to compute a solution point the considered constraint convex multiobjective optimization problem.

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1. Introduction

For a natural number m, we will use the following notations: $I_m := \{1, ..., m\}$, $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_j \geq 0, j \in I\}$, and $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m : x_j > 0, j \in I\}$. For $y, z \in \mathbb{R}^m$, $z \succeq y$(or $y \preceq z$) means that $z - y \in \mathbb{R}^m_+$, and $z \succ y$(or $y \prec z$) means that $z - y \in \mathbb{R}^m_{++}$. for the operator $H : \mathbb{R}^n \to \mathbb{R}^m$, by using above relations, we consider the (unconstraint) multiobjective minimization problem as

$$\min_{y \in \mathbb{R}^n} H(y), \tag{1.1}$$

where $H(y) := (h_1(y), ..., h_m(y))$, and $h_i : \mathbb{R}^n \to \mathbb{R}$ are the components functions of H.

Observe that the multiobjective optimization is the process of simultaneously optimizing two or more real-valued objective functions. It is really rare to find an optimal

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solution that satisfies all objectives from the mathematical point of view (i.e., there is no ideal minimizer). So we consider the following concepts of solution: a point $x^* \in \mathbb{R}^n$ is called Pareto optimal point of H, if there exists no $y \in \mathbb{R}^n$ such that $H(y) \preceq H(x^*)$ with $H(y) \neq H(x^*)$, whereas $x^* \in \mathbb{R}^n$ is said to be a weak Pareto optimal point if there exists no $y \in \mathbb{R}^n$ such that $H(y) \prec H(x^*)$. It is clear that a Pareto optimal point is also a weak Pareto optimal point but the converse is not true, see [1]. These types of solution concepts have applications in the economy, industry, agriculture, and other fields, see [2]. In order to find the optimal point, multiobjective optimization problems are usually solved by scalarization. In general, scalarization means the replacement of a multiobjective optimization problem by a suitable single objective optimization problem which is an optimization problem with a real-valued objective function. It is a fundamental principle in multiobjective optimization that optimal solutions of these problems can be characterized as solutions of certain single objective optimization problems. Moreover, there are many algorithms for solving optimization problems, see [3–5] for more details.

Mean while the proximal point algorithm is a widely used tool for solving a variety of (single objective) convex optimization problems such as finding zeros of maximally monotone operators, fixed points of nonexpansive mappings, as well as minimizing convex functions. The algorithm works by applying successively so-called resolvent operators associated to the original object that one aims to optimize. The first instance of what came later to be known as the proximal point algorithm can be found in a short communication from 1970 of Martinet [6]. Starting with the pioneering paper of Rockafellar [7], which clearly fix some existing ideas in the previous literature and gives much more insights on the potential of the algorithm when applies to optimization problems, an important literature has grown on possible extensions and generalizations of this algorithm (see, for example the survey paper [8] and the references therein). Some attention was focused also on the case of multiobjective optimization, see [3, 4, 9, 10]. Here, we provide some, which are concerned to our work:

In 2004, Drummond et al. [10] introduced the projected gradient method for convexly constrained vector optimization. After that, Gregorio and Oliveira [11] proved the convergence of the proximal point method by using a logarithmic quadratic proximal scalarization method. Later on, Bonnel et al. [4] have proposed an extension of the proximal point method to vector optimization, i.e., when other underlying ordering cones are used instead of the non-negative orthant $\mathbb{R}^m_+$. If we restrict the analysis to the finite dimensional multiobjective setting, the method proposed in [4] generates a sequence satisfying

$$x^{k+1} \in \arg\min_w \left\{ H(y) + \frac{\lambda_k}{2}\|y - x^k\|^2 e^k | y \in \Omega_k \right\}, \tag{1.2}$$

where $\Omega_k := \{y \in \mathbb{R}^n | H(y) \preceq H(x^k)\}$, $\{\lambda_k\}$ is a bounded sequence, $\{e^k\} \subset \mathbb{R}^m_{++}$ and and argmin_w denotes the set of weak Pareto solutions. They used the following well-known scalarization approach (see, e.g., [1]) for convergence analysis of the above iterate,

$$\arg\min_w \{H(y) | y \in S\} = \bigcup_{z \in \mathbb{R}^m_+ \setminus \{0\}} \arg\min\{\langle H(y), z\rangle | y \in S\}, \tag{1.3}$$

where $S \subset H$ is a convex set and $S : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathbb{R}^m_+$-convex map, i.e., for every $x, y \in \mathbb{R}^n$, the following holds:

$$H((1-t)x + ty) \preceq (1-t)H(x) + tH(y), \quad \forall t \in [0, 1].$$

see [1]. Also, a function $H : \mathbb{R}^n \to \mathbb{R}^m$ is convex iff H is componentwise convex, see Definition 6.2 ([1], pages 29).

As it is mentioned in ([4], Remark 5) and we can also see by the equality (1.3), the difference between the presentation of the iterative step (1.2) and the following iterative step

$$x^{k+1} \in \arg\min\{\langle H(y) + \frac{\lambda_k}{2}\|y - x^k\|^2 e^k, z\rangle : y \in \Omega_k\} \tag{1.4}$$

is not substantial because every solution of the scalar subproblems (1.4) is a weak Pareto solution of the subproblem (1.2).

Next, we will concern with the concept of fixed point of the nonlinear mapping. Recall that a mapping T is said to be nonexpansive if

$$\|Ty - Tx\| \leq \|y - x\|, \quad \forall x, y \in \mathbb{R}^n.$$

The set of fixed point of T is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in \mathbb{R}^n : x = Tx\}$. The fixed point theory is a fascinating subject, with an enormous number of applications in various fields of mathematics. Fixed point theory concerns itself with a very simple and basic mathematical setting. It is one of the most powerful and fruitful tools of modern mathematics and may be considered as a core subject of nonlinear analysis.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. There have been many iterative schemes constructed and proposed in order to approximate fixed points of a nonexpansive mapping, see [12, 13]. Mann invented iterative method, see [14], and used to obtain convergence to a fixed point for nonexpansive mapping, which is defined as follows: $x^1 \in \mathbb{R}^n$ and

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k T x^k, \quad k \in \mathbb{N},$$

where $\{\alpha_k\}$ is real sequence in $(0, 1)$. In fact, we would like to point out that the problem of finding a common element of the set of solutions of single objective minimization problems and the set of fixed points in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors, for instance, (see [15, 16]) and the references therein.

The main purpose of this paper is to consider the constraint multiobjective minimization problem:

$$\min_{y \in Fix(T)} H(y), \tag{1.5}$$

where $H : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^n$. We will construct the following modified proximal point algorithm for multiobjective optimization involving Mann iterate in $\mathbb{R}^n$: Take a fixed vector $z \in \mathbb{R}^m_+ \setminus \{0\}$, $\{e^k\} \subset \mathbb{R}^m_{++}$, and $\{\alpha_k\}$ is a sequence in $(0, 1)$. For a starting point $x^1 \in \mathbb{R}^n$ we generate the sequence $\{x^k\}$ in the following manner:

$$\begin{cases} \tilde{x}^k = \arg\min\{\langle H(y) + \frac{\lambda_k}{2}\|y - x^k\|^2 e^k, z\rangle : y \in \Omega_k\}, \\ x^{k+1} = (1 - \alpha_k)\tilde{x}^k + \alpha_k T\tilde{x}^k, \end{cases} \tag{1.6}$$

where $\Omega_k := \{y \in \mathbb{R}^n | H(y) \preceq H(x^k)\}$. We will show that, under some suitable conditions, the introduced algorithm (1.6) converges to a weak Pareto optimal point of the constraint multiobjective optimization problem (1.5).

2. Preliminaries

In this section, we present some basic concepts and results that are fundamental importance for the development of our work.

The domain of $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, denoted by $\operatorname{dom} g$, is the subset of $\mathbb{R}^n$ on which g has a finite valued. A function g is said to be proper when $\operatorname{dom} g \neq \emptyset$. We say that a real valued function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous function at a point $x_0 \in \mathbb{R}^n$ if

$$g(x_0) \leq \liminf_{x \to x_0} g(x).$$

If a function is lower semicontinuous at every point of its domain, then it is simply called lower semicontinuous function. It is also noted in [4] a map $H : \mathbb{R}^n \to \mathbb{R}^m$ is called positively lower semicontinuous if, for every $z \in \mathbb{R}^m_+$, the extended-valued scalar function $x \mapsto \langle H(x), z \rangle$ is lower semicontinuous.

Next, we recall some concepts of proximal point mapping.

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the proximal point mapping $\operatorname{prox}^g_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is defined by:

$$\operatorname{prox}^g_\lambda(y) = \arg\min_{x \in \mathbb{R}^n} \left(g(x) + \frac{\lambda}{2} \|x - y\|^2 \right), \quad \forall y \in \mathbb{R}^n. \tag{2.1}$$

It was shown in [17] that the fixed point set $\operatorname{Fix}(\operatorname{prox}^g_\lambda)$ coincides with the set of minimizers of g.

Some other relevant characteristics of $\operatorname{prox}^g_\lambda$ of function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are incorporated in the following couple of lemmas:

Lemma 2.1 ([18]). *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper convex and lower semi-continuous function. For any $\lambda > 0$, the proximal point mapping $\operatorname{prox}^g_\lambda$ is nonexpansive.*

Lemma 2.2 ([19]). *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper convex and lower semi-continuous. Then, for all $x, y \in \mathbb{R}^n$ and $\lambda > 0$, the following inequality holds*

$$\frac{1}{2\lambda} \| \operatorname{prox}^g_\lambda x - y\|^2 - \frac{1}{2\lambda} \|x - y\|^2 + \frac{1}{2\lambda} \|x - \operatorname{prox}^g_\lambda x\|^2 \leq g(y) - g(\operatorname{prox}^g_\lambda x). \tag{2.2}$$

Lemma 2.3 ([18]). *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. Then the following identity holds:*

$$\operatorname{prox}^g_\lambda x = \operatorname{prox}^g_\mu \left(\frac{\lambda - \mu}{\lambda} \operatorname{prox}^g_\lambda x + \frac{\mu}{\lambda} x \right), \quad \forall x \in \mathbb{R}^n \text{ and } \lambda > \mu > 0.$$

We will end this section by recalling some auxiliary facts which will be useful proving the convergence results of our proposed iterative algorithm.

Definition 2.4. A sequence $\{x^k\} \subset \mathbb{R}^n$ is said to be Fejér monotone to a nonempty set U iff, for all $x \in U$

$$\|x^{k+1} - x\| \leq \|x^k - x\|, \quad k = 0, 1, \ldots$$

Lemma 2.5 ([20]). *Let $x, y \in \mathbb{R}^n$. Let $\alpha \in \mathbb{R}$ and $\mathbb{R}$ denote the set of real numbers. Then*

$$\|\alpha x + (1 - \alpha) y\|^2 + \alpha(1 - \alpha)\|x - y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2.$$

Lemma 2.6 ([21]). *Let $U \subset \mathbb{R}^n$ be a nonempty set and $\{x^k\} \subset \mathbb{R}^n$ be a Fejér monotone sequence to U. Then, $\{x^k\}$ is bounded. Moreover, if an accumulation point $\overline{x}$ of $\{x^k\}$ belongs to U, the whole sequence $\{x^k\}$ converges to $\overline{x}$ as k goes to $+\infty$.*

Proposition 2.7 ([22], Corollary 1). *Let $H : \mathbb{R}^n \to \mathbb{R}^m$ and $C \subset \mathbb{R}^n$ be nonempty subset. It holds*

$$\arg\min_w \{H(x)|x \in \mathbb{R}^n\} \cap C \subseteq \arg\min_w \{H(x)|x \in C\}.$$

3. Convergence Theorem

In this section, we prove the main convergence theorem of proposed iterative scheme. We will work under the following assumptions:

(A1) H is $\mathbb{R}^m_+$-convex function.
(A2) H is positively lower semicontinuous function.
(A3) T is a nonexpansive mapping.
(A4) There is $z \in \mathbb{R}^m_+ \setminus \{0\}$ such that $\Upsilon_z := \arg\min_{y\in\mathbb{R}^n}\big\{\langle H(y), z\rangle\big\} \bigcap \mathrm{Fix}(T)$ is a nonempty set.

We proceed with the following main tools.

Lemma 3.1. *Assume the the assumptions (A1)-(A4) are satisfied. Then, under the following control conditions:*

(i) $\{\lambda_k\}$ *is a bounded sequence of positive real numbers;*
(ii) $\{\alpha_k\}$ *is a sequence such that* $0 < a \le \alpha_k \le b < 1$, $\forall k \ge 1$ *and for some constant* a, b *in* $(0, 1)$*,*

the sequence $\{x^k\}$*, which is generated by the algorithm* (1.6)*, with respect to* z*, satisfies the following items*

(i) $\lim_{k\to\infty} \|x^k - x^*\|$ *exist for all* $x^* \in \Upsilon_z$*;*
(ii) $\lim_{k\to\infty} \|x^k - \tilde{x}^k\| = 0$*;*
(iii) $\lim_{k\to\infty} \|Tx^k - x^k\| = 0$.

Proof. Let $x^* \in \Upsilon_z$. So, we have $x^* = Tx^*$ and $x^* \in \arg\min_{y\in\mathbb{R}^n}\big\{\langle H(y), z\rangle\big\}$. Then for all $y \in \mathbb{R}^n$, we acquire

$$\langle H(x^*), z\rangle + \frac{\lambda_k}{2}\|x^* - x^*\|^2\langle e^k, z\rangle \le \langle H(y), z\rangle + \frac{\lambda_k}{2}\|y - x^*\|^2\langle e^k, z\rangle.$$

Let us note that, since H is positively lower semicontinuous, then we have the scalar valued function $\langle H(y), z\rangle$ is lower semicontinuous and convexity of H implies the convexity of $\langle H(y), z\rangle$, hence $\phi_z(y) = \langle H(y), z\rangle$ is proper, convex and lower semicontinuous function.

Now define $\beta_k = \lambda_k\langle e^k, z\rangle$. Note that, $\beta_k > 0$, because $\lambda_k > 0$, $z \in \mathbb{R}^m_+ \setminus \{0\}$, $\{e^k\} \subset \mathbb{R}^m_{++}$. So, we get that $x^* = \mathrm{prox}^{\phi_z}_{\beta_k} x^*$.

(i) Now, we first show that $\lim_{k\to\infty} \|x^k - x^*\|$ exists for all $x^* \in \Upsilon_z$. Noting that $\tilde{x}^k = \mathrm{prox}^{\phi_z}_{\beta_k} x^k$ for all $k \ge 1$. So, Lemma 2.1 provide us

$$\|\tilde{x}^k - x^*\| = \|\,\mathrm{prox}^{\phi_z}_{\beta_k} x^k - \mathrm{prox}^{\phi_z}_{\beta_k} x^*\| \le \|x^k - x^*\|. \tag{3.1}$$

It follows from the algorithm (1.6) and nonexpansiveness of T that

$$\begin{aligned}\|x^{k+1}-x^*\| &= \|(1-\alpha_k)\tilde{x}^k+\alpha_k T\tilde{x}^k-x^*\| \\ &\le (1-\alpha_k)\|\tilde{x}^k-x^*\|+\alpha_k\|T\tilde{x}^k-x^*\| \\ &\le (1-\alpha_k)\|\tilde{x}^k-x^*\|+\alpha_k\|\tilde{x}^k-x^*\| \\ &= \|\tilde{x}^k-x^*\| \\ &\le \|x^k-x^*\|,\ \forall k\ge 1.\end{aligned} \tag{3.2}$$

This shows that $\{\|x^k-x^*\|\}$ is decreasing and bounded below. Hence $\lim_{k\to\infty}\|x^k-x^*\|$ exists for all $x^*\in\Upsilon_z$.

(ii) In order to proceed for part (ii), we assume, without loss of any generality, that

$$\lim_{k\to\infty}\|x^k-x^*\| = c\ge 0. \tag{3.3}$$

Indeed, by (2.2), we have

$$\frac{1}{2\lambda_k}\left(\|\tilde{x}^k-x^*\|^2-\|x^k-x^*\|^2+\|x^k-\tilde{x}^k\|^2\right)\le \phi_z(x^*)-\phi_z(\tilde{x}^k).$$

Since $\phi_z(x^*)\le\phi_z(\tilde{x}^k)$ for all $k\ge 1$, it follows that

$$\|x^k-\tilde{x}^k\|^2\le\|x^k-x^*\|^2-\|\tilde{x}^k-x^*\|^2. \tag{3.4}$$

Therefore in order to prove $\lim_{k\to\infty}\|x^k-\tilde{x}^k\|=0$, it suffices to prove $\|\tilde{x}^k-x^*\|\to c$, because $\|x^k-x^*\|\to c$.
Taking lim inf on both sides of the estimate (3.2), we have

$$c\le\liminf_{k\to\infty}\|\tilde{x}^k-x^*\|. \tag{3.5}$$

On the other hand, by taking lim sup on both sides of the estimate (3.1), we get

$$\limsup_{k\to\infty}\|\tilde{x}^k-x^*\|\le\limsup_{k\to\infty}\|x^k-x^*\|=c.$$

Hence, the above estimate together with (3.5) implies that

$$\lim_{k\to\infty}\|\tilde{x}^k-x^*\|=c. \tag{3.6}$$

Therefore, from (3.4), we obtain

$$\lim_{k\to\infty}\|x^k-\tilde{x}^k\|=0. \tag{3.7}$$

(iii) Next, we prove that $\lim_{k\to\infty}\|Tx^k-x^k\|=0$. As, we observe from lemma 2.5 that

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &= \|(1-\alpha_k)\tilde{x}^k+\alpha_k T\tilde{x}^k-x^*\|^2 \\ &= \|(1-\alpha_k)(\tilde{x}^k-x^*)+\alpha_k(T\tilde{x}^k-x^*)\|^2 \\ &= (1-\alpha_k)\|\tilde{x}^k-x^*\|^2+\alpha_k\|T\tilde{x}^k-x^*\|^2-\alpha_k(1-\alpha_k)\|\tilde{x}^k-T\tilde{x}^k\|^2.\end{aligned}$$

Since T is nonexpansive, it follows that

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\le (1-\alpha_k)\|\tilde{x}^k-x^*\|^2+\alpha_k\|\tilde{x}^k-x^*\|^2-\alpha_k(1-\alpha_k)\|\tilde{x}^k-T\tilde{x}^k\|^2 \\ &\le \|\tilde{x}^k-x^*\|^2-\alpha_k(1-\alpha_k)\|\tilde{x}^k-T\tilde{x}^k\|^2 \\ &\le \|x^k-x^*\|^2-a(1-b)\|\tilde{x}^k-T\tilde{x}^k\|^2.\end{aligned}$$

This implies that

$$\|\tilde{x}^k - T\tilde{x}^k\|^2 \leq \frac{1}{a(1-b)}\big(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2\big) \to 0 \text{ (as } k \to \infty), \tag{3.8}$$

Since T nonexpansive and from (3.7), (3.8) we obtain that

$$\begin{aligned} \|\tilde{x}^k - Tx^k\| &\leq \|\tilde{x}^k - T\tilde{x}^k\| + \|T\tilde{x}^k - Tx^k\| \\ &\leq \|\tilde{x}^k - T\tilde{x}^k\| + \|\tilde{x}^k - x^k\| \to 0 \; (as \;\; k \to \infty). \end{aligned} \tag{3.9}$$

Now, we can prove that $\lim_{k\to\infty} \|x^k - Tx^k\| = 0$. From (3.7) and (3.9), we obtain

$$\|x^k - Tx^k\| \leq \|x^k - \tilde{x}^k\| + \|\tilde{x}^k - Tx^k\| \to 0 \; (as \;\; k \to \infty). \tag{3.10}$$

■

Now we are in position to present our main theorem.

Theorem 3.2. *Assume the the assumptions (A1)-(A4) are satisfied. Then, under the following control conditions:*

(i) $\{\lambda_k\}$ *is a bounded sequence of positive real numbers such that* $\lambda_k \geq \lambda > 0$*, for some positive real number* λ*;*

(ii) $\{\alpha_k\}$ *is a sequence such that* $0 < a \leq \alpha_k \leq b < 1$*,* $\forall k \geq 1$ *and for some constant* a, b *in* $(0, 1)$*,*

the sequence $\{x^k\}$*, which is generated by the algorithm* (1.6)*, with respect to* z*, converges to a weak Pareto optimal point of the constraint multiobjective optimization problem* (1.5)*.*

Proof. In fact, it follows from (3.7) and Lemma 2.3 that

$$\begin{aligned} \|\operatorname{prox}_\lambda^{\phi_z} x^k - x^k\| &\leq \|\operatorname{prox}_\lambda^{\phi_z} x^k - \tilde{x}^k\| + \|\tilde{x}^k - x^k\| \\ &= \|\operatorname{prox}_\lambda^{\phi_z} x^k - \operatorname{prox}_{\beta_k}^{\phi_z} x^k\| + \|\tilde{x}^k - x^k\| \\ &= \|\operatorname{prox}_\lambda^{\phi_z} x^k - \operatorname{prox}_\lambda^{\phi_z}\left(\frac{\beta_k - \lambda}{\beta_k}\operatorname{prox}_{\beta_k}^{\phi_z} x^k + \frac{\lambda}{\beta_k}x^k\right)\| + \|\tilde{x}^k - x^k\| \\ &\leq \|x^k - (1 - \frac{\lambda}{\beta_k})\operatorname{prox}_{\beta_k}^{\phi_z} x^k - \frac{\lambda}{\beta_k}x^k\| + \|\tilde{x}^k - x^k\| \\ &\leq (1 - \frac{\lambda}{\beta_k})\|x^k - \tilde{x}^k\| + \|\tilde{x}^k - x^k\| \to 0 \; (as \;\; k \to \infty) \end{aligned} \tag{3.11}$$

Moreover, by (3.2), we have that $\{x^k\}$ is Fejér convergent to Υ_z. So, it guarantees that $\{x^k\}$ is bounded. Then there exists a subsequence $\{x^{k_i}\} \subset \{x^k\}$ such that $x^{k_i} \to p^*$. By (3.10),

$$\|x^{k_i} - Tx^{k_i}\| \to 0.$$

It follows that $p^* \in \operatorname{Fix}(T)$. Also, from (3.11), we have

$$\|x^{k_i} - \operatorname{prox}_\lambda^{\phi_z} x^{k_i}\| \to 0.$$

Since $\operatorname{prox}_\lambda^{\phi_z}$ is a nonexpansive mapping. Then, we get that

$$p^* \in \operatorname{Fix}(\operatorname{prox}_\lambda^{\phi_z}) = \arg\min_{y\in\mathbb{R}^n} \phi_z(y).$$

This shows that $p^* \in \Upsilon_z$. Therefore, using Lemma 2.6 with $U = \Upsilon_z$, we conclude that the whole sequence $\{x^k\}$ converges to p^* as k goes to ∞.

Further, we can see by equality (1.3) that $p^* \in \arg\min_{y\in\mathbb{R}^n}\{H(y)\}$. Hence $p^* \in \arg\min_w\{H(y)|y \in \mathbb{R}^n\} \cap \mathrm{Fix}(T)$. Finally, by Proposition 2.7, we obtain that the sequence $\{x^k\}$ converges to a weak Pareto optimal point of the constraint multiobjective optimization problem (1.5). This completes the proof.

∎

Remark 3.3. is remarked that, if we take constraint set $\mathrm{Fix}(T) = \mathbb{R}^n$, then we get multiobjective optimization problem (1.1), which is done by many authors by different methods, see for instance [3, 4, 9, 10].

The next is a simple example for checking the conclusion of Theorem 3.2.

Example 3.4. Let $H : \mathbb{R} \to \mathbb{R}^2$ be the vectorial function given by $H(y) = (h_1(y), h_2(y))$, where $h_1(y) := y$ and $h_2(y) := (y-1)^2$. We can see that

$$\langle H(y), z\rangle = \sum_{i=1}^{2} h_i(y)z_i = yz_1 + (y-1)^2 z_2,$$

and for fixed $z = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\arg\min\langle H(y), z\rangle = \frac{1}{2}$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Ty = 1 - y.$$

It is easy to check that T is a nonexpansive mapping with $\mathrm{Fix}(T) = \frac{1}{2}$ and $\langle H(y), z\rangle$ is proper convex and lower semi-continuous function. Thus by Proposition 1.3, we get that 0.5 is also a weak Pareto optimal solution of H. Moreover, it follows that $\frac{1}{2}$ is the solution of the problem (1.5).

Now, let $\alpha_k = \frac{1}{2}$, $e^k = \{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$ and also we put $x^1 = 10$ is the initial value.

Number of iterates	x^k	$\lvert x^{k+1} - x^k\rvert$
1	10.0000	9.50000
2	0.50000	0.00000
3	0.50000	0.00000
4	0.50000	0.00000
5	0.50000	0.00000
6	0.50000	0.00000
7	0.50000	0.00000
8	0.50000	0.00000
9	0.50000	0.00000
10	0.50000	0.00000

Table 1. Number of iteration

From Table 1, we see that the sequence $\{x^k\}$ converges to $\frac{1}{2}$ which is the solution of the problem (1.5).

4. Conclusion

We consider the convex constraint multiobjective optimization problem when the constrained set is a fixed point set of nonexpansive mapping. By combining the concepts of proximal method and Mann algorithm, we introduce the algorithm and provide the convergence results of the such proposed iterative algorithm to compute a solution point the considered constraint convex multiobjective optimization problem. Since the results in this paper is a suggestion an algorithm method for finding the weak Pareto optimal point of the considered problem (1.5), of course, the convergence analysis of this suggested algorithm and also the (new) updated algorithms for finding the Pareto optimal point of this problem (1.5) should be considered in the future works.

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