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On the Hyers-Ulam-Rassias Stability of an n-Dimensional Additive Functional Equation

P. Nakmahachalasint

Abstract : In this paper, we prove the Hyers-Ulam-Rassias stability of the following n-dimensional additive functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1})$$

where $x_0 \equiv x_n$ and $n > 1$.

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1 Introduction

In 1940 S.M. Ulam [8] proposed the famous Ulam stability problem of linear mappings. In 1941 D.H. Hyers [1] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies inequality $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying $\|f(x) - L(x)\| \leq \varepsilon$. The stability problem of various functional equations has been studied by a number of authors ([2]-[7]) since then.

In this paper, we propose an n-dimensional additive functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1})$$

where $x_0 \equiv x_n$ and $n > 1$, and investigate its Hyers-Ulam-Rassias stability.

2 The Solution

The following theorem establishes the equivalence of the proposed functional equation and the Cauchy functional equation.

Theorem 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation*

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}) \tag{2.1}$$

where $x_0 \equiv x_n$ and $n > 1$, for all $x_1, x_2, \dots, x_n \in X$ if and only if it satisfies the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{2.2}$$

for all $x, y \in X$.

Proof. Suppose a mapping $f : X \to Y$ satisfies the Cauchy functional equation. Then it is straightforward to show that

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i),$$

and

$$\sum_{i=1}^{n} f(x_i - x_{i-1}) = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) = 0.$$

Hence, f satisfies (2.1).

Suppose a mapping $f : X \to Y$ satisfies (2.1). Setting $x_1 = x_2 = \cdots = x_n = 0$, we can see that $f(0) = 0$. Settings $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = 0$, we have

$$f(x) = f(x) + \left(f(x) + f(-x)\right),$$

which shows that $f(-x) = -f(x)$ for all $x \in X$.
If $n = 2$, then (2.1) reduces to

$$f(x+y) = f(x) + f(y) + f(x-y) + f(y-x).$$

Noting the oddness of f, the above equation simplifies to the Cauchy functional equation.
If $n > 2$, then we substitute $(x_1, x_2, \dots, x_n) = (x, y, 0, \dots, 0)$ and (2.1) becomes

$$f(x+y) = f(x) + f(y) + \left(f(x-y) + f(y) + f(-x)\right).$$

Again, by the oddness of f, the above equation reduces to

$$f(x+y) - f(x-y) = 2f(y).$$

Observing that the right-hand side is independent of x, we have

$$f\left(\left(x + \frac{y}{2}\right) + \frac{y}{2}\right) - f\left(\left(x + \frac{y}{2}\right) - \frac{y}{2}\right) = 2f\left(\frac{y}{2}\right) = f\left(\frac{y}{2} + \frac{y}{2}\right) - f\left(\frac{y}{2} - \frac{y}{2}\right),$$

or $f(x+y) = f(x) + f(y)$ as desired. This completes the proof. □

3 The Hyers-Ulam-Rassias Stability

The following theorem gives a condition for which a linear mapping exists near an *approximately* linear mapping.

Theorem 3.1. *Let $p \neq 1$ be a positive real number and $\theta \geq 0$ be a real number. Let X be a real vector space and Y be a Banach space. If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}) - f\left(\sum_{i=1}^{n} x_i\right) \right\| \leq \delta + \theta \sum_{i=1}^{n} \|x_i\|^p \tag{3.1}$$

for all $x_1, x_2, \ldots, x_n \in X$, and $\delta = 0$ when $p > 1$, then there exists a unique linear mapping $L : X \to Y$ such that L satisfies the functional equation given by (2.1), and satisfies the inequality

$$\|f(x) - L(x)\| \leq \frac{4n}{2n-1}\delta + \frac{3\theta}{|2 - 2^p|}\|x\|^p \tag{3.2}$$

for all $x \in X$. The mapping L is given by

$$L(x) = \begin{cases} \lim_{m\to\infty} 2^{-m} f(2^m x) & \text{if } 0 < p < 1 \\ \lim_{m\to\infty} 2^{m} f(2^{-m} x) & \text{if } p > 1 \end{cases} \tag{3.3}$$

for all $x \in X$.

Proof. Setting $x_1 = x_2 = \cdots = x_n = 0$ in (3.1), we have $\|(2n-1)f(0)\| \leq \delta$. Thus,

$$\|f(0)\| \leq \frac{\delta}{2n-1}.$$

Setting $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = 0$, we have

$$\|(f(x) + (n-1)f(0)) + (f(x) + f(-x) + (n-2)f(0)) - f(x)\| \leq \delta + \theta\|x\|^p,$$

which simplifies to

$$\|(2n-3)f(0) + f(x) + f(-x)\| \leq \delta + \theta\|x\|^p.$$

If $n > 2$, we set $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_n = 0$, then

$$\|(2f(x) + (n-2)f(0)) + (f(x) + f(-x) + (n-2)f(0)) - f(2x)\| \leq \delta + 2\theta\|x\|^p,$$

or simply

$$\|-f(0) + ((2n-3)f(0) + f(x) + f(-x)) + 2f(x) - f(2x)\| \leq \delta + 2\theta\|x\|^p,$$

Thus,

$$\|2f(x)-f(2x)\| \le \|f(0)\| + (\delta+\theta\|x\|^p) + (\delta+2\theta\|x\|^p) = \|f(0)\| + 2\delta + 3\theta\|x\|^p.$$

If $n=2$, we set $x_1 = x_2 = x$ in (3.1), then

$$\|2f(x)+2f(0)-f(2x)\| \le \delta + 2\theta\|x\|^p.$$

It follows that

$$\|2f(x)-f(2x)\| \le 2\|f(0)\| + \delta + 2\theta\|x\|^p.$$

Hence, we determine that

$$\begin{aligned} \|2f(x)-f(2x)\| &\le 2\|f(0)\| + 2\delta + 3\theta\|x\|^p \\ &\le \left(\frac{2}{2n-1}+2\right)\delta + 3\theta\|x\|^p \\ &\le \frac{4n}{2n-1}\delta + 3\theta\|x\|^p. \end{aligned}$$

We first consider the case when $0<p<1$. Rewrite the above inequality as

$$\|f(x) - 2^{-1}f(2x)\| \le \frac{4n}{4n-2}\delta + \frac{3\theta}{2}\|x\|^p. \tag{3.4}$$

For every positive integer m,

$$\begin{aligned} \left\|f(x)-2^{-m}f(2^m x)\right\| &= \left\|\sum_{i=0}^{m-1}\left(2^{-i}f(2^i x) - 2^{-(i+1)}f(2^{i+1}x)\right)\right\| \\ &\le \sum_{i=0}^{m-1}\left\|2^{-i}f(2^i x) - 2^{-(i+1)}f(2^{i+1}x)\right\| \\ &= \sum_{i=0}^{m-1} 2^{-i}\left\|f(2^i x) - 2^{-1}f(2\cdot 2^i x)\right\|. \end{aligned}$$

Applying (3.4) with appropriate values of x's, we have

$$\left\|f(x)-2^{-m}f(2^m x)\right\| \le \frac{4n}{4n-2}\delta\sum_{i=0}^{m-1}2^{-i} + \frac{3\theta}{2}\|x\|^p\sum_{i=0}^{m-1}2^{i(p-1)}.$$

Consider the sequence $\{2^{-m}f(2^m x)\}$. For all positive integers $k<l$, we have

$$\begin{aligned} \|2^{-k}f(2^k x) - 2^{-l}f(2^l x)\| &= 2^{-k}\|f(2^k x) - 2^{-(l-k)}f(2^{l-k}\cdot 2^k x)\| \\ &\le 2^{-k}\left(\frac{4n}{4n-2}\delta\sum_{i=0}^{l-k-1}2^{-i} + \frac{3\theta}{2}\|2^k x\|^p\sum_{i=0}^{l-k-1}2^{i(p-1)}\right) \\ &\le \frac{4n}{4n-2}2^{-k}\delta\sum_{i=0}^{\infty}2^{-i} + \frac{3\theta}{2}2^{-k(1-p)}\|x\|^p\sum_{i=0}^{\infty}2^{i(p-1)}. \end{aligned}$$

The right-hand side of the above inequality approaches 0 as k tends to infinity. Since Y is a Banach space and $\{2^{-m}f(2^m x)\}$ is a Cauchy sequence, we let

$$L(x) = \lim_{m\to\infty} 2^{-m} f(2^m x).$$

It follows that

$$\begin{aligned}\|f(x) - L(x)\| &\leq \frac{4n}{4n-2}\delta \sum_{i=0}^{\infty} 2^{-i} + \frac{3\theta}{2}\|x\|^p \sum_{i=0}^{\infty} 2^{i(p-1)} \\ &= \frac{4n}{2n-1}\delta + \frac{3\theta}{2-2^p}\|x\|^p.\end{aligned}$$

From (3.1), we have

$$\begin{aligned}&2^{-m}\left\|\sum_{i=1}^{n} f(2^m x_i) + \sum_{i=1}^{n} f(2^m x_i - 2^m x_{i-1})) - f\left(\sum_{i=1}^{n} 2^m x_i\right)\right\| \\ &\qquad \leq 2^{-m}\left(\delta + \theta\sum_{i=1}^{n}\|2^m x_i\|^p\right) = 2^{-m}\delta + 2^{-m(1-p)}\theta\sum_{i=1}^{n}\|x_i\|^p.\end{aligned}$$

As m tends to infinity, the right-hand side of the above inequality approaches 0; hence,

$$\sum_{i=1}^{n} L(x_i) + \sum_{i=1}^{n} L(x_i - x_{i-1})) = L\left(\sum_{i=1}^{n} x_i\right)$$

which reveals that L satisfies (2.1).

To prove the uniqueness of L, suppose there is a mapping $L' : X \to Y$ such that L' satisfies (2.1) and the condition (3.2). Then, the linearity of (2.1) as asserted by Theorem 2.1 implies

$$\begin{aligned}\|L(x) - L'(x)\| &= 2^{-m}\|L(2^m x) - L'(2^m x)\| \\ &\leq 2^{-m}\|L(2^m x) - f(2^m x)\| + 2^{-m}\|L'(2^m x) - f(2^m x)\| \\ &\leq 2^{-m}\cdot 2\left(\frac{4n}{2n-1}\delta + \frac{3\theta}{2-2^p}\|x\|^p\right).\end{aligned}$$

The right-hand side of the above inequality approaches 0 as m tends to infinity; so, we conclude that $L(x) = L'(x)$ for all $x \in X$.

The proof for the case when $p > 1$ starts by replacing (3.4) with

$$\|f(x) - 2f(2^{-1}x)\| \leq \frac{4n}{2n-1}\delta + 3\theta\|2^{-1}x\|^p,$$

and the rest of the proof can be reproduced accordingly. □

The following corollary gives the Hyers-Ulam stability of the functional equation given by (2.1).

Corollary 3.2. *Let X be a real vector space and Y be a Banach space. If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}) - f\left(\sum_{i=1}^{n} x_i\right) \right\| \leq \delta$$

for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique linear mapping $L : X \to Y$ such that L satisfies the functional equation given by (2.1), and satisfies the inequality

$$\|f(x) - L(x)\| \leq \frac{4n}{2n-1}\delta$$

for all $x \in X$.

Proof. Letting $\theta = 0$ and $p = \frac{1}{2}$ in Theorem 3.1, we immediately obtain the desired result. □

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Paisan Nakmahachalasint
Department of Mathematics
Faculty of Science
Chulalongkon University
Bangkok 10330, THAILAND.
e-mail: `Paisan.N@chula.ac.th`