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On the Parametric Interest of the Option Price of Stock from Black-Scholes Equation

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Abstract In this paper, we studied the option price of stock from the Black-Scholes equation and discovered some parameter λ which is the generalizztion of the interest r. Such λ is the first that named the parametric interest which is new the results. Morever we found that such λ gives the conditions for the solution of the Black-Scholes equation which may be weak or strong solution.

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1. Introduction

It is well known that the the Black-Scholes equation plays an important role in solving the option price of the stock which is call the Black-Scholes formula (see [1, pp. 637-659]). The Black-Scholes equation is given by

$$\frac{\partial}{\partial t}u(s,t)+\frac{1}{2}\sigma^2 s^2\frac{\partial^2}{\partial s^2}u(s,t)+rs\frac{\partial}{\partial s}u(s,t)-ru(s,t)=0 \tag{1.1}$$

with the call payoff

$$u(s_T,T)=(s_T-p)^+\equiv\max(s_T-p,0) \tag{1.2}$$

for $0\leq t\leq T$ where $u(s,t)$ is the option price at time t, s_T is the price of stock at the expiration time T, s is the price of stock at time t, r is the interest rate, σ is the volatility of stock price and p is the strike price.

They obtained the solution $u(s,t)$ of (1.1) which is called the Black-Scholes formula and satisfies (1.2) of the form

$$u(s,t)=sN(d_1)-pe^{-r(T-t)}N(d_2) \tag{1.3}$$

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see [2, pp. 90-91] where

$$d_1 = \frac{\ln\left(\frac{s}{p}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{s}{p}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$.

In this work, we studied the solution of (1.1) in the other form which is the generalization of (1.3). Starting with changing the variable $u(s,t) = V(R,t)$ where $R = \ln s$. Then (1.1) is transformed to equation

$$\frac{\partial}{\partial t}V(R,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial R^2}V(R,t) + (r - \frac{\sigma^2}{2})\frac{\partial}{\partial R}V(R,t) - rV(R,t) = 0 \tag{1.4}$$

with the call payoff in (1.2)

$$V(R_T, T) = (e^{R_T} - p)^+ \tag{1.5}$$

where $R_T = \ln s_T$. Then we obtained the solution of (1.4) in the form

$$u(s,t,\lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \mathcal{L}^{-1}(\xi^\alpha) \tag{1.6}$$

where $\mathcal{L}^{-1}(\xi^\alpha)$ is the inverse Laplace transform of ξ^α and $X(\ln s_T)$ is the function of $\ln s_T$ and α is the real number that can be obtained from the equation

$$\sigma^2\alpha^2 + (3\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda) = 0 \tag{1.7}$$

with λ is the parametric interest.

Now consider the following cases.

(i) Suppose $\alpha = m$ where m is nonnegative integer then we obtained the solution in (1.6) as the weak solution of the form

$$u(s,t,\lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \delta^{(m)}(s) \tag{1.8}$$

where $\delta^{(m)}(s)$ is the Dirac-delta function with m-derivative with $\delta^{(0)}(s) = \delta(s)$ and the parametric interest λ can be obtained from (1.7) as

$$\lambda = \lambda(r,\sigma) = (m+2)r - \frac{m^2 + 3m + 2}{2}\sigma^2. \tag{1.9}$$

(ii) Suppose α is a negative real number, that is $\alpha < 0$ then we obtained the solution is (1.6) as the strong solution or the classical solution of the form

$$u(s,t,\lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \frac{s^{-\alpha-1}}{\Gamma(-\alpha)} \tag{1.10}$$

where $\Gamma(-\alpha)$ is the Gamma function of $-\alpha$. In particular, if $\alpha = -n$ for some positive integer n then (1.10) reduces to

$$u(s,t,\lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \left(\frac{s}{s_T}\right)^{n-1} \tag{1.11}$$

with the parametric interest

$$\lambda = (2-n)r - \frac{(n^2 - 3n + 2)}{2}\sigma^2. \tag{1.12}$$

We see that (1.8) and (1.10) are the other solutions of the Black-Scholes solution in (1.1). Particularly, for $n = 1$ in (1.12) we have $\lambda = r$ that means the parametric interest λ is the interest rate r of the option price of stock. It follows that from (1.11)

$$u(s,t,r) = (s_t - p)^+ e^{-r(T-t)}$$

and at $t = T$ we obtained $u(s_T, T, r) = (s_T - p)^+$ which is the call payoff in (1.2). Also for the case $n = 2$ in (1.12) we have $\lambda = 0$. It follows that from (1.11)

$$u(s,t,0) = (s_T - p)^+ e^{0(T-t)} \left(\frac{s}{s_T}\right) = (s_T - p)^+ \left(\frac{s}{s_T}\right)$$

and at $t = T$, $u(s_T, T, 0) = (s_T - p)^+ \left(\frac{s_T}{s_T}\right) = (s_T - p)^+$ which is the call payoff in (1.2).

2. Preliminaries

The following definition and lemma are needed.

Definition 2.1. Given f is piescewise continuous on the interval $0 \le t \le A$ for any positive A and if there exists the real constant K, a and M such that

$$|f(t)| \le K e^{\alpha t} \text{ for } t \ge M.$$

Then the Laplace transform of the $f(t)$, denoted by $\mathcal{L}f(t)$ is defined by

$$\mathcal{L}f(t) = F(\xi) = \int_0^\infty e^{-\xi t} f(t)dt \tag{2.1}$$

and the inverse Laplace transform of $F(\xi)$ is defined by

$$f(t) = \mathcal{L}^{-1}F(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(\xi)e^{\xi t} d\xi. \tag{2.2}$$

Lemma 2.2. [3]

(i) $\mathcal{L}\delta(t) = 1$ *where $\delta(t)$ is the Dirac-delta function.*

(ii) $\mathcal{L}\delta^{(k)}(t) = \xi^k$ *where $\delta^{(k)}(t)$ is the Dirac-delta function with k-derivative and $\delta^{(0)}(t) = \delta(t)$ and $\xi > 0$.*

(iii) $\mathcal{L}(t^p) = \dfrac{\Gamma(p+1)}{\xi^{p+1}}$ *for $p > -1$ and $\xi > 0$ where $\Gamma(p+1)$ is the Gamma function.*

If p is positive number n then $\mathcal{L}(t^n) = \dfrac{n!}{\xi^{n+1}}, \xi > 0.$

(iv) $\mathcal{L}[t^k f(t)] = (-1)^k F^{(k)}(\xi).$

(v) $\mathcal{L}[f^{(k)}(t)] = \xi^k F(\xi).$

3. Main Results

Theorem 3.1. *Recall the Black-Scholes equation in (1.1) that*

$$\frac{\partial}{\partial t}u(s,t)+\frac{1}{2}\sigma^2 s^2\frac{\partial^2}{\partial s^2}u(s,t)+rs\frac{\partial}{\partial s}u(s,t)-ru(s,t)=0 \tag{3.1}$$

and the call payoff in (1.2) that

$$u(s_T,T)=(s_T-p)^+ \tag{3.2}$$

then in (1.6)

$$u(s,t,\lambda)=(s_T-p)^+\frac{e^{-\lambda(T-t)}}{X(\ln s_T)}\mathcal{L}^{-1}(\xi^\alpha) \tag{3.3}$$

is the solution of (3.1) with the parametric interest

$$\lambda=(\alpha+2)r-\frac{\alpha^2+3\alpha+2}{2}\sigma^2. \tag{3.4}$$

In particular if $\alpha=m$ where m is nonnegative integer then (3.3) becomes

$$u(s,t,\lambda)=(s_T-p)^+\frac{e^{-\lambda(T-t)}}{X(\ln s_T)}\xi^{(m)}(s) \tag{3.5}$$

with the parametric interest

$$\lambda=(m+2)r-\frac{m^2+3m+2}{2}\sigma^2. \tag{3.6}$$

Also for the case α is negative real number, that is $\alpha<0$ then (3.3) becomes

$$u(S,t,\lambda)=(s_T-p)^+\frac{e^{-\lambda(T-t)}}{X(\ln s_T)}\frac{s^{-\alpha-1}}{\Gamma(-\alpha)}. \tag{3.7}$$

In particular, if α is negative integer and suppose $\alpha=-n$ then (3.7) reduces to

$$u(s,t,\lambda)=(s_T-p)^+e^{-\lambda(T-t)}\left(\frac{s}{s_T}\right)^{n-1} \tag{3.8}$$

with the parametric interest

$$\lambda=(2-n)r-\frac{(n^2-3n+2}{2}\sigma^2. \tag{3.9}$$

Proof. By changing the variable $u(s,t)=V(R,T)$ where $R=\ln s$ then (3.1) is transformed to the equation

$$\frac{\partial}{\partial t}V(R,t)+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}V(R,t)+(r-\frac{\sigma^2}{2})\frac{\partial}{\partial R}V(R,t)-rV(R,t)=0. \tag{3.10}$$

By the method of separation of variable, let $V(R,t)=X(R)U(t)$, then $\frac{\partial}{\partial t}V(R,t)=X(R)U'(t)$, $\frac{\partial}{\partial R}V(R,t)=X'(R)U(t)$ and $\frac{\partial^2}{\partial R^2}V(R,t)=X''(t)U(t)$ then substitute into (3.10). Then we obtained

$$X(R)U'(t)+\frac{1}{2}\sigma^2X''(R)U(t)+(r-\frac{\sigma^2}{2})X'(R)U(t)-rX(R)U(t)=0$$

or

$$\frac{U'(t)}{U(t)} + \frac{1}{2}\sigma^2 \frac{X''(R)}{X(R)} + (r - \frac{\sigma^2}{2})\frac{X'(R)}{X(R)} - r = 0.$$

Let

$$\frac{U'(t)}{U(t)} + \frac{1}{2}\sigma^2 \frac{X''(R)}{X(R)} + (r - \frac{\sigma^2}{2})\frac{X'(R)}{X(R)} - r = \lambda$$

where λ is a parameter.

Now consider $\frac{U'(t)}{U(t)} = \lambda$ then $U(t) = Ce^{\lambda t}$. Now we compute the constant C. Since $u(s,t) = V(R,t) = X(R)U(t)$ and the call payoff $u(s_T, T) = (s_T - p)^+$, hence $X(\ln s_T)U(T) = (s_T - p)^+$ and $U(T) = Ce^{\lambda T}$. It follows that $C = \frac{U(T)}{e^{\lambda T}} = \frac{(s_T - p)^+}{X(\ln s_T)e^{\lambda T}}$. Thus we have

$$U(t) = \frac{(s_T - p)^+}{X(\ln S_T)}\frac{e^{\lambda t}}{e^{\lambda T}} = \frac{(s_T - p)^+}{X(\ln S_T)}e^{-\lambda(T-t)}. \tag{3.11}$$

Next consider $\frac{1}{2}\sigma^2 \frac{X''(R)}{X(R)} + (r - \frac{\sigma^2}{2})\frac{X'(R)}{X(R)} - r + \lambda = 0$ and let $X(R) = X(\ln s) = y(s)$. Then $X'(R) = \frac{dy(s)}{dR} = \frac{dy(s)}{ds}\frac{ds}{dR} = sy'(s)$ and $X''(R) = s^2y''(s) + sy'(s)$ thus

$$\sigma^2[s^2y''(s) + sy'(s)] + (2r - \sigma^2)sy'(s) - (2r - 2\lambda)y(s) = 0$$

or

$$\sigma^2 s^2 y''(s) + 2rsy'(s) - (2r - 2\lambda)y(s) = 0. \tag{3.12}$$

The equation (3.12) is the Euler's equation of order 2. Take the Laplace transform of (2.1) to (3.12) and use (iv) and (v) of Lemma 2.2. Where $\mathcal{L}y(s) = Y(\xi)$ then

$$\sigma^2\frac{d^2}{d\xi^2}[\xi^2 Y(\xi)] + (-1)2r\frac{d}{d\xi}[\xi Y(\xi)] - (2r - 2\lambda)Y(\xi) = 0$$

or

$$\sigma^2\xi^2 Y''(\xi) + (4\sigma^2 - 2r)\xi Y'(\xi) + (2\sigma^2 - 4r + 2\lambda)Y(\xi) = 0 \tag{3.13}$$

which is also the Euler's equation of order 2. Let $y(\xi) = \xi^\alpha$ and substitute into (3.13) then

$$[\sigma^2\alpha(\alpha - 1) + (4\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda)]\xi^\alpha = 0.$$

Thus we have

$$\sigma^2\alpha^2 + (3\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda) = 0. \tag{3.14}$$

The real number α and the parametric interest λ can be obtained from (3.14). Since $Y(\xi) = \xi^\alpha$, hence $y(s) = \mathcal{L}^{-1}Y(\xi) = \mathcal{L}^{-1}(\xi^\alpha)$ where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Since $u(s,t) = V(R,t) = X(\ln s)U(t) = y(s)U(t)$, thus from (3.11)

$$u(s,t,\lambda) = (s_T - p)^+\frac{e^{-\lambda(T-t)}}{X(\ln s_T)}\mathcal{L}^{-1}(\xi^\alpha) \tag{3.15}$$

where $u(s,t,\lambda)$ is the function of s, t and λ.
Thus we obtained (3.3) as required.

Now for the case $\alpha = m$ where m is nonnegative integer and from (ii) of Lemma 2.2 $\mathcal{L}\delta^{(m)}(s) = \xi^m$. Thus $\delta^{(m)}(s) = \mathcal{L}^{-1}(\xi^m)$. It follows that

$$u(s,t,\lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \delta^{(m)}(s).$$

Thus we obtained (3.5) as required.
Next for the case $\alpha < 0$ and from (iii) of Lemma 2.2

$$\mathcal{L}s^p = \frac{\Gamma(p+1)}{\xi^{p+1}} = \Gamma(p+1)\xi^{-p-1}, \text{ for } p > -1.$$

Let $\alpha = -p-1$ then $p = -\alpha - 1$. Thus $\mathcal{L}(s^{-\alpha-1}) = \Gamma(-\alpha)\xi^\alpha$. It follows that $\mathcal{L}^{-1}(\xi^\alpha) = \dfrac{s^{-\alpha-1}}{\Gamma(-\alpha)}$. Thus from (3.5),

$$u(S,t,\lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \frac{s^{-\alpha-1}}{\Gamma(-\alpha)}.$$

Thus we obtained (3.7) as required. In particular, if α is negative integer and suppose $\alpha = -n$ then $u(s,t,\lambda) = (s_T - p)^+ \dfrac{e^{-\lambda(T-t)}}{X(\ln s_T)} \dfrac{s^{n-1}}{\Gamma(n)}$. Since the call payoff $u(s_T, T, \lambda) = (s_T - p)^+$ at $t = T$ hence $(s_T - p)^+ \dfrac{e^{-\lambda(T-T)}}{X(\ln s_T)} \dfrac{s_T^{n-1}}{\Gamma(n)} = (s_T - p)^+$. It follows that $X(\ln s_T) = \dfrac{s_T^{n-1}}{\Gamma(n)}$. Thus we have $u(s,t,\lambda) = (s_T - p)^+ \dfrac{e^{-\lambda(T-t)}}{\Gamma(n)} \dfrac{\Gamma(n)}{s_T^{n-1}} s^{n-1}$ or $u(s,t,\lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \left(\dfrac{s}{s_T}\right)^{n-1}$ with the parametric interest from (3.14) with $\alpha = -n$

$$\lambda = (2-n)r - \frac{(n^2 - 3n + 2)}{2}\sigma^2.$$

Thus we obtained (3.8) as required. Now consider the call payoff $u(s_T, T, \lambda)$ at $t = T$ then from (3.8),

$$u(s_T, T, \lambda) = (s_T - p)^+ e^{-\lambda(T-T)} \left(\frac{s_T}{s_T}\right)^{n-1} = (s_T - p)^+$$

with the same as the call payoff in (1.2)

$$u(s_T, T) = (s_T - p)^+.$$

We see that even $u(s,t)$ in (1.3) is different from $u(s,t,\lambda)$ in (1.11) but they have the same call payoff.

Now consider (3.8) with $n = 1$ and we have $\lambda = r$ in (3.9) with $n = 1$. Then (3.8) reduces to

$$u(s,t,r) = (s_T - p)^+ e^{-r(T-t)}$$

with call payoff $u(s_T, T, r) = (s_T - p)^+$ at $t = T$.

Moreover for $n = 2$ in (3.9) $\lambda = 0$ and again from (3.8)

$$u(s, t, 0) = (s_T - p)\left(\frac{s}{s_T}\right)$$

with the call payoff

$$u(s_T, T, 0) = (s_T - p)^+\left(\frac{s_T}{s_T}\right) = (s_T - p)^+.$$

We see that the parametric interest λ is the generalization of the interest rate r that appears in the option price $u(S, t)$. ■

4. Conclusion

The main point of this research is focusing on the parametric interest λ which gives the conditions of the solutions of the Black-Scholes equation to be weak or strong solution that appear in (1.8), (1.10) and (1.11) of the Introduction part.

Moreover λ is the generalization of the interest rate r. Thus from (3.9) for $n = 1$ we have $\lambda = r$. Thus we obtained from (3.8)

$$u(s, t, r) = (s_T - p)^+ e^{-r(T-t)}$$

or $(s_T - p)^+ = u(s, t, r)e^{r(T-t)}$. That means the call payoff $(s_T - p)^+$ is equal to the the option price $u(s, t, r)$ put in the bank with the interest rate r at the time $T - t$.

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