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Regularity of a Semigroup of Transformations with Restricted Range that Preserves an Equivalence Relation and a Cross-Section

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Abstract For a fixed nonempty subset Y of X, let $T(X,Y)$ be the semigroup consisting of all transformations from X into Y. Let ρ be an equivalence relation on X, $\hat{\rho}$ the restriction of ρ on Y and R a cross-section of the partition $Y/\hat{\rho}$. We define

$$T(X,Y,\rho,R) = \{\alpha \in T(X,Y) : R\alpha \subseteq R \text{ and } (a,b) \in \rho \Rightarrow (a\alpha, b\alpha) \in \rho\}.$$

Then $T(X,Y,\rho,R)$ is a subsemigroup of $T(X,Y)$. In this paper, we describe regular elements in $T(X,Y,\rho,R)$, characterize when $T(X,Y,\rho,R)$ is a regular semigroup and investigate some classes of $T(X,Y,\rho,R)$ such as completely regular and inverse from which the results on $T(X,\rho,R)$ and $T(X,Y)$ can be recaptured easily when taking $Y = X$ and ρ to be the identity relation, respectively. Moreover, the description of unit-regularity on $T(X,\rho,R)$ is obtained.

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1. Introduction

For any a nonempty set X, denote by $T(X)$ the semigroup of all transformations from X into itself with composition. There is a well-known result on $T(X)$ stated that $T(X)$ is a regular semigroup which was shown in [1]. Additionally, Alarcao [2] characterized the unit-regularity of $T(X)$ in 1980. Several kinds of subsemigroups of $T(X)$ have been considered in different years. Especially, in 2003, Araújo and Konieczny [3] investigated a subsemigroup of $T(X)$ with respect to an equivalence relation ρ on X and a cross-section

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R of the partition X/ρ (i.e., each ρ-class contains exactly one element of R), namely $T(X,\rho,R)$, which is defined as follows:

$$T(X,\rho,R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } (a,b) \in \rho \Rightarrow (a\alpha, b\alpha) \in \rho\}.$$

Moreover, the authors determined the automorphism groups of centralizers of idempotents. Furthermore, they studied Green's relations, regularity, inverse and completely regular classes of $T(X,\rho,R)$ in 2004 [4].

Let Y be any subset of a set X. A subsemigroup $Fix(X,Y)$ of $T(X)$ is defined to be the set of all transformations on X which fix all elements in Y, that is,

$$Fix(X,Y) = \{\alpha \in T(X) : a\alpha = a \text{ for all } a \in Y\}.$$

In 2003, Honyam and Sanwong [5] showed that $Fix(X,Y)$ is a regular submonoid of $T(X)$. Later in 2007, Chaiya et al. [6] also studied this semigroup. They provided necessary and sufficient conditions for $Fix(X,Y)$ to be unit-regular.

For a nonempty subset Y of X, a subsemigroup $T(X,Y)$ of $T(X)$ was first considered by Symons [7] in 1975. He defined $T(X,Y)$ as a semigroup of all transformations on X whose ranges are contained in Y, that is,

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}.$$

Furthermore, he described all the automorphisms of $T(X,Y)$ and also determined when $T(X_1,Y_1)$ is isomorphic to $T(X_2,Y_2)$. Later in 2005, Nenthein et al. [8] provided the characterization when $T(X,Y)$ is regular. In 2008, Sanwong and Sommanee [9] studied other algebraic properties of $T(X,Y)$. They determined its Green's relations and obtained a class of maximal inverse subsemigroups of $T(X,Y)$. In addition, they introduced a new subsemigroup of $T(X,Y)$, denoted by $F(X,Y)$, defined as follows:

$$F(X,Y) = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}.$$

They proved that $F(X,Y)$ is the largest regular subsemigroup of $T(X,Y)$. In 2011, Sanwong [10] determined all maximal regular subsemigroups of $F(X,Y)$ when Y is a finite set.

Recently, Pookpienlert et al. [11] gave descriptions of Green's relations on the subsemigroup $T(X,Y,\rho,R)$ of $T(X,Y)$ which is defined as follows. Let ρ be an equivalence relation on X, $\hat{\rho}$ the restriction of ρ on Y (i.e., $\hat{\rho} = \rho \cap (Y \times Y)$), R a cross-section of the partition $Y/\hat{\rho}$ and define

$$T(X,Y,\rho,R) = \{\alpha \in T(X,Y) : R\alpha \subseteq R \text{ and } (a,b) \in \rho \Rightarrow (a\alpha, b\alpha) \in \rho\}.$$

If $Y = X$, then $T(X,Y,\rho,R) = T(X,\rho,R)$; and if $\rho = \triangle$, then $T(X,Y,\triangle,Y) = T(X,Y)$ where $\triangle = \{(x,x) : x \in X\}$ is the identity relation on X. Thus their results extend the results of Araújo and Konieczny [4] and of Sanwong and Sommanee [9] on Green's relations of $T(X,\rho,R)$ and $T(X,Y)$, respectively. Furthermore, they observed that $F(X,Y) \cap T(X,Y,\rho,R)$ is a subsemigroup of $T(X,Y,\rho,R)$, denoted by F, since it contains all constant maps whose images belong to R.

Our purposes are to characterize regular elements in $T(X,Y,\rho,R)$ and provide necessary and sufficient conditions for $T(X,Y,\rho,R)$ to be regular. Moreover, we characterize when F is the largest regular subsemigroup of $T(X,Y,\rho,R)$. In addition, we present some conditions for $T(X,Y,\rho,R)$ to be never a completely regular semigroup and an inverse semigroup. Finally, we provide the characterization of the unit-regularity of $T(X,Y,\rho,R)$.

2. Preliminaries

Let S be a semigroup. An element $a \in S$ is *regular* if there exists $x \in S$ such that $a = axa$, and S is called a *regular semigroup* if every element of S is regular. Moreover, a is said to be *completely regular* if there exists $x \in S$ in which $a = axa$ and $ax = xa$. If every element in S is completely regular, then S is called a *completely regular semigroup*. Furthermore, an element a' in S is said to be an *inverse* of a if $a = aa'a$ and $a' = a'aa'$. If every element in S has a unique inverse, then S is called an *inverse semigroup*. Another version is that S is an inverse semigroup if and only if it is regular and its idempotents commute (Howie [1]).

Let S be a monoid with identity 1. An element $u \in S$ is called a *unit* if $uu' = 1 = u'u$ for some $u' \in S$. Furthermore, S is said to be *unit-regular* if for each $a \in S$, there exists a unit element $u \in S$ in which $a = aua$.

In fact, completely regular semigroups, inverse semigroups and unit-regular semigroups are regular semigroups.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Furthermore, we write functions on the right, this means that for a composition $\alpha\beta$, α is applied first. For an equivalence relation ρ on A, if $a, b \in A$ we sometimes write $a \,\rho\, b$ instead of $(a, b) \in \rho$, and define $a\rho$ to be the equivalence class that contains a, that is, $a\rho = \{b \in A : b \,\rho\, a\}$. In addition, the universal relation on A is denoted by ω. That is $\omega = A \times A$.

It is known that $\alpha \in T(X)$ is an idempotent if and only if $x\alpha = x$ for all $x \in X\alpha$. Moreover, $T(X)$ is a semigroup with an identity, the identity map. But for $T(X, Y, \rho, R)$, this is not always true as shown in the following example.

Example 2.1. Let $X = \{1, 2, 3, 4, 5\}, Y = \{1, 3\}$ and $X/\rho = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Then $Y/\hat{\rho} = \{\{1\}, \{3\}\}$ and let $R = Y$. Suppose that ε is an identity in $T(X, Y, \rho, R)$. Consider $\alpha \in T(X, Y, \rho, R)$ defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

We see that $(5\varepsilon)\alpha = 5(\varepsilon\alpha) = 5\alpha = 3$ which implies that $5 = 5\varepsilon \in Y$, a contradiction.

However, we provide necessary and sufficient conditions for $T(X, Y, \rho, R)$ possessing an identity in Section 5.

In general, $T(X, Y, \rho, R)$ is not a regular semigroup as shown in the example below.

Example 2.2. Let $X = \{1, 2, 3, 4, 5\}, Y = \{1, 2, 3, 4\}$ and $X/\rho = \{\{1, 2, 3\}, \{4, 5\}\}$. Then $Y/\hat{\rho} = \{\{1, 2, 3\}, \{4\}\}$ and let $R = \{1, 4\}$. Define $\alpha \in T(X, Y, \rho, R)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 1 & 3 \end{pmatrix}.$$

Suppose that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, Y, \rho, R)$. We see that $3 = 5\alpha = 5(\alpha\beta\alpha) = (3\beta)\alpha$ which implies that $5 = 3\beta \in Y$, a contradiction.

The following lemmas are used in characterizing the regularity of $T(X, Y, \rho, R)$.

Lemma 2.3. *If Y is a cross-section of X/ρ, then $T(X, Y, \rho, R)$ is isomorphic to $T(Y)$.*

Proof. Assume that Y is a cross-section of X/ρ. Then $Y = R$ and for each $\alpha \in T(X, Y, \rho, R)$, we have $\alpha|_Y \in T(Y)$. So we define $\varphi : T(X, Y, \rho, R) \to T(Y)$ by $\alpha\varphi = \alpha|_Y$ for all $\alpha \in T(X, Y, \rho, R)$. Now, we show that φ is an isomorphism.

φ is injective: Let $\alpha, \beta \in T(X, Y, \rho, R)$ be such that $\alpha\varphi = \beta\varphi$. Then $\alpha|_Y = \beta|_Y$. For each $a \in X$, we have $a \,\rho\, r$ for some unique $r \in R$. From $\alpha, \beta \in T(X, Y, \rho, R)$, we obtain that $(a\alpha, r\alpha), (a\beta, r\beta) \in \rho$. So $a\alpha = r\alpha$ and $a\beta = r\beta$ since Y is a cross-section of X/ρ. Thus $a\alpha = r\alpha = r\alpha|_Y = r\beta|_Y = r\beta = a\beta$ for all $a \in X$. Hence $\alpha = \beta$.

φ is surjective: Let $\alpha \in T(Y)$. We define $\beta \in T(X, Y, \rho, R)$ on each ρ-class as follows. Let $a\rho \in X/\rho$. Then there is a unique $r_a \in Y$ such that $a \,\rho\, r_a$ and define $a\beta = r_a\alpha$. Thus $\beta|_Y = \alpha$.

From the fact that $(\alpha\beta)|_Y = (\alpha|_Y)(\beta|_Y)$ for all $\alpha, \beta \in T(X, Y, \rho, R)$, we obtain that $(\alpha\beta)\varphi = (\alpha\beta)|_Y = (\alpha|_Y)(\beta|_Y) = (\alpha\varphi)(\beta\varphi)$. Therefore, $T(X, Y, \rho, R)$ is isomorphic to $T(Y)$. ∎

Lemma 2.4. [2, Proposition 5] *Let X be a nonempty set. Then $T(X)$ is unit-regular if and only if X is finite.*

Lemma 2.5. [6, Theorem 5.2] *Let Y be a fixed subset of X. Then $Fix(X, Y)$ is unit-regular if and only if $X\backslash Y$ is finite.*

Lemma 2.6. [3, Theorem 3.1] *Let $\alpha \in T(X, \rho, R)$. Then α is a unit if and only if α is a bijection.*

3. Regularity of $T(X, Y, \rho, R)$

Let Z be a nonempty subset of Y. An equivalence relation $\hat{\rho}$ on Y induces a partition $Z/\hat{\rho}$ of Z where $Z/\hat{\rho} = \{r\hat{\rho} \cap Z : r \in R \text{ and } r\hat{\rho} \cap Z \neq \emptyset\}$. For $\alpha \in T(X, Y, \rho, R)$, we define $\blacktriangledown\alpha$ and $\blacktriangledown^Y\alpha$ by

$$\blacktriangledown\alpha = \{(x\rho)\alpha : x \in X\} \text{ and } \blacktriangledown^Y\alpha = \{(r\hat{\rho})\alpha : r \in R\}.$$

Note that in Example 2.2, we have $X\alpha/\hat{\rho} = \{\{1, 2, 3\}\}$ is not a subset of $\blacktriangledown^Y\alpha = \{\{1\}, \{1, 2\}\}$ which destroys the regularity of α. The following theorem describes a regular element in $T(X, Y, \rho, R)$.

Theorem 3.1. *Let α be any element in $T(X, Y, \rho, R)$. Then α is regular if and only if $X\alpha/\hat{\rho} \subseteq \blacktriangledown^Y\alpha$.*

Proof. Assume that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, Y, \rho, R)$. Let $r\hat{\rho} \cap X\alpha \in X\alpha/\hat{\rho}$. Then $r\hat{\rho} \cap X\alpha \neq \emptyset$ and so there exists $b \in r\hat{\rho} \cap X\alpha$, hence $b \in r\hat{\rho}$ and $b = a\alpha$ for some $a \in X$. From $\beta \in T(X, Y, \rho, R)$, we obtain that $(r\hat{\rho})\beta \subseteq s\hat{\rho}$ for some $s \in R$. We prove that $r\hat{\rho} \cap X\alpha = (s\hat{\rho})\alpha$. Consider $b = a\alpha = a(\alpha\beta\alpha) = (b\beta)\alpha \in (s\hat{\rho})\alpha$, we obtain $b \in r\hat{\rho} \cap (s\hat{\rho})\alpha \neq \emptyset$, thus $(s\hat{\rho})\alpha \subseteq r\hat{\rho}$ since all elements in $(s\hat{\rho})\alpha$ belong to the same class. Hence $(s\hat{\rho})\alpha \subseteq r\hat{\rho} \cap X\alpha$. Now, if $x\alpha \in r\hat{\rho} \cap X\alpha$, we have $(x\alpha)\beta \in (r\hat{\rho})\beta \subseteq s\hat{\rho}$. It follows that $x\alpha = x(\alpha\beta\alpha) = (x\alpha\beta)\alpha \in (s\hat{\rho})\alpha$, that is, $r\hat{\rho} \cap X\alpha \subseteq (s\hat{\rho})\alpha$ and $r\hat{\rho} \cap X\alpha = (s\hat{\rho})\alpha \in \blacktriangledown^Y\alpha$ as required.

Conversely, assume that $X\alpha/\hat{\rho} \subseteq \blacktriangledown^Y\alpha$. Let $r_0 \in R$ be fixed and define $\beta \in T(X, Y, \rho, R)$ on each ρ-class as follows. Let $x\rho \in X/\rho$.

If $x\rho \cap X\alpha = \emptyset$, then define $a\beta = r_0$ for all $a \in x\rho$. So $(x\rho)\beta = \{r_0\} \subseteq r_0\hat{\rho}$.

If $x\rho \cap X\alpha \neq \emptyset$, then $x\rho \cap Y \neq \emptyset$. Let $x\rho \cap Y = r\hat{\rho}$ for some $r \in R$ and $a \in x\rho$. Since $\emptyset \neq r\hat{\rho} \cap X\alpha \in X\alpha/\hat{\rho} \subseteq \blacktriangledown^Y\alpha$, we obtain that $r\hat{\rho} \cap X\alpha = (s\hat{\rho})\alpha$ for some $s \in R$. If $a \in X\alpha \subseteq Y$, then $a \in r\hat{\rho} \cap X\alpha = (s\hat{\rho})\alpha$. We choose $b_a \in s\hat{\rho}$ (if $a = r$, we may choose $b_a = s$) such that $b_a\alpha = a$ and define $a\beta = b_a$. If $a \notin X\alpha$, we define $a\beta = s$. By the definition of β, we have $(x\rho)\beta \subseteq s\hat{\rho}$ and $r\beta = s$.

Since $x\rho$ is arbitrary, we conclude that $\beta \in T(X, Y, \rho, R)$. To see that $\alpha = \alpha\beta\alpha$, let $a \in X$. Then $a\alpha \in X\alpha$ and so $(a\alpha)\rho \cap X\alpha \neq \emptyset$. By the definition of β, we obtain

$a(\alpha\beta\alpha) = (a\alpha)\beta\alpha = b_{a\alpha}\alpha = a\alpha$. Thus $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, Y, \rho, R)$, that is, α is regular. ■

If $Y = X$ in Theorem 3.1, then $T(X, Y, \rho, R) = T(X, \rho, R), \hat{\rho} = \rho$ and $\blacktriangledown^Y\alpha = \blacktriangledown\alpha$ for all $\alpha \in T(X, Y, \rho, R)$, so we have the following corollary.

Corollary 3.2. [4, Theorem 3.1] *Let α be any element in $T(X, \rho, R)$. Then α is regular if and only if $X\alpha/\rho \subseteq \blacktriangledown\alpha$.*

If ρ in Theorem 3.1 is the identity relation, then $T(X, Y, \rho, R) = T(X, Y), R = Y$ and $x\rho = \{x\}$ for all $x \in X$. Thus $X\alpha/\hat{\rho} = \{\{r\} : r \in X\alpha\}$ and $\blacktriangledown^Y\alpha = \{\{s\alpha\} : s \in Y\}$. It follows that $X\alpha/\hat{\rho} \subseteq \blacktriangledown^Y\alpha$ is equivalent to $X\alpha \subseteq Y\alpha$. Therefore, we obtain the corollary below.

Corollary 3.3. [8, Theorem 2.1] *Let α be any element in $T(X, Y)$. Then α is regular if and only if $X\alpha \subseteq Y\alpha$.*

In general, F is not a regular subsemigroup of $T(X, Y, \rho, R)$ as shown in the example below.

Example 3.4. Let $X = \{1, 2, 3, 4, 5, 6, 7\}, Y = \{1, 2, 3, 4, 5\}$ and $X/\rho = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$. Then $Y/\hat{\rho} = \{\{1, 2, 3\}, \{4, 5\}\}$ and let $R = \{1, 4\}$. Define $\alpha \in F$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 2 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

We observe that $X\alpha/\hat{\rho} = \{\{1, 2, 3\}\} \nsubseteq \{\{1, 2\}, \{1, 3\}\} = \blacktriangledown^Y\alpha$. By Theorem 3.1, α is not regular.

However, we have every regular element $\alpha \in T(X, Y, \rho, R)$ is contained in F since $X\alpha = X\alpha\beta\alpha = (X\alpha\beta)\alpha \subseteq Y\alpha$ where $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, Y, \rho, R)$.

The lemma below is needed in describing the regularity of F.

Lemma 3.5. *If $\alpha \in T(X, Y, \rho, R)$ is regular, then there exists $\beta \in F$ such that $\alpha = \alpha\beta\alpha$.*

Proof. Assume that α is regular. Then $\alpha = \alpha\gamma\alpha$ for some $\gamma \in T(X, Y, \rho, R)$ and so $\alpha = \alpha\gamma\alpha = (\alpha\gamma\alpha)\gamma\alpha = \alpha(\gamma\alpha\gamma)\alpha$. We show that $\gamma\alpha\gamma \in F$. Since $X\gamma\alpha\gamma = (X\gamma)\alpha\gamma \subseteq Y\alpha\gamma \subseteq X\alpha\gamma = X(\alpha\gamma\alpha)\gamma = (X\alpha)\gamma\alpha\gamma \subseteq Y\gamma\alpha\gamma$, we obtain that $\gamma\alpha\gamma \in F(X, Y) \cap T(X, Y, \rho, R) = F$. So we conclude that $\alpha = \alpha\beta\alpha$ where $\beta = \gamma\alpha\gamma \in F$. ■

An equivalence relation ρ on X is a *T-relation* if there is at most one ρ-class containing two or more elements. If there is $n \geq 1$ such that each ρ-class has at most n elements, we say that ρ is *n-bounded*.

The following theorem characterizes the regularity of F.

Theorem 3.6. *F is a regular subsemigroup of $T(X, Y, \rho, R)$ if and only if $\hat{\rho}$ is 2-bounded or a T-relation. In this case, F is the largest regular subsemigroup of $T(X, Y, \rho, R)$.*

Proof. Assume that F is a regular subsemigroup of $T(X, Y, \rho, R)$. Suppose that $\hat{\rho}$ is not 2-bounded. So there is $r \in R$ such that $|r\hat{\rho}| \geq 3$. Let $a_1, a_2 \in r\hat{\rho}$ and $a_1 \neq r \neq a_2$. If R has exactly one element, we see that $\hat{\rho}$ is a T-relation. Now, suppose that R has more than one element. Let $s \in R$ be such that $s \neq r$. We prove that $|s\hat{\rho}| = 1$ by supposing that this is false, so $|s\hat{\rho}| \geq 2$. Let $b \in s\hat{\rho}\backslash\{s\}$ and define $\alpha \in F$ by

$$a\alpha = \begin{cases} a_1 & , a \in \{a_1, a_2\}; \\ a_2 & , a = b; \\ r & , \text{otherwise}. \end{cases}$$

Then we observe that $X\alpha/\hat{\rho} = \{\{r, a_1, a_2\}\}$ and either $\blacktriangledown^Y\alpha = \{\{r, a_1\}, \{r, a_2\}\}$ or $\blacktriangledown^Y\alpha = \{\{r, a_1\}, \{r, a_2\}, \{r\}\}$. It follows that $X\alpha/\hat{\rho} \nsubseteq \blacktriangledown^Y\alpha$. By Theorem 3.1, we obtain that α is not regular which is a contradiction. So $|s\hat{\rho}| = 1$ for all $s \neq r$. Hence $\hat{\rho}$ is a T-relation.

Conversely, assume that $\hat{\rho}$ is 2-bounded or a T-relation and let α be any element in F. To show that $X\alpha/\hat{\rho} \subseteq \blacktriangledown^Y\alpha$, let $r\hat{\rho} \cap X\alpha \in X\alpha/\hat{\rho}$. Since $\alpha \in F$, there exists $s \in R$ such that $r = s\alpha \in r\hat{\rho} \cap X\alpha$.

Case 1: $\hat{\rho}$ is 2-bounded. Then $r\hat{\rho}$ has at most two elements. If $r\hat{\rho} \cap X\alpha = \{r\}$, then $\emptyset \neq (s\hat{\rho})\alpha \subseteq r\hat{\rho} \cap X\alpha = \{r\}$, thus $r\hat{\rho} \cap X\alpha = (s\hat{\rho})\alpha \in \blacktriangledown^Y\alpha$. If $r\hat{\rho} \cap X\alpha = \{r, y\}$ where $y \neq r$, then $y \in X\alpha = Y\alpha$ and thus $y \in (t\hat{\rho})\alpha$ for some $t \in R$, so $r\hat{\rho} \cap X\alpha = \{r, y\} = (t\hat{\rho})\alpha \in \blacktriangledown^Y\alpha$.

Case 2: $\hat{\rho}$ is a T-relation. If $r\hat{\rho} \cap X\alpha = \{r\}$, then, as in Case 1, we have $r\hat{\rho} \cap X\alpha \in \blacktriangledown^Y\alpha$. If $r\hat{\rho} \cap X\alpha$ has at least two elements, then $|r\hat{\rho}| \geq 2$ and $|t\hat{\rho}| = 1$ for all $r \neq t \in R$ since $\hat{\rho}$ is a T-relation. This implies $|(t\hat{\rho})\alpha| = 1$ which forces $(r\hat{\rho})\alpha = r\hat{\rho} \cap X\alpha$. Thus $r\hat{\rho} \cap X\alpha \in \blacktriangledown^Y\alpha$.

From the above two cases, we conclude that α is regular by Theorem 3.1. Thus by Lemma 3.5, we have $\alpha = \alpha\beta\alpha$ for some $\beta \in F$ and hence F is a regular semigroup. Finally, we have known that every regular element in $T(X, Y, \rho, R)$ is contained in F. Therefore, F is the largest regular subsemigroup of $T(X, Y, \rho, R)$. ■

If $Y = X$, then $F = F(X, Y) \cap T(X, Y, \rho, R) = T(X) \cap T(X, \rho, R) = T(X, \rho, R)$ and $\hat{\rho} = \rho$. By Theorem 3.6, we have the following corollary.

Corollary 3.7. [4, Theorem 3.7] *The semigroup $T(X, \rho, R)$ is regular if and only if ρ is 2-bounded or a T-relation.*

If ρ is the identity relation, then $\hat{\rho}$ is 2-bounded and a T-relation. So by applying ρ to be the identity relation in Theorem 3.6, we obtain the following corollary.

Corollary 3.8. [9, Theorem 2.4] *$F(X, Y)$ is the largest regular subsemigroup of $T(X, Y)$.*

As shown in Example 2.2, $T(X, Y, \rho, R)$ is not a regular semigroup. The following theorem describes when $T(X, Y, \rho, R)$ is regular.

Theorem 3.9. *$T(X, Y, \rho, R)$ is regular if and only if one of the following statements holds:*

(i) *$|Y| = 1$ or Y is a cross-section of X/ρ;*
(ii) *$Y = X$; and ρ is 2-bounded or a T-relation.*

Proof. Let $T(X, Y, \rho, R)$ be regular. Suppose that $|Y| \geq 2$ and Y is not a cross-section of X/ρ. So there exists $x_0 \in X$ such that $x_0\rho \cap Y = \emptyset$ or $|x_0\rho \cap Y| \geq 2$. Now, we prove that $Y = X$ by supposing that this is false, so $Y \subsetneq X$.

If $x_0\rho \cap Y = \emptyset$, then since $|Y| \geq 2$, we can choose $r \in R$ and $y \in Y$ such that $r \neq y$. Define $\alpha \in T(X, Y, \rho, R)$ by

$$a\alpha = \begin{cases} y & , a \in x_0\rho; \\ r & , a \notin x_0\rho. \end{cases}$$

Then either $X\alpha/\hat{\rho} = \{\{r, y\}\}$ or $X\alpha/\hat{\rho} = \{\{r\}, \{y\}\}$; and $\blacktriangledown^Y\alpha = \{\{r\}\}$. Thus $X\alpha/\hat{\rho} \nsubseteq \blacktriangledown^Y\alpha$. By Theorem 3.1, α is not regular which is a contradiction.

If $|x_0\rho \cap Y| \geq 2$, then there exist $r, y \in x_0\rho \cap Y$ such that $r \in R$ and $r \neq y$. Since $Y \subsetneq X$, we obtain that $X \backslash Y \neq \emptyset$. So we define $\alpha \in T(X, Y, \rho, R)$ by

$$a\alpha = \begin{cases} r & , a \in Y; \\ y & , a \in X \backslash Y. \end{cases}$$

Then $X\alpha/\hat{\rho} = \{\{r, y\}\} \nsubseteq \{\{r\}\} = \blacktriangledown^Y\alpha$ which implies that α is not regular, a contradiction.

Thus $Y = X$ which implies that $T(X, \rho, R) = T(X, Y, \rho, R)$ is regular. By Corollary 3.7, we have ρ is 2-bounded or a T-relation.

Conversely, assume that the conditions hold. If $|Y| = 1$, then $T(X, Y, \rho, R)$ contains exactly one element, the constant map, and it is regular. If Y is a cross-section of X/ρ, then $T(X, Y, \rho, R)$ is isomorphic to $T(Y)$ by Lemma 2.3, and so it is regular. Finally, if $Y = X$; and ρ is 2-bounded or a T-relation, then $T(X, Y, \rho, R) = T(X, \rho, R)$ is regular by Corollary 3.7. ∎

The following corollary is a direct consequence of Theorem 3.9 by replacing ρ with the identity relation.

Corollary 3.10. [8, Corollary 2.2] *The semigroup $T(X, Y)$ is regular if and only if $|Y| = 1$ or $Y = X$.*

4. Completely Regular $T(X, Y, \rho, R)$ and Inverse $T(X, Y, \rho, R)$

Araújo and Konieczny [4] determined that $T(X, \rho, R)$ is never a completely regular semigroup (if $|X| \geq 4$) and an inverse semigroup (if $|X| \geq 3$). Here, we aim to find some conditions for $T(X, Y, \rho, R)$ to be never a completely regular semigroup and an inverse semigroup. This leads to new results on the semigroup $T(X, Y)$ when replacing ρ with the identity relation.

We start with the following lemma.

Lemma 4.1. *If $\alpha \in T(X, Y, \rho, R)$ is completely regular, then $X\alpha = X\alpha^2$.*

Proof. Let $\alpha \in T(X, Y, \rho, R)$ be completely regular. Then there exists $\beta \in T(X, Y, \rho, R)$ such that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$. Thus $\alpha = (\alpha\beta)\alpha = \beta\alpha^2$ which implies that $X\alpha = (X\beta)\alpha^2 \subseteq X\alpha^2$. And $X\alpha^2 = (X\alpha)\alpha \subseteq X\alpha$, so we obtain $X\alpha = X\alpha^2$. ∎

Theorem 4.2. *Suppose that Y satisfies one of the following statements:*

(i) $|Y| = 3$ *and* $Y \subsetneq X$;

(ii) $|Y| \geq 4$.

Then $T(X, Y, \rho, R)$ is not a completely regular semigroup.

Proof. Assume that $|Y| \geq 3$. We aim to define $\alpha \in T(X, Y, \rho, R)$ such that $X\alpha \neq X\alpha^2$. We consider two cases.

Case 1: there are at least three $\hat{\rho}$-classes. Then there are $r\hat{\rho}, s\hat{\rho}$ and $t\hat{\rho}$ such that r, s, t are all distinct elements in R. Define $\alpha \in T(X, Y, \rho, R)$ by

$$\alpha = \begin{pmatrix} X \backslash r\rho & r\rho \\ s & t \end{pmatrix}.$$

Thus $X\alpha = \{s, t\} \neq \{s\} = X\alpha^2$.

Case 2: there are at most two $\hat{\rho}$-classes. Since $|Y| \geq 3$, there exists $r \in R$ such that $r\hat{\rho}$ contains at least two elements. Let $y \in r\hat{\rho}$ be such that $y \neq r$. If $|Y| = 3$ and $Y \subsetneq X$; or $|Y| \geq 4$, then there exists $x \in X \backslash R$ such that $x \neq y$. Define $\alpha \in T(X, Y, \rho, R)$ by

$$\alpha = \begin{pmatrix} X \backslash \{x\} & x \\ r & y \end{pmatrix}.$$

Thus $X\alpha = \{r, y\} \neq \{r\} = X\alpha^2$.

From the two cases described above, we obtain that α defined in each case satisfies $X\alpha \neq X\alpha^2$. By Lemma 4.1, α is not completely regular. Therefore, $T(X, Y, \rho, R)$ is not a completely regular semigroup. ∎

For the case $|Y| = 1$ or 2, or $|Y| = |X| = 3$, we observe that if $|Y| = 1$; or $|Y| = |X| = 2$; or $|Y| = 2$, $Y \subsetneq X$, $\hat{\rho}$ is the identity relation on Y and there are two ρ-classes; or $|Y| = |X| = 3$ and $X/\rho = \{\{r, x\}, \{s\}\}$, then $T(X, Y, \rho, R)$ is completely regular. For the other cases, $T(X, Y, \rho, R)$ is not completely regular.

As a direct consequence of Theorem 4.2 and by taking $Y = X$, we obtain the following corollary.

Corollary 4.3. [4, Theorem 5.2] *If $|X| \geq 4$, then $T(X, \rho, R)$ is not a completely regular semigroup.*

Replacing ρ with the identity relation in Theorem 4.2, we obtain the following corollary.

Corollary 4.4. *If $|Y| \geq 3$, then $T(X, Y)$ is not a completely regular semigroup.*

Proof. By replacing ρ with the identity relation in Theorem 4.2, we have if $|Y| = 3$ and $Y \subsetneq X$; or $|Y| \geq 4$, then $T(X, Y) = T(X, Y, \rho, R)$ is not a completely regular semigroup. For the case $Y = X$ has three elements, $T(X, Y) = T(X)$ is not a completely regular semigroup since $\alpha = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_1 & a_2 \end{pmatrix} \in T(X)$ satisfies $X\alpha \neq X\alpha^2$. ∎

Theorem 4.5. *Suppose that Y satisfies one of the following statements:*

(i) $|Y| = 2$ *and* $Y \subsetneq X$*;*
(ii) $|Y| \geq 3$.

Then $T(X, Y, \rho, R)$ is not an inverse semigroup.

Proof. Assume that $|Y| \geq 2$. We aim to define idempotents $e, f \in T(X, Y, \rho, R)$ such that $ef \neq fe$. There are two possible cases to consider.

Case 1: there are at least two $\hat{\rho}$-classes. Let $r, s \in R$ be such that $r \neq s$. Define $e, f \in T(X, Y, \rho, R)$ by $xe = r$ and $xf = s$ for all $x \in X$. Then e, f are idempotents and $r(ef) = s \neq r = r(fe)$. Thus $ef \neq fe$.

Case 2: there is only one $\hat{\rho}$-class. Let $R = \{r\}$. Then $|r\hat{\rho}| \geq 2$. If $|Y| = 2$ and $Y \subsetneq X$, then $r\hat{\rho} = \{r, y\}$ in which $y \neq r$ and there exists $x \in X \backslash Y$. Define $e, f \in T(X, Y, \rho, R)$ by

$$e = \begin{pmatrix} X\backslash\{y\} & y \\ r & y \end{pmatrix} \text{ and } f = \begin{pmatrix} X\backslash\{x, y\} & \{x, y\} \\ r & y \end{pmatrix}.$$

Then e, f are idempotents and $x(ef) = r \neq y = x(fe)$. Thus $ef \neq fe$. If $|Y| \geq 3$, then $|r\hat{\rho}| \geq 3$ and so there are $y_1, y_2 \in r\hat{\rho}$ such that r, y_1, y_2 are all distinct. Define $e, f \in T(X, Y, \rho, R)$ by

$$e = \begin{pmatrix} X\backslash\{y_1, y_2\} & \{y_1, y_2\} \\ r & y_1 \end{pmatrix} \text{ and } f = \begin{pmatrix} X\backslash\{y_1, y_2\} & \{y_1, y_2\} \\ r & y_2 \end{pmatrix}.$$

Then e, f are idempotents and $y_1(ef) = y_2 \neq y_1 = y_1(fe)$. Hence $ef \neq fe$.

As the fact that all idempotents of an inverse semigroup commute, it follows that $T(X, Y, \rho, R)$ is not an inverse semigroup. ∎

We observe that if $|Y| = 1$; or $|Y| = |X| = 2$ and $\rho = \omega$, then $T(X, Y, \rho, R)$ is an inverse semigroup. But if $|Y| = |X| = 2$ and $\rho = \triangle$, then $T(X, Y, \rho, R)$ is not an inverse semigroup.

As a consequence of Theorem 4.5, we have the following corollaries.

Corollary 4.6. [4, Theorem 5.1] *If $|X| \geq 3$, then $T(X, \rho, R)$ is not an inverse semigroup.*

Corollary 4.7. *If $|Y| \geq 2$, then $T(X, Y)$ is not an inverse semigroup.*

Proof. By replacing ρ with the identity relation in Theorem 4.5, we have if $|Y| = 2$ and $Y \subsetneq X$; or $|Y| \geq 3$, then $T(X, Y) = T(X, Y, \rho, R)$ is not an inverse semigroup. For the case $Y = X$ has two elements, $T(X, Y) = T(X)$ is not an inverse semigroup since the two constant maps are idempotents which do not commute. ∎

5. Unit-Regularity of $T(X, Y, \rho, R)$

In this section, we characterize when $T(X, Y, \rho, R)$ possesses an identity. Then the unit-regularity of such semigroups are investigated, and this gives the description for the unit-regularity of $T(X, \rho, R)$.

Note that if $a \in X$ such that $a\rho \cap Y \neq \emptyset$, then there exists exactly one $r \in R$ such that $a \, \rho \, r$, which will be denoted by r_a.

Theorem 5.1. *$T(X, Y, \rho, R)$ has an identity if and only if one of the following statements holds:*

(i) *$|Y| = 1$ or $Y = X$;*
(ii) *Y is a cross-section of X/ρ.*

Proof. Assume that $T(X, Y, \rho, R)$ has an identity ε. Suppose that $|Y| \geq 2$ and $Y \subsetneq X$. From ε is an identity, we obtain that $\varepsilon\alpha = \alpha = \alpha\varepsilon$ for all $\alpha \in T(X, Y, \rho, R)$. Let $\alpha \in T(X, Y, \rho, R)$ be such that $X\alpha = Y$ (for example, fix $r_0 \in R$ and define

$$a\alpha = \begin{cases} a & , a \in Y; \\ r_a & , a \in X \backslash Y \text{ and } a\rho \cap Y \neq \emptyset; \\ r_0 & , a \in X \backslash Y \text{ and } a\rho \cap Y = \emptyset, \end{cases}$$

thus $\alpha \in T(X, Y, \rho, R)$ and $X\alpha = Y$). Then $Y\varepsilon = (X\alpha)\varepsilon = X(\alpha\varepsilon) = X\alpha = Y$ which implies that $X\varepsilon = Y$. Since ε is an idempotent, we have $y\varepsilon = y$ for all $y \in Y$. Now, we prove that Y is a cross-section of X/ρ, that is, $|x\rho \cap Y| = 1$ for all $x \in X$. Suppose that this is false, so there exists $x_0 \in X$ such that $x_0\rho \cap Y = \emptyset$ or $|x_0\rho \cap Y| \geq 2$.

If $x_0\rho \cap Y = \emptyset$, then we assume that $x_0\varepsilon = y_0 \notin x_0\rho$ for some $y_0 \in Y$. Since $|Y| \geq 2$, we choose $y_1 \in Y$ such that $y_1 \neq y_0$. Define $\beta \in T(X, Y, \rho, R)$ by

$$a\beta = \begin{cases} y_1 & , a \in x_0\rho; \\ a\varepsilon & , a \notin x_0\rho. \end{cases}$$

We observe that $x_0\varepsilon\beta = (x_0\varepsilon)\beta = y_0\beta = y_0\varepsilon = y_0 \neq y_1 = x_0\beta$, this leads to a contradiction since ε is an identity.

If $|x_0\rho \cap Y| \geq 2$, then there exist $y_1, y_2 \in x_0\rho \cap Y$ such that $y_1 \in R$ and $y_1 \neq y_2$. Since $Y \subsetneq X$, there exists $x_1 \in X \backslash Y$ and let $x_1\varepsilon = y_0$ for some $y_0 \in Y$. Define $\beta \in T(X, Y, \rho, R)$ by

$$a\beta = \begin{cases} y_1 & , a \in X \backslash \{x_1\}; \\ y_2 & , a = x_1. \end{cases}$$

We see that $x_1\varepsilon\beta = (x_1\varepsilon)\beta = y_0\beta = y_1 \neq y_2 = x_1\beta$ and hence $\varepsilon\beta \neq \beta$ which is a contradiction.

Therefore, $|x\rho \cap Y| = 1$ for all $x \in X$.

Conversely, if $|Y| = 1$ or $Y = X$, then $|T(X, Y, \rho, R)| = 1$ or $T(X, Y, \rho, R) = T(X, \rho, R)$, respectively and both have identities. If Y is a cross-section of X/ρ, then $T(X, Y, \rho, R)$ is isomorphic to $T(Y)$ by Lemma 2.3, and so $T(X, Y, \rho, R)$ has an identity. ■

If $|Y| = 1$, then $T(X, Y, \rho, R)$ has only one element, thus it is unit-regular. If Y is a cross-section of X/ρ, then $T(X, Y, \rho, R)$ is isomorphic to $T(Y)$. Therefore, in this case $T(X, Y, \rho, R)$ is unit-regular if and only if Y is finite by Lemma 2.4.

Now, we characterize the unit-regularity of $T(X, Y, \rho, R)$ when $Y = X$. In this case, $T(X, Y, \rho, R) = T(X, \rho, R)$.

Lemma 5.2. *If $T(X, \rho, R)$ is unit-regular and $\rho \neq \omega$, then each ρ-class has the same size.*

Proof. Assume that $T(X, \rho, R)$ is unit-regular and $\rho \neq \omega$. We show that each ρ-class has the same size, by supposing that this is false. So there exist $a, b \in X$ such that $|a\rho| \neq |b\rho|$. For convenience, we assume that $|a\rho| = I < J = |b\rho|$. Thus there are an injective map $\varphi : a\rho \to b\rho$ with $r_a\varphi = r_b$ and a surjective map $\psi : b\rho \to a\rho$ with $r_b\psi = r_a$. Now, we define $\alpha \in T(X, \rho, R)$ by

$$x\alpha = \begin{cases} x\varphi & , x \in a\rho; \\ x\psi & , x \in b\rho; \\ x & , \text{otherwise.} \end{cases}$$

Since $T(X, \rho, R)$ is unit-regular, by Lemma 2.6 there is a bijection $\beta \in T(X, \rho, R)$ such that $\alpha = \alpha\beta\alpha$. This implies $x\alpha = (x\alpha)\beta\alpha$ for all $x\alpha \in X\alpha$, that means $y\beta \in y\alpha^{-1}$ for all $y \in X\alpha$. Let $a\rho = \{a_i : i \in I\}$ and $b\rho = \{b_j : j \in J\}$. Since φ is injective, there is an injection $\theta : I \to J$ such that $a_i\varphi = b_{i\theta}$ for all $i \in I$. Then $b_{i\theta} = a_i\alpha \in X\alpha$, hence $b_{i\theta}\beta \in b_{i\theta}\alpha^{-1} = \{a_i\}$ for all $i \in I$. Since $|a\rho| < |b\rho|$, there is $b' \in b\rho\backslash(a\rho)\varphi$, so $b' \neq b_{i\theta}$ for all $i \in I$ and $b'\beta \in a\rho = \{a_i : i \in I\}$. Thus $b'\beta = a_k$ for some $k \in I$. It follows that $b'\beta = a_k = b_{k\theta}\beta$ which leads to a contradiction since β is injective. Therefore, each ρ-class has the same size. ■

Theorem 5.3. *$T(X, \rho, R)$ is unit-regular if and only if X is a finite set and one of the following statements holds.*

(i) *$\rho = \triangle$ or $\rho = \omega$;*
(ii) *each ρ-class has size two.*

Proof. Assume that $T(X, \rho, R)$ is unit-regular. It follows that $T(X, \rho, R)$ is regular and thus ρ is 2-bounded or a T-relation by Corollary 3.7. We first show that (i) or (ii) holds by supposing that $\rho \neq \omega$. By Lemma 5.2, we have each ρ-class has the same size. So we conclude that $\rho = \triangle$ or each ρ-class has size two.

Now, we prove that X is finite. If $\rho = \triangle$ or $\rho = \omega$, then $T(X, \rho, R) = T(X)$ or $T(X, \rho, R) = Fix(X, \{r_0\})$ where $R = \{r_0\}$. Since $T(X, \rho, R)$ is unit-regular, we obtain that X is a finite set by Lemmas 2.4 and 2.5. If each ρ-class has two elements, we suppose that X is an infinite set. Then choose $\{u, r_u\} = u\rho \in X/\rho$ and let $X' = X\backslash\{u, r_u\}$. So $|X/\rho| = |X'/\rho|$ since X is infinite. Thus there is a bijection $\phi : X/\rho \to X'/\rho$. Define $\alpha \in T(X, \rho, R)$ as follows. Let $a\rho \in X/\rho$. Then $a\rho = \{a, r_a\}$, so define $\{a, r_a\}\alpha =$

$\{a, r_a\}\phi \in X'/\rho$. Since ϕ is a bijection, we obtain α is an injective map onto X'. From α is unit-regular, there is a bijection $\beta \in T(X, \rho, R)$ such that $\alpha = \alpha\beta\alpha$. Since $x'\beta \in x'\alpha^{-1}$ for all $x' \in X'$, it follows that $X'\beta = X$ since α is injective which contradicts the injectivity of β. Therefore, X is a finite set.

Conversely, assume that X is finite and (i) or (ii) holds. If X is finite and (i) holds, then $T(X, \rho, R) = T(X)$ or $T(X, \rho, R) = Fix(X, \{r_0\})$ where $R = \{r_0\}$. Thus by Lemmas 2.4 and 2.5, $T(X, \rho, R)$ is unit-regular. If X is finite and each ρ-class has size two, then ρ is 2-bounded and hence $T(X, \rho, R)$ is regular by Corollary 3.7. Let $\alpha \in T(X, \rho, R)$. Then α is regular and so $X\alpha/\rho \subseteq \blacktriangledown\alpha$ by Corollary 3.2. We aim to find a bijection $\beta \in T(X, \rho, R)$ such that $\alpha = \alpha\beta\alpha$. Since $\{r\rho \cap X\alpha : r \in R \text{ and } r\rho \cap X\alpha \neq \emptyset\} = X\alpha/\rho \subseteq \blacktriangledown\alpha = \{(s\rho)\alpha : s \in R\}$, for each $r\rho \cap X\alpha \in X\alpha/\rho$, we can choose $s_r \in R$ such that $(s_r\rho)\alpha = r\rho \cap X\alpha$. Let $\mathscr{A} = \{r\rho : r \in R \text{ and } r\rho \cap X\alpha = \emptyset\}$ and $\mathscr{B} = X/\rho \backslash \{s_r\rho : (s_r\rho)\alpha = r\rho \cap X\alpha\}$. Thus $|\mathscr{A}| = |\mathscr{B}|$ since X is a finite set. Hence there is a bijection $\varphi : \mathscr{A} \to \mathscr{B}$. Now, we define $\beta \in T(X, \rho, R)$ on each ρ-class as follows. Let $r\rho = \{a, r\} \in X/\rho$ where $r \in R$.

If $r\rho \cap X\alpha \neq \emptyset$, then $r\rho \cap X\alpha \in X\alpha/\rho$ and so there exists $s_r \in R$ such that $(s_r\rho)\alpha = r\rho \cap X\alpha$ where $s_r\rho = \{b, s_r\}$ thus define $\{a, r\}\beta = \{b, s_r\}$.

If $r\rho \cap X\alpha = \emptyset$, then $r\rho \in \mathscr{A}$ and so define

$$\{a, r\}\beta = \{a, r\}\varphi.$$

Therefore, β is a bijection in $T(X, \rho, R)$ which implies that β is a unit by Lemma 2.6.

To see that $\alpha = \alpha\beta\alpha$, let $x \in X$. Then $x\alpha \in r\rho \cap X\alpha \in X\alpha/\rho$ for some $r \in R$ and so $x\alpha \in r\rho \cap X\alpha = (s_r\rho)\alpha = \{b, s_r\}\alpha$. If $x\alpha = b\alpha$, we have

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = (b\alpha)\beta\alpha = \begin{cases} b\alpha & , b\alpha = a; \\ r & , b\alpha = r, \end{cases}$$

which implies that $x\alpha\beta\alpha = b\alpha = x\alpha$. If $x\alpha = s_r\alpha$, then $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (s_r\alpha)\beta\alpha = (r\beta)\alpha = s_r\alpha = x\alpha$. ■

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References

[1] J.M. Howie, Fundamentals of Semigroup Theory, In: London Mathematical Society Monographs, New Series, 12 Oxford University Press, New York, 1995.

[2] H.D. Alarcao, Factorizable as a finiteness condition, Semigroup Forum 20 (1980) 281–282.

[3] J. Araújo, J. Konieczny, Automorphism groups of centralizers of idempotents, J. Algebra 269 (2003) 227–239.

[4] J. Araújo, J. Konieczny, Semigroups of transformations preserving an equivalence relation and a cross-section, Comm. Algebra 32 (2004) 1917–1935.

[5] P. Honyam, J. Sanwong, Semigroups of transformations with fixed sets, Quaest. Math. 36 (2013) 79–92.

[6] Y. Chaiya, P. Honyam, J. Sanwong, Maximal subsemigroups and finiteness conditions on transformation semigroups with fixed sets, Turk. J. Math. 41 (2017) 43–54.

[7] J.S.V. Symons, Some results concerning a transformation semigroup, J. Aust. Math. Soc. 19 (1975) 413–425.

[8] S. Nenthein, P. Youngkhong, Y. Kemprasit, Regular elements of some transformation semigroups, Pure Math. Appl. 16 (2005) 307–314.

[9] J. Sanwong, W. Sommanee, Regularity and Green's relations on a semigroup of transformations with restricted range, Int. J. Math. Math. Sci. 2008 (2008) Article ID 794013 https://doi.org/10.1155/2008/794013.

[10] J. Sanwong, The regular part of a semigroup of transformations with restricted range, Semigroup Forum 83 (2011) 134–146.

[11] C. Pookpienlert, P. Honyam, J. Sanwong, Green's relations on a semigroup of transformations with restricted range that preserves an equivalence relation and a cross-section, Mathematics 6 (8) (2018) https://doi.org/10.3390/math6080134.