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On Asymptotically Wijsman Deferred Statistical Equivalence of Sequence of Sets

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Abstract The concept of Wijsman deferred statistical convergence of sequences of sets was defined by authors in [M. Altınok, B. Inan, M. Küçükaslan, On deferred statistical convergence of sequences of sets in Metric space, TJMCS. (2015)]. In this paper, by considering this notation asymptotically Wijsman deferred statistical equivalence of sequences of sets is defined. Besides main properties of asymptotically Wijsman deferred statistical equivalence, some inclusion results are given under strict restrictions. The obtained theorems include some known results in literature.

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1. Introduction

In the paper [1] and [2], the concept of statistical convergence of real valued sequences is defined by Fast and Steinhaus independently. The idea of statistical convergence based on asymptotic density of the subset of natural numbers (see [3]). Over the years, under different names, statistical convergence has been studied and it is applied some problems in Fourier Analysis, Ergodic Theory and Number Theory such as [4–9] and [10], etc.

Let K be a subset of positive natural numbers $\mathbb{N}$ and $K(n)$ denotes the set

$$\{k \leq n : \ k \in K\}.$$

Asymptotic density of the subset K is denoted by $\delta(K)$ if the limit

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |K(n)|$$

exists, where $|K(n)|$ denotes the cardinality of $K(n)$.

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Definition 1.1. A real valued sequence $x = (x_n)$ is said to be *statistical convergent* to L, if for every $\varepsilon > 0$, the set

$$K(\varepsilon) := \{n : |x_n - L| \geq \varepsilon\}$$

has zero asymptotic density. In this case, we write $st - \lim_{n\to\infty} x_n = L$.

In 1980, by comparing convergence rate of nonnegative two sequences, Pobyvanets in [11] gave necessary and sufficient conditions for a nonnegative summability matrix A to have a property that, two nonnegative sequences $x = (x_n)$ and $y = (y_n)$ bounded below from zero, one has that $\frac{Ax}{Ay} \to 1$ whenever $\frac{x}{y} \to 1$.

After this paper, Fridy (in [12]) offered a new way to compare convergence rate of nonnegative sequences which are belonging l_1-space and c-space.

Later, Marouf (in [13]) continued to study this problem and gave necessary and sufficient conditions for a matrix A to be asymptotic regular matrix. In [14], under weak conditions, Jinlu gave some further results for asymptotic regular matrix. Some analogy results were established by Patterson in [15] by considering statistical convergence instead of ordinary convergence. Also, all results in [11] and [15] are extended to the setting where convergence is replaced by convergence with respect to ideal in [16] by Connor-Gümüş.

Another active research area about asymptotic equivalence is the preservation of asymptotic equivalence. This kind of research was began with a paper of Patterson and Savaş in [17] which they established necessary and sufficient conditions for sequences to be simultaneously asymptotic equivalence with respect to the statistical convergence, lacunary statistical convergence and strong lacunary convergence. By considering different summability methods, this kind of results are obtained by different authors such as [18–21] etc.

On the contrary to the convergence of point sequences, in literature, there are only well known three type convergence methods, and some generalizations of them, for sequence of sets: Wijsman, Hausdorff and Kuratowski (see [22–24]). In [25], Nuray and Rhoades defined Wijsman statistical convergence of sequence of sets. Later, in the paper [26] by using lacunary sequence, this concept is generalized to the lacunary statistical convergence and some parallel results in [25] is given by Ulusu and Nuray. Apart from these results, ideal convergence of sequences of sets has been given in [27].

Definition 1.2 ([13])**.** Two nonnegative sequence $x = (x_n)$ and $y = (y_n)$ are said to be *asymptotically equivalent* if

$$\lim_n \frac{x_n}{y_n} = 1. \tag{1.1}$$

It is denoted by $x \sim y$.

By combination of Definition 1.1 and Definition 1.2 asymtotically statistical equivalent with multiple L of two nonnegative sequences is defined by Patterson in [15] as follows:

Definition 1.3 ([15])**.** Two nonnegative sequences $x = (x_n)$ and $y = (y_n)$ are said to be *asymptotically equivalent with multiple* L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \quad \left| \frac{x_k}{y_k} - L \right| \geqslant \varepsilon \right\} \right| = 0, \tag{1.2}$$

exists and it is denoted by $x \overset{S_L}{\sim} y$.

Also, if $L=1$ in (1.2), the sequences x and y are simply called *asymptotically statistical equivalent* and it is denoted by $x \overset{S}{\sim} y$.

In this paper, our aim to give a typical generalization of Definition 1.3 by considering deferred statistical density which is defined in [28], and to obtained many general results than literature.

Let us recall deferred Cesàro mean. In 1932, R. P. Agnew in [29] defined the deferred Cesàro mean $D_{p,q}$ of a sequence $x=(x_n)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_k,$$

where $\{p(n)\}_{n\in\mathbb{N}}$ and $\{q(n)\}_{n\in\mathbb{N}}$ are sequences of positive natural numbers under which

$$p(n) < q(n) \quad \text{and} \quad \lim_{n\to\infty} q(n) = \infty. \tag{1.3}$$

Let (X,ρ) be a metric space. For any nonempty closed subsets $A_k, T \subseteq X$, we say that the sequence $A=(A_k)$ is *Wijsman convergent to the set* T if

$$\lim_{k\to\infty} d_x(A_k) = d_x(T), \tag{1.4}$$

exists for each $x\in X$. It is denoted by $W-\lim A_k = T$.

In (1.4), the symbol $d_x(B)$ denotes the distance of the point $x\in X$ to the set B such that

$$d_x(B) := \inf\{\rho(x,a) \ : \ a\in B\}.$$

Definition 1.4 ([30])**.** A sequence $A=(A_k)$ is said to be *Wijsman strongly deferred Cesàro summable to the set* T if for each $x\in X$,

$$\lim_{n\to\infty} \frac{1}{q-p} \sum_{k=p+1}^{q} |d_x(A_k) - d_x(T)| = 0$$

holds. In this case, we write $WD-\lim_{k\to\infty} A_k = T$.

Deferred density of $K\subset\mathbb{N}$ is defined as if the limit exists:

$$\delta_D(K) := \lim_{n\to\infty} \frac{1}{q-p} \big|\{p<k\le q : k\in K\}\big|.$$

Definition 1.5 ([30])**.** A sequence $A=(A_k)$ is said to be *Wijsman deferred statistically convergent to a set* T if for every $\varepsilon>0$ and $x\in X$,

$$\lim_{n\to\infty} \frac{1}{q-p} |\{p<k\le q \ : |d_x(A_k) - d_x(T)| \ge \varepsilon\}| = 0,$$

hold. In this case, we write $WDS-\lim_{k\to\infty} A_k = T$.

Asymptotically equivalent and asymptotically statistical equivalent of sequences of sets is defined by Ulusu and Nuray in [31] as follow:

Definition 1.6 ([31])**.** Let (X,d) be a metric space. For any nonempty closed subsets $A=(A_k)$, $B=(B_k)\subseteq X$ such that $d_x(A_k)>0$ and $d_x(B_k)>0$, for each $x\in X$. We say that the sequences $A=(A_k)$ and $B=(B_k)$ are

(i) *asymptotically equivalent(in the Wijsman sense) with mutiple* L if for each $x \in X$,

$$\lim_{n\to\infty} \frac{d_x(A_k)}{d_x(B_k)} = L, \tag{1.5}$$

and it is denoted by $A \overset{W_L}{\sim} B$.

(ii) *asymptotically statistical equivalent(in the Wijsman sense) with mutiple* L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\} \right| = 0, \tag{1.6}$$

and it is denoted by $A \overset{WS_L}{\sim} B$.

Definition 1.7. Let (X, d) be a metric space. For any non empty closed subsets $A = (A_k)$, $B = (B_k) \subseteq X$ such that $d_x(A_k) > 0$ and $d_x(B_k) > 0$ for each $x \in X$. We say that the sequences $A = (A_k)$ and $B = (B_k)$ are

(i) *asymptotically deferred equivalent(in the Wijsman sense) with mutiple* L if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{q-p} \sum_{k=p+1}^{q} \left(\frac{d_x(A_k)}{d_x(B_k)} - L \right) = 0 \tag{1.7}$$

and it is denoted by $A \overset{WD_L}{\sim} B$.

(ii) *asymptotically deferred statistical equivalent(in the Wijsman sense) with mutiple* L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{q-p} \left| \left\{ p < k \leq q : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\} \right| = 0 \tag{1.8}$$

and it is denoted by $A \overset{WDS_L}{\sim} B$.

Example 1.8. Let $X = \mathbb{R}^2$ and d be a metric on $\mathbb{R}^2$ and $A = (A_k)$, $B = (B_k)$ be sequences of sets as follows:

$$A_k := \begin{cases} \left\{ (x,y) : x^2 + (y-1)^2 = \frac{1}{k} \right\}, & \textit{if } p < k \leq q, k = m^2, \; m = 1, 2, ..., \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

and

$$B_k := \begin{cases} \left\{ (x,y) : x^2 + (y+1)^2 = \frac{1}{k} \right\}, & \textit{if } p < k \leq q, k = m^2, \; m = 1, 2, ..., \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$

For any $(x_0, 0) \in \mathbb{R}^2$, we have $d_{(x_0,0)}(A_k) = d_{(x_0,0)}(B_k)$. Since

$$\lim_{n\to\infty} \frac{1}{q-p} \left| \left\{ p < k \leq q : \left| \frac{d_{(x_0,0)}(A_k)}{d_{(x_0,0)}(B_k)} - 1 \right| \geq \varepsilon \right\} \right| = 0,$$

then, the sequences $A = (A_k)$ and $B = (B_k)$ are asymptotically Wijsman deferred statistical equivalence, i.e., $A \overset{WDS}{\sim} B$.

Remark 1.9. It is clear from the Definition 1.7 that

(i) If $q(n) = n$ and $p(n) = n - 1$, then (1.7) is coincide with (1.5).

(ii) If $q(n) = n$ and $p(n) = 0$, then (1.8) is coincide with (1.6).

(iii) If we consider $q(n) = k_n$ and $p(n) = k_{n-1}$ (for any lacunary sequence of non-negative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$), then (1.8) is coincide the definition of asymptotically lacunary statistical equivalence which is given by Patterson and Savaş in [17] and some version of it is studied by Braha and Ulusu- Savaş in [18] and [32], respectively.

(iv) If $q(n) = \lambda_n$ and $p(n) = 0$ (where λ_n is a strictly increasing sequence of natural numbers such that $\lim_n \lambda_n = \infty$), then (1.8) is coincide $\lambda-$statistical equivalence of sequences which is given by Osikievich in [33].

(iv) If $q(n) = n$ and $p(n) = n - \lambda_n$ (where (λ_n) is a nondecrasing sequence of real numbers such that $\lambda_0 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ for all $n \in \mathbb{N}$), then (1.8) is coincide $\lambda-$density which was defined by Mursaleen in [34] and was studied by Hazarika-Esi in [20, 35].

2. WDS_L-Equivalence of Sequences of Sets

Throughout the paper, we consider $p = \{p(n)\}_{n\in\mathbb{N}}$ and $q = \{q(n)\}_{n\in\mathbb{N}}$ are sequences of positive natural numbers satisfying the conditions (1.3). Also, it is assumed, for any nonempty closed subsets $A = (A_k)$, $B = (B_k) \subseteq X$ such that $d_x(A_k) > 0$ and $d_x(B_k) > 0$ hold for each $x \in X$ and $k \in \mathbb{N}$.

The notation $A \prec B$ will be used if $A_n \subseteq B_n$ holds for all $n \in \mathbb{N}$.

Theorem 2.1. *Let $A = (A_k)$ and $B = (B_k)$ and $C = (C_k)$ be sequences of nonempty closed sets. If $A \overset{WDS_L}{\sim} B$ and $A \prec C$, then $C \overset{WDS_L}{\sim} B$.*

Proof. Assume that $A \overset{WDS_L}{\sim} B$ and $A \prec C$. Let $x \in X$ be an arbitrary fixed point. Since $A \prec C$, then

$$d_x(C_k) \leq d_x(A_k)$$

hold for all $n \in \mathbb{N}$. Therefore, the inequality

$$\left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \leq \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|$$

holds for all sufficiently large $n \in \mathbb{N}$, then the inclusion

$$\left\{p < k \leq q : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} \subseteq \left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}$$

is true.

Hence, for any $\varepsilon > 0$, following inequality

$$\frac{1}{q-p}\left|\left\{p < k \leq q : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \leq \frac{1}{q-p}\left|\left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|$$

holds. If we take limit when $n \to \infty$, desired result is obtained. ∎

Theorem 2.2. *Let $A = (A_k)$ and $B = (B_k)$ and $C = (C_k)$ be sequences of nonempty closed sets. If $A \overset{WDS_L}{\sim} B$ and $C \prec B$, then $A \overset{WDS_L}{\sim} C$.*

Proof. From the assumption the inequality $d_x(B_k) \le d_x(C_k)$ hold for all $k \in \mathbb{N}$. So for sufficiently large $n \in \mathbb{N}$, we have

$$\left|\frac{d_x(A_k)}{d_x(C_k)} - L\right| \le \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|.$$

Hence, for any $\varepsilon > 0$ the following inclusion

$$\left\{p < k \le q : \left|\frac{d_x(A_k)}{d_x(C_k)} - L\right| \ge \varepsilon\right\} \subset \left\{p < k \le q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \ge \varepsilon\right\}$$

holds. From this inclusion, we have

$$\frac{1}{q-p}\left|\left\{p < k \le q : \left|\frac{d_x(A_k)}{d_x(C_k)} - L\right| \ge \varepsilon\right\}\right| \le \frac{1}{q-p}\left|\left\{p < k \le q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \ge \varepsilon\right\}\right|.$$

If we take limit when $n \to \infty$, we obtaine $A \overset{WDS_L}{\sim} C$. ■

Theorem 2.3. *Let $A = (A_k)$, $B = (B_k)$ and $C = (C_k)$ be sequences of nonempty closed sets. If $A \overset{WDS_L}{\sim} B$, then $A \cup C \overset{WDS_L}{\sim} B$ and $A \overset{WDS_L}{\sim} B \cap C$ hold.*

Proof. For any sequence of sets $C = (C_k)$ we have $A_k \subset A_k \cup C_k$ and $B_k \cap C_k \subset B_k$ for all $k \in \mathbb{N}$. It means that $A \prec A \cup C$ and $B \cap C \prec B$. So, proof is clear from the Theorem 2.1 and Theorem 2.2. So it is omitted here. ■

Definition 2.4. If $A = (A_k)$ satisfies a property P for all $k \in \mathbb{N}$ except a set which has zero deferred density, then it is said that the sequence $A = (A_k)$ has the property P deferred almost all $k \in \mathbb{N}$ and it is denoted by "*d.a.a.k*".

Following Theorems are the typically generalization of Theorem 2.1 and Theorem 2.2.

Theorem 2.5. *Let $A = (A_k)$, $B = (B_k)$ and $C = (C_k)$ be sequences of nonempty closed sets. If $A \overset{WDS_L}{\sim} B$ and $A \prec C$ (d.a.a.k), then $C \overset{WDS_L}{\sim} B$.*

Proof. Let us consider $M = \{k : C_k \subset A_k\}$. From the assumption, $\delta_D(M) = 0$ holds. Therefore, following inequality

$$\left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \le \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right|$$

holds *d.a.a.k*. Then, we have

$$\frac{1}{q-p}\left|\left\{p < k \le q : \left|\frac{d_x(C_k)}{d_x(B_k)} - L\right| \ge \varepsilon\right\}\right| \le \frac{1}{q-p}\left|\left\{p < k \le q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \ge \varepsilon\right\}\right| + \frac{1}{q-p}|M|.$$

By taking limit when $n \to \infty$, we obtained $C \overset{WDS_L}{\sim} B$. ■

Theorem 2.6. *Let $A = (A_k)$, $B = (B_k)$ and $C = (C_k)$ be sequences of nonempty closed sets. If $A \overset{WDS_L}{\sim} B$ and $B \prec C$ (d.a.a.k), then $A \overset{WDS_L}{\sim} C$.*

Proof. The proof can be obtain by following the proof of Theorem 2.5. So it is omitted here. ■

Theorem 2.7. *Let $A=(A_k)$, $B=(B_k)$ and $C=(C_k)$ be sequences of nonempty closed sets. If $A \overset{WDS_L}{\sim} B$ and $A=C$ (d.a.a.k), then $C \overset{WDS_L}{\sim} B$.*

Proof. Take $M := \{k: \ A_k \neq C_k\}$. From the assumption we have $\delta_D(M)=0$. So, for any $\varepsilon>0$, the following inclusion

$$\begin{aligned}\left\{p<k\leq q:\left|\frac{d_x(C_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\} &= \left\{p<k\leq q:\left|\frac{d_x(C_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cap\left(M^C\cup M\right)\\ &\subseteq\left(\left\{p<k\leq q:\left|\frac{d_x(C_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cap M^C\right)\\ &\quad\cup\left(\left\{p<k\leq q:\left|\frac{d_x(C_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cap M\right)\\ &\subseteq\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cup M\end{aligned}$$

holds. Hence,

$$\begin{aligned}\frac{1}{q-p}\left|\left\{p<k\leq q:\left|\frac{d_x(C_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right| \ &\leq \ \frac{1}{q-p}\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\\ &\quad+\frac{1}{q-p}\left|M\right|\end{aligned}$$

holds. After taking limit when $n\to\infty$, desired result is obtained. ■

Theorem 2.8. *Let $A=(A_k)$, $B=(B_k)$ and $C=(C_k)$ be sequences of nonempty closed sets. If $A \overset{WDS_L}{\sim} B$ and $B=C$ (d.a.a.k), then $A \overset{WDS_L}{\sim} C$.*

Proof. Denote the set $M := \{k: \ B_k \neq C_k\}$ such that we have $\delta_D(M)=0$. From this fact $d_x(B_k)=d_x(C_k)$ $(d.a.a.k)$ satisfied for any $x\in X$. Therefore, following inclusion

$$\begin{aligned}\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\} &= \left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\cap\left(M^C\cup M\right)\\ &\subseteq\left(\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cap M^C\right)\\ &\quad\cup\left(\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\cap M\right)\end{aligned}$$

holds. Since

$$\left[\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cap M^C\right]\subseteq\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}$$

and

$$\left[\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\cap M\right]\subseteq M,$$

then we have

$$\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\subseteq\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cup M.$$

Hence,

$$\frac{1}{q-p}\left|\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}\right| \ \leq \ \frac{1}{q-p}\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(C_k)}-L\right|\geq\varepsilon\right\}$$

$$+\frac{1}{q-p}\left|M\right|$$

holds. After taking limit when $n\rightarrow\infty$,desired result is obtained. ∎

3. Comparison of WD_L and WDS_L-Equivalence

In this section, WD_L-equivalence and WDS_L-equivalence will be compared. Also, it will be shown that WD_L-equivalence is equal WDS_L-equivalence under some conditions.

Theorem 3.1. *Let $A=(A_k)$ and $B=(B_k)$ be sequences of nonempty closed sets. Then, $A\overset{WD_L}{\sim}B$ implies $A\overset{WDS_L}{\sim}B$.*

Proof. Assume that $A\overset{WD_L}{\sim}B$ i.e.,

$$\lim_{n\rightarrow\infty}\frac{1}{q-p}\sum_{k=p+1}^{q}\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|=0.$$

For an arbitrary $\varepsilon>0$, the following inequality

$$\begin{aligned}\frac{1}{q-p}\sum_{k=p+1}^{q}\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right| &= \frac{1}{q-p}\left(\sum_{\substack{k=p+1\\ \left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon}}^{q}+\sum_{\substack{k=p+1\\ \left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|<\varepsilon}}^{q}\right)\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\\ &\geq \frac{1}{q-p}\sum_{\substack{k=p+1\\ \left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon}}^{q}\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\\ &\geq \varepsilon\frac{1}{q-p}\left|\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|\end{aligned}$$

holds. If we take limit when $n\rightarrow\infty$, then we obtain

$$\lim_{n\rightarrow\infty}\frac{1}{q-p}\left|\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|=0.$$

This gives the proof. ∎

Corollary 3.2. *If $A\overset{W_L}{\sim}B$ then $A\overset{WDS_L}{\sim}B$.*

Remark 3.3. The converse of Theorem 3.1 and Corollary 3.2 are not true, in general.

Let $x_0\in X$ be an arbitrary fixed point and take any arbitrary sequnce of sets $C=(C_k)$ such that $d_{x_0}(C_k)=k$ holds for all $k\in\mathbb{N}$. Also, let $D=(D_k)$ be a sequence of sets such that $d_{x_0}(C_k)d_{x_0}(D_k)=1$ holds for all $k\in\mathbb{N}$.

Now, we are ready to give counter example: Let sequences of sets $A=(A_k)$ and $B=(B_k)$ as follows:

$$A_k:=\begin{cases} C_k, & \left[\sqrt{q(n)}\right]-m_0<k\leq\left[\sqrt{q(n)}\right], \quad n=1,2,3,..,\\ E, & \text{otherwise}\end{cases}$$

and

$$B_k := \begin{cases} D_k, & \left[\sqrt{q(n)}\right] - m_0 < k \leq \left[\sqrt{q(n)}\right], \quad n = 1, 2, 3, ..., \\ E, & \text{otherwise} \end{cases}$$

respectively, where $q(n)$ is a strictly monotone increasing sequence and m_0 is an arbitrary fixed natural number. The symbol $[.]$ is the integer part of inside number.

Therefore, we have

$$d_{x_0}(A_k) := \begin{cases} k, & \left[\sqrt{q(n)}\right] - m_0 < k \leq \left[\sqrt{q(n)}\right], \quad n = 1, 2, 3, .., \\ d_{x_0}(E), & \text{otherwise} \end{cases}$$

and

$$d_{x_0}(B_k) := \begin{cases} k^{-1}, & \left[\sqrt{q(n)}\right] - m_0 < k \leq \left[\sqrt{q(n)}\right], \quad n = 1, 2, 3, .., \\ d_{x_0}(E), & \text{otherwise.} \end{cases}$$

Hence, it is clear that $A \overset{WDS_L}{\sim} B$ for $L = d_{x_0}(E)$ but $A \overset{WD_L}{\nsim} B$.

Theorem 3.4. *If $A = (A_k)$ and $B = (B_k) \in l_\infty$, then $A \overset{WDS_L}{\sim} B$ implies $A \overset{WD_L}{\sim} B$, where l_∞ denotes the set of all bounded sequences(in the Wijsman sense).*

Proof. It is clear from Theorem 3.1 that $A \overset{WD_L}{\sim} B$ implies $A \overset{WDS_L}{\sim} B$. Now assume that the sequences $A = (A_k)$ and $B = (B_k)$ are from ℓ_∞ and satisfying $A \overset{WDS_L}{\sim} B$. Then, there exists $M > 0$ such that

$$\left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \leq M$$

holds for all $k \in \mathbb{N}$. So, for any $\varepsilon > 0$, following inequality

$$\begin{aligned} \frac{1}{q-p} \sum_{k=p+1}^{q} \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| &= \left(\sum_{k \in N} + \sum_{k \in N^c} \right) \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \\ &\leq \frac{M}{q-p} \left| \left\{ p < k \leq q : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\} \right| + \varepsilon \end{aligned}$$

is satisfied where

$$N := \left\{ p < k \leq q : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\}.$$

Hence, after taking limit when $n \to \infty$ desired result is obtained. ∎

Definition 3.5 ([29]). A method $D_{p,q}$ is called properly deferred when $p = \{p(n)\}$ and $q = \{q(n)\}$ satisfy in addition to (1.3), the condition

$$\left\{ \frac{p(n)}{q(n) - p(n)} \right\}_{n \in \mathbb{N}}$$

is bounded.

In the following theorem, it is shown that WS_L-equivalence implies WDS_L- equivalence.

Theorem 3.6. *In order that $A \overset{WS_L}{\sim} B$ implies $A \overset{WDS_L}{\sim} B$ if and only if the method $D_{p,q}$ is properly deferred.*

Proof. Since $A \overset{WS_L}{\sim} B$, then we have

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\leq n:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|=0.$$

Therefore, following limit

$$\lim_{n\to\infty}\frac{1}{q}\left|\left\{k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|=0$$

because $q(n)\to\infty$, $n\to\infty$. It is clear from set comparison that the following inequality

$$\left|\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|\leq\left|\left\{k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|$$

holds for every $\varepsilon>0$. Hence,

$$\begin{aligned}\frac{1}{q-p}&\left|\left\{p<k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|\\&\leq\frac{q}{q-p}\frac{1}{q}\left|\left\{k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|\\&=\left(1+\frac{p}{q-p}\right)\left[\frac{1}{q}\left|\left\{k\leq q:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|\right].\end{aligned}$$

After taking limit when $n\to\infty$, we obtain desired result if and only if $D_{p,q}$ is properly deferred. ■

Theorem 3.7. *If $A \overset{WDS_L}{\sim} B$ w.r.t an arbitrary p and $q=n$, then $A \overset{WS_L}{\sim} B$ hold.*

Proof. Let us assume that $A \overset{WDS_L}{\sim} B$ with respect to $q=n$ and arbitrary p. For any $n\in\mathbb{N}$, there is a $h\in\mathbb{N}$ such that $n^{h+1}=0$ and the inequality

$$p(n)=n^{(1)}>p\left(n^{(1)}\right)=n^{(2)}>p\left(n^{(2)}\right)=n^{(3)}>...>p\left(n^{(h-1)}\right)=n^{(h)}\geq 1$$

holds. Therefore, the set $\left\{k\leq n:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}$ can be represent as

$$\left\{k\leq n^{(1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cup\left\{n^{(1)}<k\leq n:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}.$$

By the same way the first set in the union can be represent as

$$\left\{k\leq n^{(2)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cup\left\{n^{(2)}<k\leq n^{(1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}.$$

After finite step (at most h step),

$$\begin{aligned}&\left\{k\leq n^{(h-1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\\&\quad=\left\{k\leq n^{(h)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\cup\left\{n^{(h)}<k\leq n^{(h-1)}:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\end{aligned}$$

is obtained. Therefore,

$$\frac{1}{n}\left|\left\{k\leq n:\left|\frac{d_x(A_k)}{d_x(B_k)}-L\right|\geq\varepsilon\right\}\right|=\sum_{m=0}^{h}\frac{n^{(m)}-n^{(m+1)}}{n}U_m,$$

where

$$U_m := \frac{1}{n^{(m)} - n^{(m+1)}} \left| \left\{ n^{(m+1)} < k \leq n^{(m)} : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\} \right|.$$

If we consider a matrix $S := (s_{n,m})$ as

$$s_{n,m} := \begin{cases} \frac{n^{(m)} - n^{(m+1)}}{n}, & m = 0, 1, 2, ..., h, \\ 0, & \text{otherwise}, \end{cases}$$

then the sequence

$$\left\{ \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d_x(A_k)}{d_x(B_k)} - L \right| \geq \varepsilon \right\} \right| \right\}_{n \in \mathbb{N}}$$

is $(s_{n,m})$ transformation of the sequence (U_m). Since the matrix $S = (s_{n,m})$ satisfies Silverman-Toeplitz Theorem (see in [36]) and from assumption on $A = (A_k)$ and $B = (B_k)$ then we have desired result. ■

Combining Theorem 3.6 and Theorem 3.7 we can give following theorem without proof:

Theorem 3.8. *WDS_L-asymptotically equivalence w.r.t. any p and $q = n$ is equivalent to WS_L-equivalence if and only if $\left\{ \frac{p}{n-p} \right\}$ is bounded for all $n \in \mathbb{N}$.*

If we consider the method as

$$D_n^{\theta} := \frac{S_{[\theta n]+1} + S_{[\theta n]+2} + ... + S_n}{n - [\theta n]},$$

where θ is a constant $0 \leq \theta < 1$. Then, as a Corollary of Theorem 3.8, the following result can be given:

Corollary 3.9. *$A \overset{WD_n^{\theta} S_L}{\sim} B$ if and only if $A \overset{WS_L}{\sim} B$.*

4. Comparison of WDS_L-Equivalence for Any Sequences p and q

Take into consider $p' = \{p'(n)\}$ and $q' = \{q'(n)\}$ be any sequences of positive natural numbers such that

$$p(n) \leq p'(n) < q'(n) \leq q(n) \tag{4.1}$$

hold for all $n \in \mathbb{N}$ besides (1.3). Denote by the associated sets

$$E := \{p(n) : n \in \mathbb{N}\}, E' := \{p'(n) : n \in \mathbb{N}\},$$
$$F := \{q(n) : n \in \mathbb{N}\} \text{ and } F' := \{q'(n) : n \in \mathbb{N}\}.$$

Theorem 4.1. *If the set $F' \setminus F$ is finite and*

$$\lim_{n \to \infty} \frac{q(n) - q'(n)}{q'(n) - p(n)} < \infty$$

holds, then $A \overset{WDS_L}{\backsim} B$ w.r.t. p and q implies $A \overset{WDS_L}{\backsim} B$ w.r.t. p and q'.

Proof. Since $F' \setminus F$ is finite, then there is a number $n_0 \in \mathbb{N}$ such that the inclusion

$$\{q' : n > n_0\} \subset \{q : n \in \mathbb{N}\}$$

holds. So, there is a strictly increasing sequence $j = \{j(n)\}$ such that $q'(n) = q(j(n))$ hold for all $n \geq n_0$. Therefore, for sufficiently large $n \in \mathbb{N}$, following inequality

$$\begin{aligned}
&\frac{1}{q'-p}\left|\left\{p < k \leq q' \ : \ \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad = \frac{1}{q(j(n)) - p}\left|\left\{p < k \leq q(j(n)) : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad \leq \frac{q-p}{q'-p}\cdot\frac{1}{q-p}\left|\left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad = \left(\frac{q-q'}{q'-p}+1\right)\frac{1}{q-p}\left|\left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

holds. Under the assumption, if we take limit where $n \to \infty$, we have desired result. ■

Theorem 4.2. *If the set $F \setminus F'$ is finite and*

$$\liminf_{n\to\infty}\frac{q'(n) - p(n)}{q(n) - p(n)} > 0$$

holds, then $A \overset{WDS_L}{\backsim} B$ w.r.t. p and q' implies $A \overset{WDS_L}{\backsim} B$ w.r.t. p and q.

Proof. It can be proved by following Theorem 4.1. So, proof is omitted here. ■

Corollary 4.3. *If $F \triangle F'$ is finite, then $A \overset{WDS_L}{\backsim} B$ w.r.t. p and q if and only if $A \overset{WDS_L}{\backsim} B$ w.r.t. p and q'.*

Theorem 4.4. *If $E' \setminus E$ is finite and*

$$\liminf_{n\to\infty}\frac{q(n) - p'(n)}{q(n) - p(n)} > 0$$

holds, then $A \overset{WDS_L}{\backsim} B$ w.r.t. p and q implies $A \overset{WDS_L}{\backsim} B$ w.r.t. p' and q.

Proof. If $E' \setminus E$ is finite, then there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\{p' : n \geq n_0\} \subset \{p : n \in \mathbb{N}\}$$

holds. It means that there is a strictly increasing sequence $j = (j_n)$ of natural numbers such that

$$p'(n) = p(j_n).$$

Therefore, following inequality

$$\begin{aligned}
&\frac{1}{q-p'}\left|\left\{p' < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad = \frac{1}{q-p}\left|\left\{p(j_n) < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad \leq \frac{q-p}{q-p'}\cdot\frac{1}{q-p}\left|\left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

holds. If we take limit when $n \to \infty$, we obtain the proof of theorem. ∎

Theorem 4.5. *The sequence $p'(n)$ and $q'(n)$ are satisfied (4.1) such that the set $\{k : p(n) < k \leq p'(n)\}$ and $\{k : q'(n) < k \leq q(n)\}$ are finite for all $n \in \mathbb{N}$. Then, $A \overset{WDS_L}{\sim} B$ w.r.t. p' and q' implies $A \overset{WDS_L}{\sim} B$ w.r.t. p and q.*

Proof. Assume that $A \overset{WDS_L}{\sim} B$ (*w.r.t.* p' and q'). So, for an arbitrary $\varepsilon > 0$, we have the following inequality

$$\begin{aligned}
&\tfrac{1}{q-p}\left|\left\{p < k \leq q : \left|\tfrac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{q'-p'}\left|\left\{p < k \leq p' : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad + \frac{1}{q'-p'}\left|\left\{p' < k \leq q' : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad + \frac{1}{q'-p'}\left|\left\{q' < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{m_1}{q'-p'} + \frac{1}{q'-p'}\left|\left\{p' < k \leq q' : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| + \frac{m_2}{q'-p'},
\end{aligned}$$

where

$$m_1 := |\{k : p < k \leq p'\}|, \; m_2 := |\{k : q' < k \leq q\}|.$$

On taking limit when $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{1}{q-p}\left|\left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| = 0.$$

Thus $A \overset{WDS_L}{\sim} B$ (*w.r.t.* p and q). ∎

Theorem 4.6. *The sequence $p'(n)$ and $q'(n)$ are satisfying (4.1) such that*

$$\lim_{n\to\infty} \frac{q(n) - p(n)}{q'(n) - p'(n)} = d > 0.$$

Then, $A \overset{WDS_L}{\sim} B$ w.r.t. p and q implies $A \overset{WDS_L}{\sim} B$ w.r.t. p' and q'.

Proof. It is clear from the inclusion

$$\left\{p' < k \leq q' : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\} \subset \left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}$$

that the following inequality

$$\begin{aligned}
&\frac{1}{q'-p'}\left|\left\{p' < k \leq q' : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\qquad \leq \frac{q-p}{q'-p'} \cdot \frac{1}{q-p}\left|\left\{p < k \leq q : \left|\frac{d_x(A_k)}{d_x(B_k)} - L\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

holds. After taking limit when $n \to \infty$ the desired result is obtained. ∎

References

[1] H. Fast, Sur la convergece statistique, Colloq. Math. 2 (1951) 241–244.

[2] H. Steinhaus, Sur la convergence ordinaire et la convergence asymtotique, Colloq. Math. 2 (1951) 73–74.

[3] C.R. Buck, The measure theoretic approch to density, Amer. J. Math. 68 (1946) 560–580.

[4] J.S. Connor, The statistical and strong p-Cesaro convergence of sequences, Analysis 8 (1988) 47–63.

[5] J.S. Connor, R-type summability methods, Cauchy criteria, p-sets and statistical convergence, Proc. Amer. Math. Soc. 115 (1992) 319–327.

[6] P. Erdös, G. Tenenbaum, Sur les densities de certaines suites d'entries, Proc. London Math. Soc. 59 (1989) 417–438.

[7] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.

[8] J.A. Fridy, H.I. Miller, A matrix characterization of statistical convergence, Analysis 11 (1991) 59–66.

[9] E. Kaya, M. Küçükaslan, R. Wagner, On statistical convergence and statistical monotonicity, Annales Univ. Sci. Budapest., Sect. Comp. 39 (2013) 257–270.

[10] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.

[11] J.P. Pobyvanets, Asymptotic equivalence of some linear transformation defined by a nonnegative matrix and reduced to generalized equivalence in the sense of Cesaro and Abel, Mat. Fiz. 28 (123) (1980) 83–87.

[12] J.A. Fridy, Minimal rates of summability, Can. J. Math. 4 (1978) 808–816.

[13] M. Marouf, Asymtotic equivalence and summability, Internat. J. Math. Sci. 16 (4) (1993) 755–762.

[14] L. Jinlu, Asymptotic equivalence of sequences and summability, Internat. J. Math. and Math. Sci. 20 (4) (1997) 749–758.

[15] R.F. Patterson, On Asymototically statistically equivalent sequence, Demonstratio Math. 36 (1) (2003) 149–153.

[16] J. Connor, H. Gümüş, Summability matrices that preserve asymptotic equivalence for ideal convergence, Sarajevo J. Math. 12 (24) (2016) 107–124.

[17] R.F. Patterson, E. Savas, On Asymptotically lacunary statistical equivalent sequences, Thai J. Math. 4 (2) (2006) 267–272.

[18] N.L. Braha, On asymptotically Δ^m-lacunary statistical equivalent sequences, Appl. Math. Comput. 219 (2012) 280–288.

[19] A. Esi, On almost asymptotically lacunary statistical equivalent sequences induced probabilistic norms, Analysis 34 (1) (2014) 19–28.

[20] B. Hazarika, A. Esi, N.L. Braha, On asymptotically Wijsman lacunary $\sigma-$ statistical convergence of set sequences, J. Math. Anal. 4 (3) (2013) 33–46.

[21] V. Kumer, A. Sharma, On asymptotically generalized statistical equivalent sequences via ideals, Tamkang J. Math. 43 (2012) 469–478.

[22] J.P. Aubin, H. Frankowska, Set Valued Analysis, Birkhauser, Basel, 1986.

[23] C. Kuratowski, Topology, Academic Press, New York, 1986.

[24] R.A. Wijsman, Convergence of sequences of convex of sets, Bull. Am. Math. Soc. 70 (1964) 186–188.

[25] F. Nuray, B.E. Rhoades, Statistical convergence of sequences of sets, Fasc. Math. 49 (2012) 87–99.

[26] U. Ulusu, F. Nuray, Lacunary statistical convergence of sequences of sets, Progress in Applied Mathematics 4 (2) (2012) 99–109.

[27] B. Inan, M. Küçükaslan, Ideal convergence of sequence of sets, Contemporary Analysis and Applied Mathematics 3 (2) (2015) 184–212.

[28] M. Yılmaztürk, Ö. Mızrak, M. Küçükaslan, Deferred statistical cluster points of real valued sequences, Univers. J. Appl. Math. 1 (1) (2013) 1–6.

[29] R.P. Agnew, On deferred Cesaro mean, Ann. of Math. 33 (1932) 413–421.

[30] M. Altınok, B. Inan, M. Küçükaslan, On deferred statistical convergence of sequences of sets in metric space, TJMCS. (2015) Article ID 20150050.

[31] U. Ulusu, F. Nuray, On Asymptotically lacunary statistical equivalent set sequences, J. Math. 2013 (2013) Article ID 310438.

[32] U. Ulusu, E. Savaş, An extension of asymptotically lacunary statistical set sequences, J. Inequal. Appl. 134 (2014) 1–8.

[33] J.A. Osikiewicz, Summability of Matrix Sub Methods and Spliced Sequences, Ph.D. Thesis, Kent State University, 1997.

[34] M. Mursaleen, λ-Statistical convergence, Math. Slovaca 50 (1) (2000) 111–115.

[35] B. Hazarika, A. Esi, On $\lambda-$asymptotically Wijsman generalized statistical convergence of sequences of sets, Tatra Mt. Math. Publ. 56 (2013) 67–77.

[36] G.H. Hardy, Divergent Series, Oxford University Press, Oxford, 1949.