On Geometry of Volume Elements and Fractional Differentiable Manifolds

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Abstract In this paper, the area and volume on fractional differentiable manifolds are discussed. Further, some examples are given.

1. Introduction

Fractional calculus is an important and useful branch of mathematics having a broad range of applications at almost every branch of sciences. Techniques of fractional calculus have been employed at the modeling of many different phenomena in engineering, physics and in mathematics. Problem in fractional calculus is not only important but also quite challenging which usually involves hard mathematical solution techniques. However a general solution theory for almost each problem in this area has yet to be established. Each application has developed its own approaches and implementations. As a consequence, a single standard method for the problems in fractional calculus has not emerged yet. Therefore, funding reliable and efficient solution techniques along with fast implementation methods are significantly important and active research areas.

Further, It is also realized that the operators of fractional integration and derivation have physical and geometric interpretations, which essentially streamline along their utilization for related issues in various fields of science (see [1–5]). Moreover, the fractional differential calculus on a differential manifold is studied in ([6, 7]). Even though fractional calculus is a highly useful and important topic, however the research on the geometric interpretation and applications are limited and not many in current literature. Thus in the this study we focus on the area and volume on fractional differentiable manifolds and discussed some related properties. We also give some examples.

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2. Preliminaries

Let M be an n-dimensional differential manifold (U, x_i) a local coordinate system on M and $U_0 = \{x \in U : 0 \leq x_i \leq b_i, i = 1, 2, ..., n\}$, see [6].

For a function $f : U_0 \to \mathbb{R}$ the fractional derivative with respect to x_i:

$$\partial_i^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \partial_{x_i}^m \int_0^{x_i} \frac{f(x_1, ..., x_{i-1}, s, x_{i+1}, ..., x_n) - f(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n)}{(x_i - s)^{\alpha - m + 1}} ds,$$

where $\partial_{x_i}^m = \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial x_i} \circ ... \circ \frac{\partial}{\partial x_i}$ (m times, i is fixed, $\alpha \geq 0$).

For $\alpha \in (0, 1), \gamma > -1$,

$$\partial_i^\alpha (x_i)^\gamma = \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \gamma - \alpha)} x_i^{\gamma - \alpha} \quad ; \partial_i^\alpha (x_j)^\alpha = \Gamma(1 + \alpha) \delta_i^j.$$

A fractional vector field $U \subset M$ is an object of the form $X^\alpha = X_i^\alpha \partial_i^\alpha$, where $X_i^\alpha \in \Im_U(M) \quad i = 1, ..., n$.

We denote by χ_U^α the fractional vector fields on $U.\chi_U^\alpha$ is generated by the operators $\partial_i^\alpha, i = 1, 2, ..., n$. If $c \colon x = x(t), t \in I$ is a parametrized curve in U, then the fractional tangent vector field of c is

$$x^\alpha(t) = \frac{1}{\Gamma(1 + \alpha)} \partial_t^\alpha x_i(t) \partial_i^\alpha.$$

A fractional covariant derivative is given by

$$\nabla_{X^\alpha}^\alpha Y^\alpha = X_i^\alpha (\partial_i^\alpha Y_j^\alpha + \Gamma_{ik}^{\alpha j} Y_k^\alpha) \partial_j^\alpha,$$

where $X^\alpha, Y^\alpha \in \chi_U^\alpha$ and $\Gamma_{ik}^{\alpha j}$ the functions defining the coefficients of a fractional linear connection on M. They are determined by the relations

$$\nabla_{\partial_i^\alpha}^\alpha \partial_k^\alpha = \Gamma_{ik}^{\alpha j} \partial_j^\alpha.$$

3. Specialized Definition of Fractional Curvature

Let $y(x)$ any smooth path. Fractional curvature measures the rate which the tangent line turns per unit distance moved a long the path, which implies that it is the rate of change of the path. Let p_1 and p_2 be two points on a curve, separated by an arc of length $\triangle s$ (Figure 1). Then, the average fractional curvature of the arc from p_1 to p_2 is

$$\frac{\triangle \varphi}{\triangle s}$$

where $\triangle \varphi = \varphi_1 - \varphi_2$, is the angle turned through by the tangent lines L_α, $0 < \alpha < 1$ moving from p_1 to p_2. Then,

$$\tilde{C} = \lim_{\triangle x \to 0} \frac{\triangle \varphi}{\triangle s} = \frac{d^\alpha \varphi}{d^\alpha s},$$

is the fractional curvature at point p_1 and

$$\frac{d^\alpha \varphi}{d^\alpha s} = \frac{d^\alpha \varphi}{d^\alpha x} \frac{d^\alpha x}{d^\alpha s}$$

and since
$$(d^\alpha s)^2 = (d^\alpha x)^2 + (d^\alpha y)^2,$$

then we have
$$\frac{d^\alpha x}{d^\alpha s} = \frac{1}{\frac{d^\alpha s}{d^\alpha x}} = \frac{1}{\sqrt{1 + (\frac{d^\alpha y}{d^\alpha x})^2}}.$$

Since $dy = \dfrac{1}{\alpha!}\dfrac{d^\alpha y}{dx^\alpha}dx^\alpha$, and $dx = \dfrac{1}{\alpha!(1-\alpha)!}x^{1-\alpha}dx^\alpha$, then

$$\tan\varphi = \frac{dy}{dx} = (1-\alpha)!x^{\alpha-1}\frac{d^\alpha y}{dx^\alpha}$$

where $0 < \alpha < 1$, then

$$\varphi = \arctan\left((1-\alpha)!x^{\alpha-1}\frac{d^\alpha y}{dx^\alpha}\right).$$

Moreover, $\dfrac{d^\alpha}{d^\alpha x} = \dfrac{d^\alpha}{dx^\alpha}\dfrac{dx^\alpha}{d^\alpha x}$, and $\dfrac{dx^\alpha}{d^\alpha x} = (1-\alpha)!x^{\alpha-1}$, then we obtain

$$\frac{d^\alpha \varphi}{d^\alpha x} = \frac{d^\alpha}{dx^\alpha}\left(\arctan\left((1-\alpha)!x^{\alpha-1}y^{(\alpha)}\right)\right)(1-\alpha)!x^{\alpha-1}$$

$$= \frac{((1-\alpha)!)^2}{\Gamma(1-\alpha)}x^{\alpha-1}\left(\frac{\arctan((1-\alpha)!x^{\alpha-1}y^{(\alpha)})}{(1-\alpha)!x^{\alpha-1}y^{(\alpha)}}\right)^{1-\alpha}\frac{(-1)^\alpha x^{-1}y^{(\alpha)} + \Gamma(1-\alpha)x^{\alpha-1}y^{(\beta)}}{\left(1 + ((1-\alpha)!x^{\alpha-1}y^{(\alpha)})^2\right)^\alpha}$$

$$= \frac{((1-\alpha)!)^2}{\Gamma(1-\alpha)}x^{\alpha-1}\left(\frac{\arctan((1-\alpha)!x^{\alpha-1}y^{(\alpha)})}{(1-\alpha)!x^{\alpha-1}y^{(\alpha)}}\right)^{1-\alpha}\frac{e^{i\alpha\pi}x^{-1}y^{(\alpha)} + \Gamma(1-\alpha)x^{\alpha-1}y^{(\beta)}}{\left(1 + ((1-\alpha)!x^{\alpha-1}y^{(\alpha)})^2\right)^\alpha}.$$

Then,

$$\tilde{C} = \frac{d^\alpha \varphi}{d^\alpha x}\frac{d^\alpha x}{d^\alpha s}$$

$$= \frac{((1-\alpha)!)^2}{\Gamma(1-\alpha)}x^{\alpha-1}\left(\frac{\arctan((1-\alpha)!x^{\alpha-1}y^{(\alpha)})}{(1-\alpha)!x^{\alpha-1}y^{(\alpha)}}\right)^{1-\alpha}$$

$$\times \frac{e^{i\alpha\pi}x^{-1}y^{(\alpha)} + \Gamma(1-\alpha)x^{\alpha-1}y^{(\beta)}}{\left(1 + ((1-\alpha)!x^{\alpha-1}y^{(\alpha)})^2\right)^\alpha}\left[1 + \left(\frac{d^\alpha y}{d^\alpha x}\right)^2\right]^{\frac{-1}{2}}$$

$$= \frac{((1-\alpha)!)^2}{\Gamma(1-\alpha)}x^{\alpha-1}\left(\frac{\arctan((1-\alpha)!x^{\alpha-1}y^{(\alpha)})}{(1-\alpha)!x^{\alpha-1}y^{(\alpha)}}\right)^{1-\alpha}$$

$$\times \frac{e^{i\alpha\pi}x^{-1}y^{(\alpha)} + \Gamma(1-\alpha)x^{\alpha-1}y^{(\beta)}}{\left(1 + ((1-\alpha)!x^{\alpha-1}y^{(\alpha)})^2\right)^{\alpha+\frac{1}{2}}},$$

where $y^{(\alpha)} = \dfrac{d^\alpha y}{dx^\alpha}$ and $y^\beta = \dfrac{d^\alpha}{dx^\alpha}\dfrac{d^\alpha}{dx^\alpha}y$.
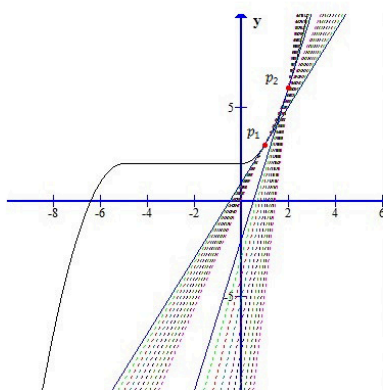
FIGURE 1.   Graph of tangent lines $L_\alpha, 0 < \alpha < 1$ moving from p_1 to p_2

Example 3.1. The fractional curvature of the function $y = x^{3\alpha}$ at (1.1) is computed as follows:

$y^{(\alpha)} = \frac{(3\alpha)!}{(2\alpha)!} x^{2\alpha}$, then $y^{(\alpha)} = \frac{(3\alpha)!}{(2\alpha)!}$ at $x = 1$,

$y^{(\beta)} = \frac{(3\alpha)!}{\alpha!} x^{\alpha}$, then $y^{(\beta)} = \frac{(3\alpha)!}{\alpha!}$ at $x = 1$.

Also, $\arctan\left((1-\alpha)! x^{\alpha-1} y^{(\alpha)} \right) = \arctan\left[\frac{(1-\alpha)!(3\alpha)!}{(2\alpha)!} \right]$ at $x = 1$ and

$$\frac{e^{i\alpha\pi} x^{-1} y^{(\alpha)} + \Gamma(1-\alpha) x^{\alpha-1} y^{(\beta)}}{\left(1 + ((1-\alpha)! x^{\alpha-1} y^{(\alpha)})^2\right)^{\alpha+\frac{1}{2}}} = \left(\frac{e^{i\alpha\pi}(3\alpha)!}{(2\alpha)!} + \frac{(3\alpha)!\Gamma(1-\alpha)}{\alpha!} \right) \left(1 + \left(\frac{(3\alpha)!(1-\alpha)!}{(2\alpha)!} \right)^2 \right)^{-\left(\alpha+\frac{1}{2}\right)}$$
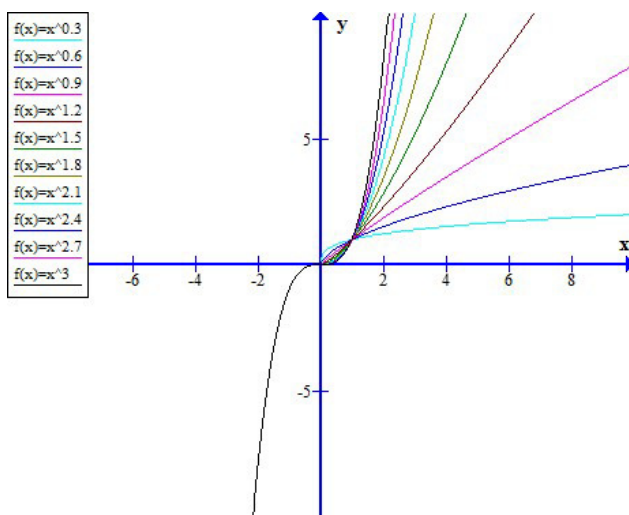
at $x = 1$.

FIGURE 2.   Graph of $y = x^{3\alpha}$ $(0 < \alpha \leq 1)$

4. Area on a Fractional Differentiable Manifolds

The differential of area on the manifolds produced by $M(u_1, u_2)$ characterized as (see [7]):

$$dA = |dM_{u_1} \wedge dM_{u_2}|$$

$$dM_{u_i} = \frac{1}{\alpha!} \frac{\partial^\alpha M}{\partial u_i^\alpha} (du_i)^\alpha, i = 1, 2,$$

then we have

$$dA = \frac{1}{(\alpha!)^2} \left| \frac{\partial^\alpha M}{\partial u_1^\alpha} \wedge \frac{\partial^\alpha M}{\partial u_2^\alpha} \right| (du_1)^\alpha (du_2)^\alpha.$$

Therefore, we obtain

$$A = \frac{1}{(\alpha!)^2} \int_D \left| \frac{\partial^\alpha M}{\partial u_1^\alpha} \wedge \frac{\partial^\alpha M}{\partial u_2^\alpha} \right| (du_1)^\alpha (du_2)^\alpha.$$

Example 4.1. Let us compute the area of helicoid is given by

$$M(u, v) = (u^\alpha \cos^\alpha v, u^\alpha \sin^\alpha v, v^\alpha), u \geq 0, v \in \mathbb{R},$$

then

$$\frac{\partial^\alpha M}{\partial u^\alpha} = (\alpha! \cos^\alpha v, \alpha! \sin^\alpha v, 0)$$

$$\frac{\partial^\alpha M}{\partial v^\alpha} = ((-1)^\alpha \alpha! u^\alpha \sin^\alpha v, \alpha! u^\alpha \cos^\alpha v, \alpha!)$$

and

$$\frac{\partial^\alpha M}{\partial u^\alpha} \wedge \frac{\partial^\alpha M}{\partial v^\alpha} = ((\alpha!)^2 \sin^\alpha v, -(\alpha!)^2 \cos^\alpha v, (\alpha!)^2 u^\alpha (\cos^{2\alpha} v - (-1)^\alpha \sin^{2\alpha} v))$$

$$\left| \frac{\partial^\alpha M}{\partial u^\alpha} \wedge \frac{\partial^\alpha M}{\partial v^\alpha} \right| = (\alpha!)^2 \left[\sin^{2\alpha} v + \cos^{2\alpha} v + u^{2\alpha} (\cos^{2\alpha} v - (-1)^\alpha \sin^{2\alpha} v)^2 \right]^{\frac{1}{2}}$$

therefore,

$$A = \int_D \left[\sin^{2\alpha} v + \cos^{2\alpha} v + u^{2\alpha} (\cos^{2\alpha} v - (-1)^\alpha \sin^{2\alpha} v)^2 \right]^{\frac{1}{2}} (du)^\alpha (dv)^\alpha.$$

Hence

$$A = \int_D \left[\sin^{2\alpha} v + \cos^{2\alpha} v + u^{2\alpha} (\cos^{2\alpha} v - e^{i\alpha\pi} \sin^{2\alpha} v)^2 \right]^{\frac{1}{2}} (du)^\alpha (dv)^\alpha.$$

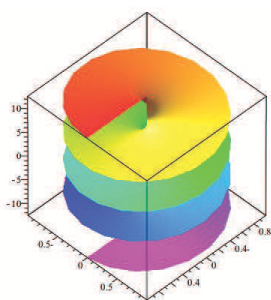|	$\left\lvert \dfrac{\partial^\alpha M}{\partial u^\alpha} \wedge \dfrac{\partial^\alpha M}{\partial v^\alpha}\right\rvert$
$\alpha = 0.1$	$0.9050682874 \left[\sin^{.2} v + \cos^{.2} v + u^{.2}(\cos^{.2} v - i\sin^{.2} v)^2\right]^{\frac{1}{2}}$
$\alpha = 0.2$	$0.8430338395 \left[\sin^{.4} v + \cos^{.4} v + u^{.4}(\cos^{.4} v - \sin^{.4} v)^2\right]^{\frac{1}{2}}$
$\alpha = 0.3$	$0.8054536507 \left[\sin^{.6} v + \cos^{.6} v + u^{.6}(\cos^{.6} v - i\sin^{.6} v)^2\right]^{\frac{1}{2}}$
$\alpha = 0.4$	$0.7872370818 \left[\sin^{.8} v + \cos^{.8} v + u^{.8}(\cos^{.8} v - \sin^{.8} v)^2\right]^{\frac{1}{2}}$
$\alpha = 0.5$	$0.7853981635 \left[\sin v + \cos v + u(\cos v - i\sin v)^2\right]^{\frac{1}{2}}$
$\alpha = 0.6$	$0.7983696794 \left[\sin^{1.2} v + \cos^{1.2} v + u^{1.2}(\cos^{1.2} v - \sin^{1.2} v)^2 v\right]^{\frac{1}{2}}$
$\alpha = 0.7$	$0.8256243469 \left[\sin^{1.4} v + \cos^{1.4} v + u^{1.4}(\cos^{1.4} v - i\sin^{1.4} v)^2 v\right]^{\frac{1}{2}}$
$\alpha = 0.8$	$0.8674757289 \left[\sin^{1.6} v + \cos^{1.6} v + u^{1.6}(\cos^{1.6} v - \sin^{1.6} v)^2\right]^{\frac{1}{2}}$
$\alpha = 0.9$	$0.9249935154 \left[\sin^{1.8} v + \cos^{1.8} v + u^{1.8}(\cos^{1.8} v - i\sin^{1.8} v)^2\right]^{\frac{1}{2}}$
$\alpha = 1$	$\left[1 + u^2\right]^{\frac{1}{2}}$

FIGURE 3. $\alpha = 1$

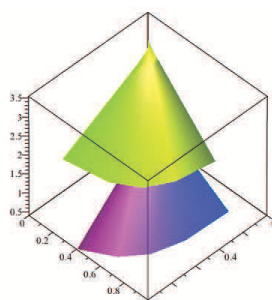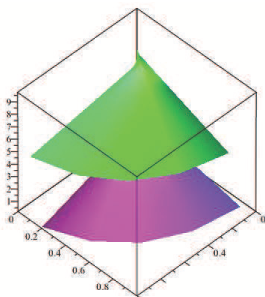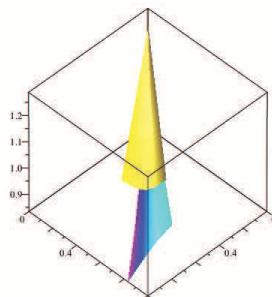FIGURE 4. $\alpha = 0.5$

FIGURE 5. $\alpha = 0.9$

FIGURE 6. $\alpha = 0.1$

Example 4.2. Consider

$$M(u,v) = (u^\alpha \cos^\alpha v, u^\alpha \sin^\alpha v).$$

Now, if $\alpha = 1$, then this is circle with radius u. If $\alpha = \frac{1}{2}$, it is a diamond with vertices $(\pm u, 0)$ and $(0, \pm u)$. Thus the regular parametrization of torus and then the area of it is given by

$$A = \frac{1}{(\alpha!)^2} \int_D \left\lvert \frac{\partial^\alpha M}{\partial u^\alpha} \wedge \frac{\partial^\alpha M}{\partial v^\alpha}\right\rvert (du)^\alpha (dv)^\alpha.$$

Then,

$$\frac{\partial^\alpha M}{\partial u^\alpha} = ((\alpha)!\cos^\alpha v, (\alpha)!\sin^\alpha v)$$

$$\frac{\partial^\alpha M}{\partial v^\alpha} = ((-1)^\alpha u^\alpha (\alpha)!\sin^\alpha v, u^\alpha (\alpha)!\cos^\alpha v)$$

$$\left|\frac{\partial^\alpha M}{\partial u^\alpha} \times \frac{\partial^\alpha M}{\partial v^\alpha}\right| = (\alpha)![u^\alpha(\cos^{2\alpha} v - (-1)^\alpha \sin^{2\alpha} v)]$$

therefore,

$$A = \frac{1}{(\alpha)!}\int_D \left[u^\alpha(\cos^{2\alpha} v - (-1)^\alpha \sin^{2\alpha} v)\right](du)^\alpha (dv)^\alpha.$$

Hence

$$A = \frac{1}{(\alpha)!}\int_D \left[u^\alpha(\cos^{2\alpha} v - e^{i\alpha\pi} \sin^{2\alpha} v)\right](du)^\alpha (dv)^\alpha.$$

Then, we have the following table and the figure.

|	$\dfrac{\partial^\alpha M}{\partial u^\alpha} \wedge \dfrac{\partial^\alpha M}{\partial v^\alpha}$
$\alpha = 0.1$	$0.9513507699[u^{0.1}(\cos^{0.2} v - (0.9510565163 + 0.3090169944i)\sin^{0.2} v)]$
$\alpha = 0.2$	$0.9181687424[u^{0.2}(\cos^{0.4} v - (0.8090169944 + 0.5877852523i)\sin^{0.4} v)]$
$\alpha = 0.3$	$0.8974706963[u^{0.3}(\cos^{0.6} v - (0.5877852523 + 0.8090169944i)\sin^{0.6} v)]$
$\alpha = 0.4$	$0.8872638175[u^{0.4}(\cos^{0.8} v - (0.3090169944 + 0.9510565163i)\sin^{0.8} v)]$
$\alpha = 0.5$	$0.8862269255[u^{0.5}(\cos v - i\sin v)]$
$\alpha = 0.6$	$0.8935153493[u^{0.6}(\cos^{1.2} v - (-0.3090169944 + 0.9510565163i)\sin^{1.2} v)]$
$\alpha = 0.7$	$0.9086387329[u^{0.7}(\cos^{1.4} v - (-0.5877852523 + 0.8090169944i)\sin^{1.4} v)]$
$\alpha = 0.8$	$0.9313837710[u^{0.8}(\cos^{1.6} v - (-0.8090169944 + 0.5877852523i)\sin^{1.6} v)]$
$\alpha = 0.9$	$0.9617658319[u^{0.9}(\cos^{1.8} v - (-0.9510565163 + 0.3090169944i)\sin^{1.8} v)]$
$\alpha = 1$	u
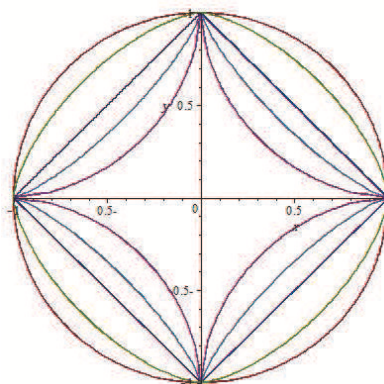
FIGURE 7.   Graph of $M(u,v) = (u^\alpha \cos^\alpha v, u^\alpha \sin^\alpha v)$, $\alpha = 1$, $\alpha = 0.3$, $\alpha = 0.4$, $\alpha = 0.5$, $\alpha = 0.8$

5. Volume Element in Fractional Riemannian Manifolds

Similarly, to the above section, in this section we study volume. Since

$$g_{ij} = e_i e_j = \frac{1}{(\alpha!)^2}\tilde{e}_i\tilde{e}_j = \frac{1}{(\alpha!)^2}\tilde{g}_{ij},$$

then

$$\sqrt{\det g} = \frac{1}{(\alpha!)^n}\sqrt{\det\tilde{g}}(dx_1)^\alpha(dx_2)^\alpha...(dx_n)^\alpha$$

therefore, the volume element in n-dimensional fractional differentiable Riemannian manifold with metric $F = \tilde{g}d^\alpha x_i d^\alpha x_j$ is defined by

$$\frac{1}{(\alpha!)^n}\sqrt{\det\tilde{g}}(dx_1)^\alpha(dx_2)^\alpha...(dx_n)^\alpha.$$

Then,

$$\tilde{V} = \frac{1}{(\alpha!)^n}\int_A \sqrt{\det\tilde{g}}(dx_1)^\alpha(dx_2)^\alpha...(dx_n)^\alpha,$$

where A is a domain in the fractional Riemannian manifolds. In particular, if $n = 1$, the volume is just length and if $n = 2$, the volume is area. Now, if $\alpha = 1$, one can recover the classical well known formula.

5.1. Invariance of Volume Element under Changing of Coordinates

Let $y_1, y_2, ..., y_n$ be new coordinates: $x_1 = y_i, x_2 = y_i, ..., x_n = y_i, i = 1, 2, ..., n$ and $\tilde{g}_{\hat{i}\hat{j}}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{\hat{i}\hat{j}}(y) = \frac{\partial x_i^\alpha}{\partial y_{\hat{i}}^\alpha}\tilde{g}_{ij}(x(y))\frac{\partial x_j^\alpha}{\partial y_{\hat{j}}^\alpha}, \tag{5.1}$$

then

$$\sqrt{\det\tilde{g}_{ij}(x)}(dx_1)^\alpha(dx_2)^\alpha...(dx_n)^\alpha = \sqrt{\det\tilde{g}_{\hat{i}\hat{j}}(y)}(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha. \tag{5.2}$$

This follows from (5.1). Namely

$$\sqrt{\det\tilde{g}_{\hat{i}\hat{j}}(y)}(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha = \sqrt{\det\left(\frac{\partial x_i^\alpha}{\partial y_{\hat{i}}^\alpha}\tilde{g}_{ij}(x(y))\frac{\partial x_j^\alpha}{\partial y_{\hat{j}}^\alpha}\right)}(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha.$$

By using the fact that $\det(A_1 A_2 A_3) = \det(A_1)\det(A_2)\det(A_3)$ and by letting $\det\left(\frac{\partial x_i^\alpha}{\partial y_{\hat{i}}^\alpha}\right) = \det\left(\frac{\partial x_j^\alpha}{\partial y_{\hat{j}}^\alpha}\right)$, then we have

$$\sqrt{\det\tilde{g}_{\hat{i}\hat{j}}(y)}(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha$$

$$= \sqrt{\left(\det\left(\frac{\partial x_i^\alpha}{\partial y_{\hat{i}}^\alpha}\right)\right)^2}\sqrt{\det\tilde{g}_{ij}(x(y))}(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha$$

$$= \sqrt{\det\tilde{g}_{ij}x(y)}\det\left(\frac{\partial x_i^\alpha}{\partial y_{\hat{i}}^\alpha}\right)(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha.$$

Now, note that

$$\det\left(\frac{\partial x_i^\alpha}{\partial y_i^\alpha}\right)(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha = (dx_1)^\alpha(dx_2)^\alpha...(dx_n)^\alpha,$$

then

$$\sqrt{\det \tilde{g}_{ij}x(y)}\det\left(\frac{\partial x_i^\alpha}{\partial y_i^\alpha}\right)(dy_1)^\alpha(dy_2)^\alpha...(dy_n)^\alpha = \sqrt{\det \tilde{g}_{ij}x(y)}(dx_1)^\alpha(dx_2)^\alpha...(dx_n)^\alpha.$$

Thus, we come to (5.2).

Example 5.1. "Segment of the sphere". Consider spherical coordinates $\{r, \theta, \phi\}$ in a domain $A = \{x_1, x_2, x_3 \colon x_1 > 0, x_2 > 0, x_3 > 0\}$ of $\mathbb{R}^3$ defined as

$$
\begin{aligned}
x_1 &= r^\alpha \sin^\alpha \theta \cos^\alpha \phi, \\
x_2 &= r^\alpha \sin^\alpha \theta \sin^\alpha \phi, \\
x_3 &= r^\alpha \cos^\alpha \theta.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
\frac{\partial^\alpha x_1}{\partial r^\alpha} &= \alpha! \sin^\alpha \theta \cos^\alpha \phi, \\
\frac{\partial^\alpha x_1}{\partial \theta^\alpha} &= \alpha! r^\alpha \cos^\alpha \theta \cos^\alpha \phi, \\
\frac{\partial^\alpha x_1}{\partial \phi^\alpha} &= (-1)^\alpha \alpha! r^\alpha \sin^\alpha \theta \sin^\alpha \phi.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{\partial^\alpha x_2}{\partial r^\alpha} &= \alpha! \sin^\alpha \theta \sin^\alpha \phi, \\
\frac{\partial^\alpha x_2}{\partial \theta^\alpha} &= \alpha! r^\alpha \cos^\alpha \theta \sin^\alpha \phi, \\
\frac{\partial^\alpha x_2}{\partial \phi^\alpha} &= \alpha! r^\alpha \sin^\alpha \theta \cos^\alpha \phi.
\end{aligned}
$$

Also, we get

$$
\begin{aligned}
\frac{\partial^\alpha x_3}{\partial r^\alpha} &= \alpha! \cos^\alpha \theta, \\
\frac{\partial^\alpha x_3}{\partial \theta^\alpha} &= (-1)^\alpha \alpha! r^\alpha \sin^\alpha \theta, \\
\frac{\partial^\alpha x_3}{\partial \phi^\alpha} &= 0.
\end{aligned}
$$

Now, if we let

$$X = (r^\alpha \sin^\alpha \theta \cos^\alpha \phi, r^\alpha \sin^\alpha \theta \sin^\alpha \phi, r^\alpha \cos^\alpha \theta),$$

then it follows that

$$\frac{\partial^\alpha X}{\partial r^\alpha} = \alpha!(\sin^\alpha \theta \cos^\alpha \phi, \sin^\alpha \theta \sin^\alpha \phi, \cos^\alpha \theta),$$

$$\frac{\partial^\alpha X}{\partial \theta^\alpha} = \alpha!(r^\alpha \cos^\alpha \theta \cos^\alpha \phi, r^\alpha \cos^\alpha \theta \sin^\alpha \phi, (-1)^\alpha r^\alpha \sin^\alpha \theta),$$

$$\frac{\partial^\alpha X}{\partial \phi^\alpha} = \alpha!((-1)^\alpha r^\alpha \sin^\alpha \theta \sin^\alpha \phi, r^\alpha \sin^\alpha \theta \cos^\alpha \phi, 0).$$

$\tilde{g}_{11} = \langle \frac{\partial^\alpha X}{\partial r^\alpha}, \frac{\partial^\alpha X}{\partial r^\alpha} \rangle = (\alpha!)^2 \sin^{2\alpha} \theta (\cos^{2\alpha} \phi + \sin^{2\alpha} \phi) + \cos^{2\alpha} \theta.$

$\tilde{g}_{12} = \langle \frac{\partial^\alpha X}{\partial r^\alpha}, \frac{\partial^\alpha X}{\partial \theta^\alpha} \rangle = (\alpha!)^2 r^\alpha (\sin^\alpha \theta \cos^\alpha \theta (\cos^{2\alpha} \phi + \sin^{2\alpha} \phi) + (-1)^\alpha \sin^\alpha \theta \cos^\alpha \theta) = \tilde{g}_{21}.$

$\tilde{g}_{13} = \langle \frac{\partial^\alpha X}{\partial r^\alpha}, \frac{\partial^\alpha X}{\partial \phi^\alpha} \rangle = (\alpha!)^2 r^\alpha \sin^{2\alpha} \theta (\sin^\alpha \phi \cos^\alpha \phi + (-1)^\alpha \cos^\alpha \phi \sin^\alpha \phi) = \tilde{g}_{31}.$

Similarly, we have

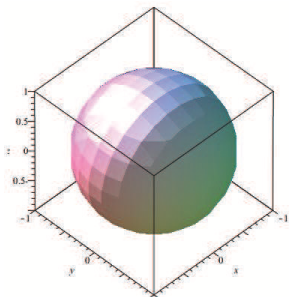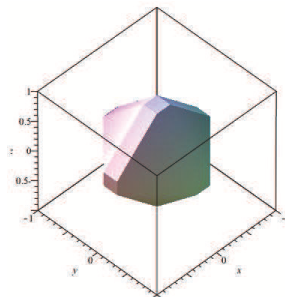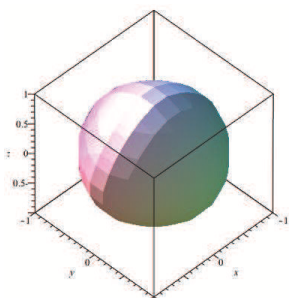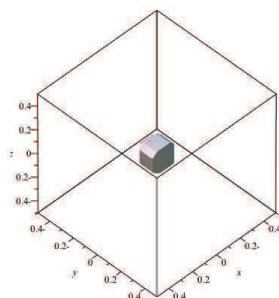$$\tilde{g}_{22} = \langle \frac{\partial^\alpha X}{\partial \theta^\alpha}, \frac{\partial^\alpha X}{\partial \theta^\alpha} \rangle = (\alpha!)^2 r^{2\alpha} (\cos^{2\alpha} \theta (\cos^{2\alpha} \phi + \sin^{2\alpha} \phi) + (-1)^{2\alpha} \sin^{2\alpha} \theta).$$

$$\tilde{g}_{23} = \langle \frac{\partial^\alpha X}{\partial \theta^\alpha}, \frac{\partial^\alpha X}{\partial \phi^\alpha} \rangle$$

$$= (\alpha!)^2 r^{2\alpha} (\sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi + (-1)^\alpha \sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi) = \tilde{g}_{32}.$$

$$\tilde{g}_{33} = \langle \frac{\partial^\alpha X}{\partial \phi^\alpha}, \frac{\partial^\alpha X}{\partial \phi^\alpha} \rangle = (\alpha!)^2 r^{2\alpha} \sin^{2\alpha} \theta (\cos^{2\alpha} \phi + (-1)^{2\alpha} \sin^{2\alpha} \phi).$$

Then, in general we have

$\det(\tilde{g_{ij}})$

$$= (\alpha!)^6 \left\{ r^{16\alpha^4} \left[\sin^{2\alpha} \theta (\cos^{2\alpha} \theta (\cos^{2\alpha} \phi + \sin^{2\alpha} \phi) + (-1)^{2\alpha} \sin^{2\alpha} \theta)(\cos^{2\alpha} \phi + (-1)^{2\alpha} \sin^{2\alpha} \phi) \right. \right.$$

$$\left. - (\sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi + (-1)^\alpha \sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi)^2 \right]^2$$

$$+ r^{4\alpha^4} \sin^{4\alpha^2} \theta \left[(\sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi + (-1)^\alpha \sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi) \right.$$

$$(\sin^\alpha \phi \cos^\alpha \phi + (-1)^\alpha \cos^\alpha \phi \sin^\alpha \phi) - (\sin^\alpha \theta \cos^\alpha \theta (\cos^{2\alpha} \phi + \sin^{2\alpha} \phi) + (-1)^\alpha \sin^\alpha \theta \cos^\alpha \theta)$$

$$\left. (\cos^{2\alpha} \phi + (-1)^{2\alpha} \sin^{2\alpha} \phi) \right]^2$$

$$+ r^{4\alpha^4} \left[(\sin^\alpha \theta \cos^\alpha \theta (\cos^{2\alpha} \phi + \sin^{2\alpha} \phi) + (-1)^\alpha \sin^\alpha \theta \cos^\alpha \theta) \right.$$

$$(\sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi + (-1)^\alpha \sin^\alpha \theta \sin^\alpha \phi \cos^\alpha \theta \cos^\alpha \phi)$$

$$\left. \left. - \sin^{2\alpha} \theta (\cos^{2\alpha} \theta (\cos^{2\alpha} \phi + \sin^{2\alpha} \phi) + (-1)^{2\alpha} \sin^{2\alpha} \theta)(\sin^\alpha \phi \cos^\alpha \phi + (-1)^\alpha \cos^\alpha \phi \sin^\alpha \phi) \right]^2 \right\}$$

$= H.$

Then, the volume element is given by

$$d\tilde{V} = \frac{1}{(\alpha!)^2} \sqrt{H} (d\theta)^\alpha (d\phi)^\alpha.$$

FIGURE 8. $\alpha = 1$

FIGURE 9. $\alpha = 0.5$

FIGURE 10. $\alpha = 0.8$

FIGURE 11. $\alpha = 0.2$

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