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A New Family of Semicircular and Circular Arc Tan-Exponential Type Distributions

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Abstract In this paper, an attempt is made to construct a new family of two parameter semicircular model, we call this as Semicircular Arc Tan-Exponential Type distribution, by applying simple projection on Arc Tan-Exponential Type distribution ([Y. Phani, S.V.S. Girija, A.V.D. Rao, Arc Tan-Exponential Type distribution induced by stereographic projection/ bilinear transformation on modified wrapped exponential distribution, Journal of Applied Mathematics, Statistics and Informatics (JAMSI) 9 (1) (2013) 69–74]) for modeling semicircular data, probability density and cumulative distribution functions of said model are presented and their graphs are plotted for various values of parameters. It is extended to the l-axial Arc Tan-Exponential Type distribution by simple transformation for modeling any arc of arbitrary length.

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Keywords: circular models; characteristic function; probability distribution; projection

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1. Introduction

Angular / circular data are very common in the areas of Biology, Geology, Meteorology, Earth Science, Political Science, Economics, Computer Science, etc. Full circular models are prevalent in most of the text books (Fisher, 1993 [1]; Jammalamadaka and Sen Gupta, 2001 [2]; Mardia and Jupp, 2000 [3]). The authors like Gradshteyn and Ryzhik (2007) [4], Ahn and Kim (2008) [5] and Yedlapalli et al. (2013) [6] noticed that for some cases angular data do not require full circular models for modeling. For example, when sea turtles emerge from the ocean in search of a nesting site on dry land, a random variable having values on a semicircle is sufficient for modeling such data. Similarly, when an aircraft is lost but its departure and its initial headings are known, a semicircular random

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variable is sufficient for such angular data. A few more examples of semicircular data are available in Ugai et al. (1977) [7].

Guardiola (2004) [4] obtained the semicircular normal distribution by using a simple projection and Byoung et al. (2008) [5] developed a family of the semicircular Laplace distributions for modeling semicircular data by simple projection, Phani et al. (2013) [6] constructed some semicircular distributions by applying Inverse Stereographic projection, Dattatreya Rao et al. (2016) [8] developed a circular logistic distribution by applying inverse stereographic projection. In this paper we developed a family of Semicircular Arc Tan-Exponential Type distribution (**SCATETD**) by projecting Arc Tan-Exponential Type distribution (Phani et al., 2013 [9]) over a semicircular segment. We plotted the graphs of the density function and distribution function for various values of parameters. We derive the characteristic function for some of the members of the proposed family, and also we extend this family for l-axial also.

2. Arc Tan-Exponential Type Distributions

Definition 2.1. Let X be a continuous random variable and $\alpha > 0$ and $\lambda > 0$ be real numbers. Then X is said to have an *Arc Tan-Exponential Type distribution* with parameters α and λ, if the probability density and cumulative distribution functions are given respectively by

$$f_X\left(x;\alpha,\lambda\right) = \frac{\alpha\lambda\text{cosech}\left(\pi\lambda\right)\exp\left(-2\lambda\tan^{-1}\left(\frac{x}{\alpha}\right)\right)}{\left(x^2+\alpha^2\right)} \tag{2.1}$$

and

$$F_X\left(x;\alpha,\lambda\right) = \frac{\text{cosech}\left(\pi\lambda\right)\left[\exp\left(\pi\lambda\right)-\exp\left(-2\tan^{-1}\left(\frac{x}{\alpha}\right)\right)\right]}{2} \tag{2.2}$$

where $-\infty < x < \infty$, $\alpha > 0$ and $\lambda > 0$.

By applying simple projection defined by $x = \nu\tan\left(\theta\right)$, $\nu > 0$, which leads to a Semicircular Arc Tan-Exponential Type distribution, whose probability density and cumulative distribution functions are given by

$$g\left(\theta\right) = \frac{\sigma\lambda\text{cosech}\left(\lambda\pi\right)\sec^2\left(\theta\right)}{\left(\sigma^2+\tan^2\left(\theta\right)\right)}\exp\left(-2\lambda\tan^{-1}\left(\frac{\tan\left(\theta\right)}{\sigma}\right)\right) \tag{2.3}$$

and

$$G\left(\theta\right) = \frac{\text{cosech}\left(\pi\lambda\right)}{2}\left[\exp\left(\pi\lambda\right)-\exp\left(-2\lambda\tan^{-1}\left(\frac{\tan\left(\theta\right)}{\sigma}\right)\right)\right] \tag{2.4}$$

where $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$, $\lambda > 0$ and $\sigma = \frac{\alpha}{\nu} > 0$ and it is denoted by $\textbf{SCATETD}\left(\lambda,\sigma\right)$.

Case(1) When $\sigma = 1$, i.e., $\nu \to \alpha$, we get

$$g\left(\theta\right) = \lambda\text{cosech}\left(\pi\lambda\right)\exp\left(-2\lambda\theta\right), \text{ where } -\frac{\pi}{2} < \theta \leq \frac{\pi}{2},$$

we call it as Semicircular Exponential-Type distribution.

Case(2) When $\sigma = 1$, i.e., $\nu \to \alpha$ and $\lambda \to 0^+$, we get

$$g(\theta) = \frac{1}{\pi}, \text{ where } -\frac{\pi}{2} < \theta \leq \frac{\pi}{2},$$

which is a Semicircular uniform distribution.

2.1. Semicircular Exponential Type Distribution

Definition 2.2. A random variable X_{SC} on the Semicircle is said to have the *Semicircular Exponential-Type distribution* with parameter $\lambda > 0$ denoted by **SCETD**(λ), if the probability density, the cumulative distribution and the characteristic functions are respectively given by

(i)

$$g(\theta) = \lambda \text{cosech}(\pi\lambda) \exp(-2\lambda\theta) \tag{2.5}$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\lambda > 0$,

(ii)

$$G(\theta) = \frac{\lambda}{2} \text{cosech}(\pi\lambda) [\exp(\pi\lambda) - \exp(-2\lambda\theta)] \tag{2.6}$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\lambda > 0$ and

(iii)

$$\varphi_p = \alpha_p + i\beta_p, \; p = 0, \pm1, \pm2, \pm3, ...,$$

where

$$\alpha_p = \left(\tfrac{4\lambda^2}{4\lambda^2+p^2}\right) \cos\left(\tfrac{p\pi}{2}\right) + \left(\tfrac{2\lambda p}{4\lambda^2+p^2}\right) \sin\left(\tfrac{p\pi}{2}\right) \coth \pi\lambda$$

$$\beta_p = \left(\tfrac{2\lambda p}{4\lambda^2+p^2}\right) \cos\left(\tfrac{p\pi}{2}\right) - \left(\tfrac{4\lambda^2}{4\lambda^2+p^2}\right) \sin\left(\tfrac{p\pi}{2}\right) \coth \pi\lambda$$

i.e.

$$\alpha_p = \begin{cases} (-1)^{\left(\frac{p-1}{2}\right)} \left(\tfrac{2\lambda p}{4\lambda^2+p^2}\right) \coth \pi\lambda, & \text{if p is odd} \\ (-1)^{\left(\frac{p}{2}\right)} \left(\tfrac{4\lambda^2}{4\lambda^2+p^2}\right), & \text{if p is even} \end{cases}$$

$$\beta_p = \begin{cases} (-1)^{\left(\frac{p+1}{2}\right)} \left(\tfrac{4\lambda^2}{4\lambda^2+p^2}\right) \coth \pi\lambda, & \text{if p is odd} \\ (-1)^{\left(\frac{p}{2}\right)} \left(\tfrac{2\lambda p}{4\lambda^2+p^2}\right), & \text{if p is even.} \end{cases} \tag{2.7}$$

Graphs of pdf and cdf of Semicircular Exponential Type Distribution for Various Values of the Parameter

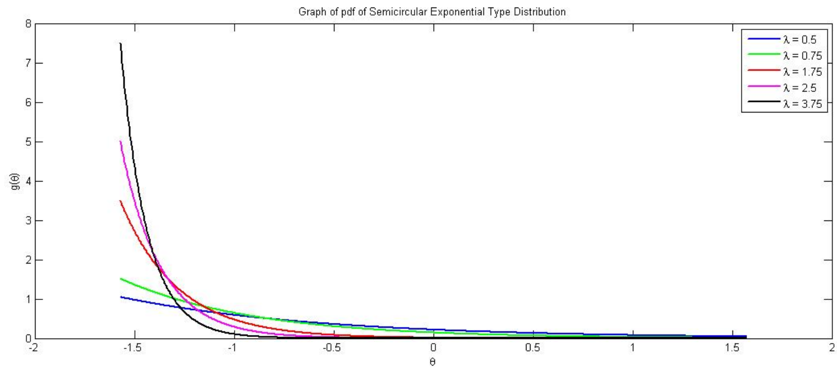

FIGURE 1. Graph of pdf of Semicircular Exponential Type Distribution

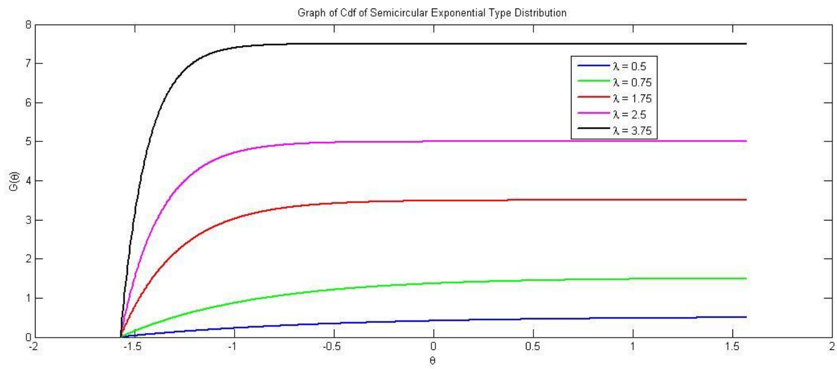

FIGURE 2. Graph of cdf of Semicircular Exponential Type Distribution

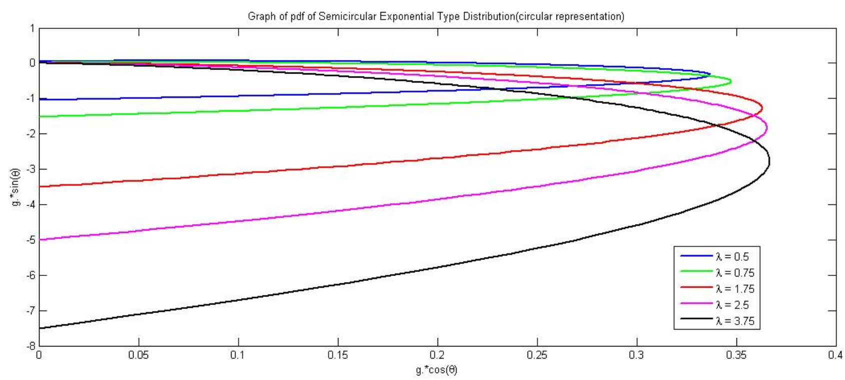

FIGURE 3. Graph of pdf of Semicircular Exponential Type Distribution (Circular Representation)

3. Extension to l-Axial Distribution

We extend the proposed family to the l -axial distribution, which is applicable to any arc of arbitrary length say $\frac{2\pi}{l}$ for $l = 1, 2, 3, \ldots$. To construct the l -axial Arc Tan-Exponential Type distribution, we consider the density function of Semicircular Arc Tan-Exponential Type distribution and use the transformation $\phi = \frac{2\pi}{l}$, $l = 1, 2, 3, \ldots$. The probability density function of ϕ is given by

$$g(\phi) = \frac{l\sigma\lambda \operatorname{cosech}(\lambda\pi)\sec^2\left(\frac{l\phi}{2}\right)}{2\left(\sigma^2 + \tan^2\left(\frac{l\phi}{2}\right)\right)} \exp\left(-2\lambda\tan^{-1}\left(\frac{\tan\left(\frac{l\phi}{2}\right)}{\sigma}\right)\right), \; -\frac{\pi}{l} < \phi < \frac{\pi}{l}. \tag{3.1}$$

Case (1): When $l = 1$ in (3.1), we get a density function

$$g(\phi) = \frac{\sigma\lambda \operatorname{cosech}(\lambda\pi)\sec^2\left(\frac{\phi}{2}\right)}{2\left(\sigma^2 + \tan^2\left(\frac{\phi}{2}\right)\right)} \exp\left(-2\lambda\tan^{-1}\left(\frac{\tan\left(\frac{\phi}{2}\right)}{\sigma}\right)\right), \; -\pi < \phi < \pi. \tag{3.2}$$

We call it as Circular Arc Tan-Exponential Type distribution.

Case (2): When $l = 1$ and $\sigma = 1$ in (3.1), we get a density function

$$g(\phi) = \frac{\lambda \operatorname{cosech}(\lambda\pi)}{2}\exp(-\lambda\phi), \quad -\pi < \phi < \pi. \tag{3.3}$$

We call it as Circular Exponential Type distribution.

Case (3): When $l = 2$ in (3.1), we get a density function

$$g(\phi) = \frac{\sigma\lambda \operatorname{cosech}(\lambda\pi)\sec^2(\phi)}{\sigma^2 + \tan^2(\phi)} \exp\left(-2\lambda\tan^{-1}\left(\frac{\tan(\phi)}{\sigma}\right)\right), \; -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \tag{3.4}$$

Which is same as Semicircular Arc Tan-Exponential Type distribution.

3.1. Circular Exponential Type Distribution

Definition 3.1. A random variable X_s on the circle is said to have the *Circular Exponential Type distribution* with parameter $\lambda > 0$ denoted by **CETD**(λ), if the probability density, the cumulative distribution and the characteristic function are respectively given by

$$\text{i)} \quad g(\theta) = \frac{\lambda \operatorname{cosech}(\lambda\pi)}{2}\exp(-\lambda\theta), \quad -\pi < \theta < \pi, \tag{3.5}$$

$$\text{ii)} \quad G(\theta) = \frac{\lambda}{2}\operatorname{cosech}(\lambda\pi)\left[\exp(\pi\lambda) - \exp(-2\lambda\theta)\right], \quad -\pi < \theta < \pi \tag{3.6}$$

and

$$\text{iii)} \quad \varphi_p = \alpha_p + i\beta_p, \; p = 0, \pm1, \pm2, \pm3, \ldots \tag{3.7}$$

where

$$\alpha_p = \left(\frac{4\lambda^2}{4\lambda^2+p^2}\right)\cos\left(\frac{p\pi}{2}\right) + \left(\frac{2\lambda p}{4\lambda^2+p^2}\right)\sin\left(\frac{p\pi}{2}\right)\coth(\pi\lambda),$$

$$\beta_p = \left(\frac{2\lambda p}{4\lambda^2+p^2}\right)\cos\left(\frac{p\pi}{2}\right) - \left(\frac{4\lambda^2}{4\lambda^2+p^2}\right)\sin\left(\frac{p\pi}{2}\right)\coth(\pi\lambda).$$

Characteristics of Semicircular Exponential Type Distribution

Parameter(s)	**Value(s)**
Trigonometric moments	$\alpha_p = \begin{cases} (-1)^{\left(\frac{p-1}{2}\right)}\left(\frac{2\lambda p}{4\lambda^2+p^2}\right)\coth(\pi\lambda) \text{ if p is odd} \\ (-1)^{\left(\frac{p}{2}\right)}\left(\frac{4\lambda^2}{4\lambda^2+p^2}\right) \text{ if p is even}\end{cases}$ $\beta_p = \begin{cases} (-1)^{\left(\frac{p+1}{2}\right)}\left(\frac{4\lambda^2}{4\lambda^2+p^2}\right)\coth(\pi\lambda) \text{ if p is odd} \\ (-1)^{\left(\frac{p}{2}\right)}\left(\frac{2\lambda p}{4\lambda^2+p^2}\right) \text{ if p is even}\end{cases}$
ρ_p and μ_p^0	$\rho_p = \frac{2\lambda}{\sqrt{(4\lambda^2+p^2)}}\left[\sqrt{\cos^2\left(\frac{p\pi}{2}\right)+\sin^2\left(\frac{p\pi}{2}\right)\coth^2(\pi\lambda)}\right]$ $\mu_p^0 = \tan^{-1}\left(\frac{p\cos\left(\frac{p\pi}{2}\right)-2\lambda\sin\left(\frac{p\pi}{2}\right)\coth(\pi\lambda)}{2\lambda\cos\left(\frac{p\pi}{2}\right)+p\sin\left(\frac{p\pi}{2}\right)\coth(\pi\lambda)}\right)$
Resultant length	$\rho = \frac{2\lambda\coth(\pi\lambda)}{\sqrt{4\lambda^2+1}}$
Mean direction	$\mu_0 = 2\pi + \tan^{-1}(2\lambda)$
Circular variance	$V_0 = \frac{\sqrt{4\lambda^2+1}-2\lambda\coth(\pi\lambda)}{\sqrt{4\lambda^2+1}}$
Circular standard deviation	$\sigma_0 = \sqrt{-2\ln(1-V_0)} = \sqrt{\ln(4\lambda^2+1)-2\ln(2\lambda\coth(\pi\lambda))}$
Central trigonometric moments	$\alpha_p^* = \rho_p\cos\left(\mu_p^0 - p\mu_0\right)$ $\beta_p^* = \rho_p\sin\left(\mu_p^0 - p\mu_0\right)$
Skewness	$\gamma_1^0 = \frac{\beta_2^*}{V_0^{3/2}}$ where $\beta_2^* = \left(\frac{\lambda}{1+\lambda^2}\right)\sin\left(\tan^{-1}\left(\frac{1}{\lambda}\right) - 4\pi - 2\tan^{-1}(2\lambda)\right)$
Kurtosis	$\gamma_2^0 = \frac{\alpha_2^*-(1-V_0)^4}{V_0^2}$ where $\alpha_2^* = \left(\frac{\lambda}{1+\lambda^2}\right)\cos\left(\tan^{-1}\left(\frac{1}{\lambda}\right) - 4\pi - 2\tan^{-1}(2\lambda)\right)$

4. Conclusion

In this paper, we investigated a family of semicircular and circular Arc Tan-Exponential Type distributions, which follows from inducing simple projection on Arc Tan-Exponential Type distribution. The density and distribution functions of this distribution admit explicit forms, as do characteristic function. As this distribution is asymmetric, it is hoped that it offers a better option for modeling skew circular as well as l-axial data.

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