Thai Journal of Mathematics
Volume 18 Number 2 (2020)
Pages 745–749

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Some Differential Inequalities II

H. Aaisha Farzana[1,*] and M. P. Jeyaraman[2]

[1] *Department of Mathematics with CA, Agurchand Manmull Jain College, Meenambakkam, Chennai, India*
e-mail : h.aaisha@gmail.com

[2] *Department of Mathematics, L. N. Government College, Ponneri, Chennai 601204, TamilNadu, India*
e-mail : jeyaraman_mp@yahoo.co.in

Abstract In this paper, we derive some interesting relations associated with some differential inequalities in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. Some interesting applications of the main results are also obtained.

MSC: 30C45; 30C50; 30C55

Keywords: analytic function; starlike function; convex function; differential subordination mappings

Submission date: 12.05.2017 / Acceptance date: 16.08.2018

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in $\mathbb{U}$. For n a positive integer and $a \in \mathbb{C}$,

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$.

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disc $\mathbb{U}$.

Definition 1.1. If f and g are two analytic functions in $\mathbb{U}$, we say that f *is said to be subordinate to* g, written symbolically as $f \prec g$, if there exists a *Schwarz function* w, which (by definition) is analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.

If the function g is univalent in $\mathbb{U}$, then we have the following equivalence (c.f [1, 2]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.2. Let Q denote the set of all functions q that are analytic and injective on $\partial\mathbb{U} \backslash E(q)$, where

*Corresponding author.

Published by The Mathematical Association of Thailand.

$$E(q) = \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$. Further, let the subclass of Q for $Q(0) \equiv a$ be denoted by $Q(a)$ and $Q(1) \equiv Q_1$.

In the present paper, we obtain some interesting relations associated with some differential inequalities in $\mathbb{U}$. These relations extend and generalize earlier results. Some applications of the main results are also obtained.

2. Preliminaries

To prove our results, we need the following results due to Miller and Mocanu [2].

Lemma 2.1. [2, p. 24] *Let $q \in Q$, with $q(0) = a$, and let $p(z) = a+a_n z^n + a_{n+1}z^{n+1}+\ldots$ be analytic in $\mathbb{U}$ with $p(z) \not\equiv a$ and $k \geq 1$. If p is not subordinate to q, then there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(q)$ and $k \geq n \geq 1$ for $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$,*
(i) $p(z_0) = q(\zeta_0)$,
(ii) $z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$.

Lemma 2.2. [2, p. 26] *Let $p \in \mathcal{H}[a, n]$, with $p(z) \not\equiv a$ and $k \geq 1$. If $z_0 \in \mathbb{U}$ and*

$$Re(p(z_0)) = min\{Re(p(z)) : |z| \leq |z_0|\},$$

then

$$z_0 p'(z_0) \leq -\frac{k}{2}\frac{|p(z_0) - a|^2}{Re(a - p(z_0))}.$$

3. Main Results

Unless and otherwise mentioned throughout the paper $\sigma \geq 0$, $0 \leq \beta \leq 1$, $a \in \mathbb{C}$ with $Re(a) > 0$ and all the powers are the principal ones.

Theorem 3.1. *Let $p(z)$ be an analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$\left|Im\left([ap(z) - \lambda]\left[\frac{zp'(z)}{p(z)} + ap(z) - 1\right]\right)\right| < \frac{\mathcal{T}}{(Re(a)\sqrt{(\lambda + 2\lambda Re(a) + 2Re(a))\lambda})}, \tag{3.1}$$

where

$$\mathcal{T} = \lambda\big(2|a|\lambda Re(a) + 2Re(a)|a| + \lambda|a| + \sqrt{((\lambda + 2\lambda Re(a) + 2Re(a))\lambda)}Im(a)\big), \tag{3.2}$$

$(\lambda > 0)$ *then* $Re(ap(z)) > 0$.

Proof. Let us define both $q(z)$ and $h(z)$ as follows

$$q(z) = ap(z)$$

and

$$h(z) = \frac{a + \bar{a}z}{1 - z},\ Re(a) > 0,$$

where $q(z)$ and $h(z)$ are analytic functions in $\mathbb{U}$ with $g(0) = h(0) = a \in \mathbb{C}$ and $h(\mathbb{U}) = \{w : Re(w) > 0\}$.

Now, we suppose that $q(z) \not\prec h(z)$, then by using Lemma 2.1 there exists $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash\{1\}$ such that

$$q(z_0) = h(\zeta_0) = i\beta \text{ and } z_0 q'(z_0) = k\zeta_0 h'(\zeta_0),\ k > 1$$

Also, we note that

$$z_0 q'(z_0) = \frac{-|q(z_0) - a|^2}{2Re(a - q(z_0))}.$$

Now,

$$\begin{aligned}
&\left| Im\left((ap(z_0) - \lambda)\left(\frac{z_0 p'(z_0)}{p(z_0)} + ap(z_0) - 1 \right)\right)\right| \\
&= \left| Im\left(\frac{k\zeta_0 h'(\zeta_0}{h(\zeta_0)}(h(\zeta_0) - \lambda) + h(\zeta_0)(h(\zeta_0) - \lambda) - (h(\zeta_0) - \lambda) \right)\right| \\
&\geq \left| Im\left(\frac{\zeta_0 h'(\zeta_0}{h(\zeta_0)}(h(\zeta_0) - \lambda) + h(\zeta_0)(h(\zeta_0) - \lambda) - (h(\zeta_0) - \lambda) \right)\right| \\
&= \left| Im\left(\frac{-|i\beta - a|^2}{2Re(a)i\beta}(i\beta - \lambda) + i\beta(i\beta - \lambda) - (i\beta - \lambda) \right)\right| \\
&= \left| \frac{-\lambda(|a|^2 + \beta^2 - 2Im(a)\beta)}{2\beta Re(a)} - \lambda\beta - \beta \right| \\
&\geq \frac{-\lambda(|a|^2 + \beta^2 - 2Im(a)\beta)}{2\beta Re(a)} - \lambda\beta - \beta = Q(\beta),
\end{aligned}$$

where $Q(\beta)$ is a function of β, and it attains a minimum at

$$\beta^* = \frac{-\sqrt{(\lambda + 2\lambda Re(a) + 2Re(a))\lambda)}|a|}{(\lambda + 2\lambda Re(a) + 2Re(a))}.$$

Therefore,

$$\begin{aligned}
\left| Im\left((ap(z_0) - \lambda)\left(\frac{z_0 p'(z_0)}{p(z_0)} + ap(z_0) - 1 \right)\right)\right| &\geq Q(\beta^*) \\
&= \frac{\mathcal{T}}{(Re(a)\sqrt{((\lambda + 2\lambda Re(a) + 2Re(a))\lambda))}}
\end{aligned}$$

where $\mathcal{T}$ is defined by (3.2) and hence, we obtain a contradiction to the assumption (3.1). Therefore, $q(z) \prec h(z)$ and $Re(ap(z)) > 0$. ■

Theorem 3.2. *Let $B(z)$ be a complex valued function in $\mathbb{U}$ with $Re(aB(z)) \leq Im(ap(z))$.If $p(z)$ is an analytic function in $\mathbb{U}$ with $p(0) = 1$ and*

$$Re\left(1 + \frac{zp'(z)B(z)}{p(z)^2}\right) > \frac{-K}{2|a|^2 Re(a)^2}, \tag{3.3}$$

where

$$\begin{aligned}
K = {} & |a|(1 + Re(a)^6) + 2Im(a)Re(a)|a|^2 \\
& + |a|Im(a)^2 Re(a)^2(2Re(a)^2 + Im(a)^2) - 2Re(a)^2|a|^2,
\end{aligned} \tag{3.4}$$

then $Re(ap(z)) > 0$.

Proof. Let us define both $g(z)$ and $h(z)$ as follows

$$g(z) = ap(z)$$

and

$$h(z) = \frac{a + \bar{a}z}{1 - z}, Re(a) > 0$$

where $g(z)$ and $h(z)$ are analytic functions in $\mathbb{U}$ with $g(0) = h(0) = a \in \mathbb{C}$ and $h(\mathbb{U}) = \{w : Re(w) > 0\}$.
Now, suppose $g(z) \not\prec h(z)$ by Lemma 2.1 there exists $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash\{1\}$ such that

$$g(z_0) = h(\zeta_0) = i\gamma \text{ and } z_0 g'(z_0) = k\zeta_0 h'(\zeta_0).$$

Also, from Lemma 2.2 we have

$$z_0 g'(z_0) \leq -\frac{k|i\gamma - a|^2}{2Re(a)}, k \geq 1.$$

Next,

$$\begin{aligned} &Re\left(1 + \frac{z_0 p'(z_0)B(z_0)}{p(z_0)^2}\right) \qquad (3.5)\\ &= 1 + Re\left(\frac{z_0 g'(z_0)aB(z_0)}{g(z_0)^2}\right)\\ &= 1 + Re\left(\frac{k\zeta_0 h'(\zeta_0)aB(z_0)}{h(\zeta_0)^2}\right)\\ &\leq 1 - Re\left(\frac{k|i\gamma - a|^2 aB(z_0)}{2Re(a)(i\gamma)^2}\right)\\ &\leq 1 + \frac{(|a|^2 - 2Im(a)\gamma + \gamma^2)Re(aB(z_0))}{2Re(a)\gamma^2}\\ &\leq 1 + \frac{(|a|^2 - 2Im(a)\gamma + \gamma^2)\gamma}{2Re(a)\gamma^2} = f(\gamma), \end{aligned}$$

where the function $f(\gamma)$ attains the maximum at $|a|$. Therefore,

$$Re\left(1 + \frac{z_0 p'(z_0)B(z_0)}{p(z_0)^2}\right) \leq f(|a|) = \frac{-K}{2|a|^2 Re(a)^2},$$

where K is defined by (3.4). Hence we obtain a contradiction to (3.3). Therefore $g(z) \prec h(z)$ or $Re(ap(z)) > 0$. ∎

4. Applications and Examples

Letting $a = 1$ in Theorem 3.1, we have the following Corollary

Corollary 4.1. *Let $p(z)$ be an analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$\left|Im\left((p(z) - \lambda)\left(\frac{zp'(z)}{p(z)} + p(z) - 1\right)\right)\right| < \sqrt{(3\lambda + 2)\lambda} \qquad (\lambda > 0),$$

then $Re(p(z)) > 0$.

By taking $p(z) = \frac{zf'(z)}{f(z)}$ in the above Corollary, we get the following result due to Xu and Yang [3, Theorem 2, pp 586]

Corollary 4.2. *If $f \in \mathcal{A}$ satisfies $f'(z)f(z) \neq 0$ in $0 < |z| < 1$ and*

$$\left| Im\left(\frac{z^2 f''(z)}{f(z)} - \frac{\lambda z f''(z)}{f'(z)}\right)\right| < \sqrt{(3\lambda+2)\lambda} \textit{ for } (\lambda > 0),$$

then $Re(\frac{zf'(z)}{f(z)}) > 0$.

Letting $\lambda = 1$ in the above Corollary 4.2, we have the following result which is an improvement of the result which were earlier proved by Lin and Owa [4] and Obradovic [5] respectively

Corollary 4.3. *If $f \in \mathcal{A}$ satisfies $f'(z)f(z) \neq 0$ in $0 < |z| < 1$ and*

$$\left| Im\left(\frac{z^2 f''(z)}{f(z)} - \frac{z f''(z)}{f'(z)}\right)\right| < \sqrt{5},$$

then $Re(\frac{zf'(z)}{f(z)}) > 0$.

Letting $p(z) = \frac{zf'(z)}{f(z)}$ and $B(z) = 1$ in Theorem 3.2, we obtain the following Corollary

Corollary 4.4. *If $f \in \mathcal{A}$ satisfies $\frac{zf'(z)}{f(z)} \neq 0$ and $Re\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) > 0$, then $Re(\frac{zf'(z)}{f(z)}) > 0$.*

By taking $p(z) = f'(z)$ and $B(z) = 1$ in Theorem 3.2, we obtain the following Corollary

Corollary 4.5. *If $f \in \mathcal{A}$ and $Re\left(1 + \frac{zf''(z)}{f'(z)^2}\right) > 0$, then $Re(f'(z)) > 0$, hence f is univalent in $\mathbb{U}$.*

Acknowledgements

The work of the first author was supported by the *Department of Science and Technology*, India with reference to the sanction order no. SR/DST-WOS A/MS-10/2013(G).

References

[1] S.S. Miller, P.T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28 (1981) 157–171.

[2] S.S. Miller, P.T. Mocanu, Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics 225, Marcel Dekker Inc., New York and Basel, 2000.

[3] N. Xu and D. Yang, Some criteria for starlikeness and strongly starlikeness, Bull. Korean Math. Soc. 42 (3) (2005) 579–590.

[4] L.J. Lin, S. Owa, Properties of the Salagean operator,Georgian Mathematical Journal 5 (4) (1998) 361–366.

[5] M. Obradovic, Ruscheweyh derivatives and some classes of univalent functions, In: Current Topics in Analytic Function Theory, (H.M. Srivastava and S. Owa, Editors), World Sci. Publishing, River Edge, NJ (1992), 220–233.