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Some Common Fixed Point Theorem for Geraghty's Type Contraction Mapping with Two T-Metrics in T-Metric Spaces with Graph

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Abstract The purpose of this paper is to present some existence and uniqueness results for common fixed point theorems for Geraghty's type contraction mappings with two T-metrics in T-metric spaces endowed with a directed graph. In addition, Our results generalize those presented in [J. Martínez-Moreno, W. Sintunavarat, Y.J. Cho, Common fixed point theorems for Geraghty's type contraction mappings using the monotone property with two metrics, Fixed Point Theory Appl. 2015 (2015) 174].

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1. Introduction and Preliminaries

Geraghty [1] introduced an interesting class Θ of functions $\theta : [0, \infty) \to [0, 1)$ satisfying that:

$$\theta(t_n) \to 1 \quad \Longrightarrow \quad t_n \to 0,$$

and obtained some results which is a generalization of the Banach's contraction principle in 1973.

Recently, Martíneez-Moreno et al. [2] gave some new common fixed point theorems for Geraghty's type contraction mappings employing the monotone property with two metrics by using d-compatibility and g-uniform continuity defined as follows.

Definition 1.1 ([3])**.** Let (X, d) be a metric space, and let $f, g : X \to X$ be two mappings. The mappings g and f are said to be *d-compatible* if

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$.

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Definition 1.2 ([3]). Let (X,d) and (Y,d') be two metric spaces, and let $f : X \to Y$ and $g : X \to X$ be two mappings. A mapping f is said to be *g-uniformly continuous* on X if, for any real number $\epsilon > 0$, there exists $\delta > 0$ such that $d'(fx, fy) < \epsilon$ whenever $x, y \in X$ and $d(gx, gy) < \delta$. If g is the identity mapping, then f is said to be *uniformly continuous* on X.

Let (X,d) be a metric space, and Δ be a diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

Throughout the paper we shall say that G with the above-mentioned properties satisfies standard conditions.

The fixed point theorem using the context of metric spaces endowed with a graph was initiated by Jachymski [4], which generalizes the Banach contraction principle to mappings on a metric spaces with a graph. Also, the definitions of G-continuous and the property A were given in [4].

Definition 1.3 ([4]). A mapping $f : X \to X$ is called *G-continuous* if for any $x \in X$ such that there exists a sequence $\{x_n\}$ in X, $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $f(x_n) \to f(x)$.

Definition 1.4 ([4]). Let (X,d) be a metric space, and suppose that G is a directed graph. We say that the triple (X,d,G) has the property A, if for any sequence $\{x_n\}$ in X with $x_n \to x$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$.

Definition 1.5 ([5]). Let (X,d) be a complete metric space, and let $E(G)$ be the set of the edges of the graph. We say that $E(G)$ satisfies the transitivity property if and only if, for all $x, y, a \in X$,

$$(x,a), (a,y) \in E(G) \Rightarrow (x,y) \in E(G).$$

Since then, many authors have studied the problem of existence of a fixed point for single-valued mappings and multi-valued mappings in several spaces with a graph, see [6–13].

Definition 1.6 ([14]). A binary normed operation is a mapping $\diamond : [0,\infty) \times [0,\infty) \to [0,\infty)$ which satisfies the following conditions:

(1) $\diamond$ is associative and commutative;
(2) $\diamond$ is continuous;
(3) $a \diamond 0 = a$ for all $a \in [0,\infty)$;
(4) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,\infty)$.

In 2011, S.Sedghi et.al. [14] introduced the concept of T-metric spaces as follows.

Definition 1.7 ([14]). Let X be a nonempty set. A T-metric on X is a function $T : X^2 \to \mathbb{R}$ that satisfies the following condition for each $x, y, z \in X$

(1) $T(x,y) \ge 0$ and $T(x,y) = 0$ if and only if $x = y$;
(2) $T(x,y) = T(y,x)$;
(3) $T(x,y) \le T(x,z) \diamond T(y,z)$.

The 3-tuple $(X, T, \diamond)$ is called a *T-metric space*.

Example 1.8. Every ordinary metric d is a T-metric with $a \diamond b = a + b$.

Definition 1.9 ([14]). Let $(X, T, \diamond)$ be a T-metric space.

(1) A sequence $\{x_n\}$ in X converges to x if $T(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $T(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.
(3) The T-metric space $(X, T, \diamond)$ is said to be *complete* if every Cauchy sequence is convergent.

Definition 1.10 ([14]). Let $(X, T, \diamond)$ be a T-metric space. T is said to be continuous if

$$\lim_{n\to\infty} T(x_n, y_n) = T(x, y),$$

wherever

$$\lim_{n\to\infty} T(x_n, x) = \lim_{n\to\infty} T(y_n, y) = 0.$$

Lemma 1.11 ([14]). *Let $(X, T, \diamond)$ be a T-metric space. Then T is a continuous function.*

The aim of this paper is to present some existence and uniqueness results for common fixed point theorems for θ contraction mappings with two T-metrics endowed with a directed graph. Furthermore, by using our main results, we are able to generalize the results obtained in [2].

2. Main Results

We introduce the concept of g-Cauchy and edge preserving which are an effective tool as follows:

Definition 2.1 ([3]). Let $(X, T, \diamond)$ be a T-metric space, and let $f, g : X \to X$ be two mappings. The mappings g and f are said to be T-*compatible* if

$$\lim_{n\to\infty} T(gfx_n, fgx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$.

Definition 2.2. Let $(X, T, \diamond)$ and $(Y, T', \diamond)$ be two T-metric spaces, and let $f : X \to Y$ and $g : X \to X$ be two mappings. The mapping f is said to be g-*Cauchy* on X if, for any sequence $\{x_n\}$ in X such that $\{gx_n\}$ is a Cauchy sequence in $(X, T, \diamond)$, then $\{fx_n\}$ is Cauchy sequence in $(Y, T', \diamond)$.

Definition 2.3. Let $(X, T, \diamond)$ be a T-metric space, and suppose that G is a directed graph. A mapping $f : X \to X$ is called G-*continuous* if for any $x \in X$ such that there exists a sequence $\{x_n\}$ in X, $T(x_n, x) \to 0$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $T(f(x_n), f(x)) \to 0$ as $n \to \infty$.

Definition 2.4. Let $(X, T, \diamond)$ be a T-metric space, and suppose that G is a directed graph. We say that the triple (X, T, G) has the property A, if for any sequence $\{x_n\}$ in X with $T(x_n, x) \to 0$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$.

Definition 2.5. Let G be a directed graph, and let $f, g : X \to X$ be two mapping. We say that f is *g-edge preserving w.r.t G* if

$$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G).$$

We now introduce a new class of the Geraghty type contractions in the following definition.

Definition 2.6. Let $(X, T, \diamond)$ be a T-metric space endowed with a directed graph G, and let $f, g : X \to X$ be given mappings. The pair (f, g) is called a *θ-contraction w.r.t T* if :

(1) f is g-edge preserving w.r.t G;
(2) there exists two functions $\theta \in \Theta$ such that for all $x, y \in X$ such that $(gx, gy) \in E(G)$,

$$T(fx, fy) \leq \theta(M(gx, gy))M(gx, gy), \tag{2.1}$$

where $M(gx, gy) = \max\left\{T(gx, gy), T(gx, fx), T(gy, fy)\right\}$.

Let (X, d) be a metric space endowed with a directed graph G satisfying the standard conditions, and let two mappings $f, g : X \to X$ be given.

We define important subsets of X as follows

$$X(f, g) := \{u \in X : (gu, fu) \in E(G)\},$$

$$C(f, g) := \{u \in X : fu = gu\},$$

i.e., the set of all coincidence points of mappings f and g, and

$$Cm(f, g) := \{u \in X : fu = gu = u\},$$

i.e., the set of all common fixed points of mappings f and g.

Let T', T be two T-metrics on X. By $T < T'$ (resp., $T \leq T'$), we mean $T(x, y) < T'(x, y)$ (resp., $T(x, y) \leq T'(x, y)$) for all $x, y \in X$.

Now we are ready to present and prove the main results.

Theorem 2.7. *Let $(X, T', \diamond)$ be a complete T-metric space endowed with a directed graph G, and let T be another T-metric on X. Suppose that $f, g : X \to X$ and (f, g) is a θ-contraction w.r.t T. Suppose that*

(1) *$g : (X, T', \diamond) \to (X, T', \diamond)$ is continuous ;*
(2) *$f(X) \subseteq g(X)$ and $(g(X), T', \diamond)$ be a complete T-metric space ;*
(3) *$E(G)$ satisfies the transitivity property;*
(4) *if $T \ngeq T'$, assume that $f : (X, T, \diamond) \to (X, T', \diamond)$ is* g-Cauchy *on X;*
(5) *$f : (X, T', \diamond) \to (X, T', \diamond)$ is G-continuous, and f and g are T'-compatible.*

Then, under these conditions,

$$X(f, g) \neq \emptyset \text{ if and only if } C(f, g) \neq \emptyset.$$

Proof. ($\Leftarrow$) Suppose that $C(f, g) \neq \emptyset$. Let $u \in C(f, g)$. We have $fu = gu$. Then $(gu, fu) = (gu, gu) \in \Delta \subset E(G)$. Hence $(gu, gu) = (gu, fu) \in E(G)$ which means that $u \in X(f, g)$ and thus $X(f, g) \neq \emptyset$.

($\Rightarrow$) Suppose now $X(f,g) \neq \emptyset$. Let $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$. By the assumption that $f(X) \subseteq g(X)$ and $f(x_0) \in X$, it easy to construct a sequences $\{x_n\}$ in X for which

$$gx_n = fx_{n-1},$$

for all $n \in \mathbb{N}$. If $gx_{n_0} = gx_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0-1} is a coincidence point of the mappings g and f. Therefore, we assume that, for each $n \in \mathbb{N}$, $gx_n \neq gx_{n-1}$ holds.

Since $(gx_0, fx_0) = (gx_0, gx_1) \in E(G)$ and f is edge preserving w.r.t g, we have $(fx_0, fx_1) = (gx_1, gx_2) \in E(G)$. Continue inductively, we obtain that $(gx_{n-1}, gx_n) \in E(G)$ for each $n \in \mathbb{N}$. Hence it follows from the contractive condition that

$$\begin{aligned} T(gx_{n+1}, gx_{n+2}) &= T(fx_n, fx_{n+1}) \\ &\leq \theta(M(gx_n, gx_{n+1}))M(gx_n, gx_{n+1}) \\ &< M(gx_n, gx_{n+1}). \end{aligned} \tag{2.2}$$

On the other hand, we get

$$\begin{aligned} M(gx_n, gx_{n+1}) &= \max\left\{T(gx_n, gx_{n+1}), T(gx_n, fx_n), T(gx_{n+1}, fx_{n+1})\right\} \\ &= \max\left\{T(gx_n, gx_{n+1}), T(gx_{n+1}, gx_{n+2})\right\}. \end{aligned}$$

If $M(gx_n, gx_{n+1}) = T(gx_{n+1}, gx_{n+2})$, then by (2.2), we obtain that

$$T(gx_{n+1}, gx_{n+2}) < T(gx_{n+1}, gx_{n+2})$$

which is a contradiction. So, for all $n \geq 1$, we have

$$M(gx_n, gx_{n+1}) = T(gx_n, gx_{n+1}). \tag{2.3}$$

Notice that in view of (2.2), we have

$$T(gx_{n+1}, gx_{n+2}) < T(gx_n, gx_{n+1}), \ \forall n \in \mathbb{N}.$$

Hence, we deduce that the sequence $\{T(gx_n, gx_{n+1})\}$ is nonnegative and increasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} T(gx_n, gx_{n+1}) = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, due to (2.2), we have

$$\frac{T(gx_{n+1}, gx_{n+2})}{T(gx_n, gx_{n+1})} = \frac{T(gx_{n+1}, gx_{n+2})}{M(gx_n, gx_{n+1})} \leq \theta(M(gx_n, gx_{n+1})) < 1.$$

It follows that $\lim_{n\to\infty} \theta(M(gx_n, gx_{n+1})) = 1$. Owing to the fact that $\theta \in \Theta$, we get $\lim_{n\to\infty} T(gx_n, gx_{n+1}) = \lim_{n\to\infty} M(gx_n, gx_{n+1}) = 0$, a contradiction. So, we conclude that

$$\lim_{n\to\infty} T(gx_n, gx_{n+1}) = 0. \tag{2.4}$$

We assert that $\{gx_n\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{gx_n\}$ is not a Cauchy sequence. Thus, there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, there exists $n(k), m(k) \in \mathbb{N}$ such that $n(k) > m(k) \geq k$ with the smallest number satisfying the condition below

$$T(gx_{n(k)}, gx_{m(k)}) \geq \epsilon \ \text{ and } \ T(gx_{n(k)-1}, gx_{m(k)}) < \epsilon.$$

Then, we have

$$\begin{aligned}\epsilon &\le T(gx_{m(k)}, gx_{n(k)})\\ &\le T(gx_{m(k)}, gx_{n(k)-1}) \diamond T(gx_{n(k)-1}, gx_{n(k)})\\ &\le \epsilon \diamond d(gx_{n(k)-1}, gx_{n(k)}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality. By (2.4), we have

$$\lim_{n\to\infty} \epsilon \diamond T(gx_{n(k)-1}, gx_{n(k)}) = \epsilon \diamond 0 = \epsilon$$

and

$$\lim_{n\to\infty} T(gx_{m(k)}, gx_{n(k)}) = \epsilon > 0. \tag{2.5}$$

By the transitivity property of $E(G)$, we get $(gx_{m(k)}, gx_{n(k)}) \in E(G)$ for all k. Thus, we have

$$\begin{aligned}T(gx_{m(k)+1}, gx_{n(k)+1}) &= T(fx_{m(k)}, fx_{n(k)})\\ &\le \theta(M(gx_{m(k)}, gx_{n(k)}))M(gx_{m(k)}, gx_{n(k)}),\end{aligned}$$

where

$$\begin{aligned}M(gx_{m(k)}, gx_{n(k)}) &= \max\left\{T(gx_{m(k)}, gx_{n(k)}), T(gx_{m(k)}, fx_{m(k)}), T(gx_{n(k)}, fx_{n(k)})\right\}\\ &= \max\left\{T(gx_{m(k)}, gx_{n(k)}), T(gx_{m(k)}, gx_{m(k)+1}), T(gx_{n(k)}, gx_{n(k)+1})\right\}.\end{aligned}$$

Hence, we conclude that

$$\frac{T(gx_{m(k)+1}, gx_{n(k)+1})}{M(gx_{m(k)}, gx_{n(k)})} \le \theta(M(gx_{m(k)}, gx_{n(k)})) < 1. \tag{2.6}$$

Keeping (2.4),(2.5) in mind and letting $k \to \infty$, we derive that

$$\lim_{k\to\infty} M(gx_{m(k)}, gx_{n(k)}) = \epsilon > 0.$$

By inequality (2.6), we get

$$\lim_{k\to\infty} \theta(M(gx_{m(k)}, gx_{n(k)})) = 1$$

and hence $\lim_{k\to\infty} M(gx_{m(k)}, gx_{n(k)}) = 0$, a contradiction. So, we conclude that $\{gx_n\}$ is a Cauchy sequence in $(X, T, \diamond)$.

Next, we claim that $\{gx_n\}$ is a Cauchy sequence with respect to T'.

If $T \ge T'$, it is trivial. Thus, suppose $T \ngeq T'$. Let $\varepsilon > 0$. Since $\{gx_n\}$ is a Cauchy sequence in $(X, T, \diamond)$ and f is *g-Cauchy* on X, we have $\{fx_n\}$ is Cauchy sequence in $(X, T', \diamond)$. Then there exists $N_0 \in \mathbb{N}$ with

$$T'(gx_{n+1}, gx_{m+1}) = T'(fx_n, fx_m) < \varepsilon,$$

whenever $n, m \ge N_0$. So $\{gx_n\}$ is a Cauchy sequence with respect to T'.

Since $(g(X), T', \diamond)$ is a complete T-metric space, there exists $u = gx \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = u.$$

Now, since $f:(X,T',\diamond)\to(X,T',\diamond)$ is G-continuous, and f and g are T'-compatible, we have

$$\lim_{n\to\infty} T'(gfx_n, fgx_n)=0. \tag{2.7}$$

Using the triangle inequality, we have

$$T'(gu,fu)\le T'(gu,gfx_n)\diamond T'(gfx_n,fgx_n)\diamond T'(fgx_n,fu).$$

Letting $n\to\infty$, from (2.7), f is G-continuous and the continuity of g and $\diamond$, we have

$$\lim_{n\to\infty} T'(gu,gfx_n)\diamond T'(gfx_n,fgx_n)\diamond T'(fgx_n,fu)=0\diamond 0\diamond 0=0.$$

It follows that $T'(gu,fu)=0$ which implies that $gu=fu$. So u is a coincidence point of f and g. ■

If $T=T'$, we have the following theorem.

Theorem 2.8. *Let $(X,T,\diamond)$ be a complete T-metric space endowed with a directed graph G. Suppose that $f,g:X\to X$ and (f,g) is a θ-contraction w.r.t T. Moreover, suppose that:*

1. *g is continuous;*
2. *$f(X)\subseteq g(X)$ and $(g(X),T,\diamond)$ be a complete T-metric space;*
3. *$E(G)$ satisfies the transitivity property;*
4. *assume that (a) f is G-continuous and f and g are T-compatible or (b) (X,T,G) has the property A.*

Then, under these conditions,

$$X(f,g)\neq\emptyset \;\text{ if and only if }\; C(f,g)\neq\emptyset.$$

Proof. In order to avoid the repetition, following from the same proof in Theorem 2.7,we can only consider (b) of the condition (3). Since $\{gx_n\}$ is a Cauchy sequence in $(X,T,\diamond)$ and $(g(X),T,\diamond)$ is a complete T-metric space, there exists $u\in X$ such that

$$\lim_{n\to\infty} gx_n=gu=\lim_{n\to\infty} fx_n. \tag{2.8}$$

Now, we show that u is a coincidence point of f and g. Suppose, on the contrary, that $fu\neq gu$. Then $T(fu,gu)>0$. Since (X,T,G) has the property A, we have $(gx_n,gu)\in E(G)$ for each $n\in\mathbb{N}$. We have

$$T(gu,fu)\le T(gu,fx_{n(k)})\diamond T(fx_{n(k)},fu)$$

which implies that

$$\begin{aligned}T(gu,fu) &\le T(gu,fx_{n(k)})\diamond T(fx_{n(k)},fu)\\ &\le T(gu,fx_{n(k)})\diamond\theta(M(gx_{n(k)},gu))M(gx_{n(k)},gu).\end{aligned} \tag{2.9}$$

Letting $k\to\infty$ in inequality (2.9), by the property of ϕ, $\diamond$ is continuous and (2.8), we obtain that

$$\begin{aligned}\lim_{k\to\infty} T(gu,fu) &\le \lim_{k\to\infty}\{T(gu,fx_{n(k)})\diamond\theta(M(gx_{n(k)},gu))M(gx_{n(k)},gu)\}\\ &= 0\diamond\lim_{k\to\infty}\theta(M(gx_{n(k)},gu))M(gx_{n(k)},gu)\\ &= \lim_{k\to\infty}\theta(M(gx_{n(k)},gu))M(gx_{n(k)},gu),\end{aligned} \tag{2.10}$$

where

$$M(gx_{n(k)}, gu) = \max\left\{T(gx_{n(k)}, gu), T(gx_{n(k)}, fx_{n(k)}), T(gu, fu)\right\}.$$

From (2.8), we obtain that

$$\lim_{k\to\infty} M(gx_{n(k)}, gu) = T(gu, fu) > 0.$$

From (2.10), we obtain that $\lim_{k\to\infty}\theta(M(gx_{n(k)}, gu)) = 1$ so $\lim_{k\to\infty} M(gx_{n(k)}, gu) = T(gu, fu) = 0$, a contradiction. Therefore $fu = gu$. Consequently, we conclude that f and g have a coincidence point. ■

Theorem 2.9. *In addition to the hypotheses of Theorem 2.7 (Theorem 2.8), assume that*

(K) *for any $x, y \in C(f, g)$ such that $gx \neq gy$, we have $(gx, gy) \in E(G)$.*

If $X(f, g) \neq \emptyset$, then $Cm(f, g) \neq \emptyset$.

Proof. Theorem 2.7 implies that there exists a coincidence point $x \in X$, that is, $gx = fx$. Suppose that there exists another coincidence point $y \in X$ such that $gy = fy$. Assume that $gx \neq gy$. By assumption (K), $(gx, gy) \in E(G)$, we have

$$\begin{aligned} T(fx, fy) &\leq \theta(M(gx, gy))M(gx, gy) \\ &< M(gx, gy) = T(fx, fy), \end{aligned}$$

which is a contradiction. Therefore, $gx = gy$. Starting from $x_0 = x$, choose the sequences $\{x_n\}$ satisfying $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$. Taking into account the properties of coincidence points, it is easy to see that it can be done so that $x_n = x$, i.e.,

$$gx_n = fx,$$

for all $n \in \mathbb{N}$. Now, let $p = gx$. Hence we have $gp = ggx = gfx$. By the definition of the sequence $\{x_n\}$, we have $gx_n = fx = fx_{n-1}$ for all $n \in \mathbb{N}$ so

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$$

with respect to T'. Since g and f are T'-compatible, we have

$$\lim_{n\to\infty} T'(gfx_n, fgx_n) = 0,$$

that is, $gfx = fgx$. Therefore, we have $gp = gfx = fgx = fp$. This implies that p is another coincidence point of the mappings f and g. By the property we have just proved, it follows that $fp = gp = gx = p$ and so p is a common fixed point of g and f. This completes the proof. ■

Let Φ denote the class of all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(ϕ_1): ϕ is nondecreasing;
(ϕ_2): ϕ is continuous;
(ϕ_3): $\phi(t) = 0 \Leftrightarrow t = 0$;
(ϕ_4): $\phi(t \diamond s) \leq \phi(t) \diamond \phi(s)$.

Definition 2.10. Let $(X, T, \diamond)$ be a T-metric space endowed with a directed graph G, and let $f, g : X \to X$ be given mappings. The pair (f, g) is called a *θ-ϕ-contraction w.r.t T* if :

(1) f is g-edge preserving w.r.t G;
(2) there exist two functions $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ such that $(gx, gy) \in E(G)$,

$$\phi(T(fx, fy)) \leq \theta(T(gx, gy))\phi(T(gx, gy)). \tag{2.11}$$

Applying the similar argument as in the proof of Theorem 2.7 and 2.8 , we have the following theorem.

Theorem 2.11. *Let $(X, T', \diamond)$ be a complete T-metric space endowed with a directed graph G, and let T be another T-metric on X. Suppose that $f, g : X \to X$ and (f, g) is a θ-ϕ-contraction w.r.t T. Suppose that*

(1) *$g : (X, T', \diamond) \to (X, T', \diamond)$ is continuous;*
(2) *$f(X) \subseteq g(X)$ and $(g(X), T', \diamond)$ be a complete T-metric space;*
(3) *$E(G)$ satisfies the transitivity property;*
(4) *if $T \ngeq T'$, assume that $f : (X, T, \diamond) \to (X, T', \diamond)$ is g-Cauchy on X;*
(5) *$f : (X, T', \diamond) \to (X, T', \diamond)$ is G-continuous, and f and g are T'-compatible.*

Then, under these conditions,

$$X(f, g) \neq \emptyset \text{ if and only if } C(f, g) \neq \emptyset.$$

Theorem 2.12. *Let $(X, T, \diamond)$ be a complete T-metric space endowed with a directed graph G. Suppose that $f, g : X \to X$ and (f, g) is a θ-ϕ-contraction w.r.t d. Moreover, suppose that:*

(1) *g is continuous;*
(2) *$f(X) \subseteq g(X)$ and $(g(X), T, \diamond)$ be a complete T-metric space;*
(3) *$E(G)$ satisfies the transitivity property;*
(4) *assume that (a) f is G-continuous and f and g are T-compatible or (b) (X, T, G) has the property A.*

Then, under these conditions,

$$X(f, g) \neq \emptyset \text{ if and only if } C(f, g) \neq \emptyset.$$

Remark 2.13. Put $E(G) = \{(x, y) \in X \times X : x \preceq y\}$, $\phi(t) = t$ and $a \diamond b = a + b$ in Theorem 2.11. In this case, we obtain the results of [2].

Example 2.14. Let $X = [0, \infty) \subseteq \mathbb{R}$ and the T-metrics $T, T' : X \times X \to [0, \infty)$ be defined by $T(x, y) = (x - y)^2$ and $T'(x, y) = L(x - y)^2$ where L is a real number such that $L \in (0, 1)$ and $a \diamond b = (\sqrt{a} + \sqrt{b})^2$.

Now, we consider $E(G)$ given by

$$E(G) = \{(x, y) : x = y \text{ or } [x, y \in [0, 1/9] \text{ with } x \leq y]\},$$

where $\leq$ is the usual order.

Consider the mappings $f : X \to X$ and $g : X \to X$ defined by

$$gx = x^2, \quad fx = x^4,$$

for all $x \in X$, respectively.

Next, we show that the conditions (1)–(2) in Definition 2.10 hold as follows:

(1) Let $(gx, gy) \in E(G)$,
if $gx = gy$ then $fx = fy$ and $(fx, fy) \in E(G)$,

if $gx, gy \in E(G)$ with $gx \leq gy$, then we obtain $gx = x^2$, $gy = y^2 \in [0, 1/9]$ and $x^2 = gx \leq gy = y^2$, we have $fx = x^4 \leq fy = y^4$ and $fx, fy \in [0, 1/9]$. This implies that $(fx, fy) \in E(G)$;

(2) Let $\theta \in \Theta$ be defined by

$$\theta(t) = \begin{cases} \frac{1}{20}, & \text{if } 0 \leq t < 1, \\ t^2 + 4, & \text{if } t \geq 1. \end{cases}$$

Let x, y be arbitrary points in X and $(gx, gy) \in E(G)$. If $gx = gy$, we have $x = y$ and hence the contractive condition (2.1) holds for this case. In another case, we have

$$gx = x^2, \ gy = y^2 \in [0, 1/9] \text{ with } gx \leq gy.$$

Then we obtain $x^2 + y^2 \in [0, 2/9]$ and $x \leq y$. Also, we have

$$\begin{aligned} T(fx, fy) &= (x^4 - y^4)^2 \\ &= (x^2 + y^2)^2 (x^2 - y^2)^2 \\ &\leq \frac{1}{20}(x^2 - y^2)^2 \\ &= \theta((x^2 - y^2)^2)(x^2 - y^2)^2 \\ &= \theta(T(gx, gy))T(gx, gy) \\ &\leq \theta(T(gx, gy))M(gx, gy), \end{aligned}$$

where $M(gx, gy) = \max\left\{T(gx, gy), T(gx, fx), T(gy, fy)\right\}$.

Therefore, (f, g) is a θ-contraction w.r.t T.

Next, we show that the conditions (1)–(5) in Theorem 2.7 hold as follows:

(1) We can easily check that $g : (X, T', \diamond) \to (X, T', \diamond)$ is continuous;

(2) By the definition of f and g, we can see that $f(X) = g(X)$ and it is easy to see that $(g(X), T', \diamond)$ is a complete T-metric space;

(3) It is easy to see that $E(G)$ satisfies the transitivity property;

(4) It is easy to see that $T \geq T'$.So, we have nothing to show this condition;

(5) We will prove that $f : (X, T', \diamond) \to (X, T', \diamond)$ is G-continuous, and f and g are T'-compatible. It is easy to see that $f : (X, T', \diamond) \to (X, T', \diamond)$ is G-continuous. So we will only show that f and g are T'-compatible. Suppose that $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = a,$$

for some $a \in X$. Now, we have

$$T'(gfx_n, fgx_n) = L(x_n^8 - x_n^8)^2 = 0,$$

for all $n \in \mathbb{N}$. This implies that $T'(gfx_n, fgx_n) \to 0$ as $n \to \infty$.

We have $0 \in X$ such that $(0, 0) = (g0, f0) \in E(G)$, then $X(f, g) \neq \emptyset$. Consequently, all the conditions of Theorem 2.7 hold. Therefore, g and f have a coincidence point and, further, the points 0 and 1 are common fixed points of the mappings g and f.

Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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