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Numerical Solution of a Partial Differential Equation Model of Heat Flow through the Boundary Surfaces of Poultry Shed

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Abstract : This paper provides a practical overview of the Crank-Nicolson method for obtaining numerical solutions to one-dimensional, time-dependent heat conduction problems. The method has been used to determine the heat flow through the boundary surfaces of a poultry shed for the two cases of constant air temperatures outside the shed and for known time varying air temperatures outside the shed. The heat flow has also been determined for several different wall materials. The results can be used to select suitable materials for shed walls that can help maintain the good health of the poultry in the shed.

Keywords : Finite Difference, Crank-Nicolson, Heat Conduction, Poultry Shed.

1 Introduction

A well-designed poultry shed is an important factor in maintaining the temperature conditions necessary for the good health of the poultry in the shed. Two factors influencing the temperature in the shed are the outside air temperature and the conduction of heat through the wall. Another important factor is, of course, proper ventilation of the shed. In this paper we look at the problem of the heat conduction through the wall.

The mathematical model of heat conduction through the wall of a poultry shed involves a set of coupled ordinary and partial differential equations subject to complicated boundary conditions. Gordan and Zarmi [1], [2], Feuermann et al. [3] and Bhandori and Bansal [4] have solved the differential equations governing heat flows but still assume steady-state heat transfer and zero heat capacitance in building elements. Boland [5], [6] have simplified the solution of the differential equations which describe heat flows in domestic dwellings and have given an analytic solution of the differential equations describing heat flow in houses. In this paper, we use the Crank-Nicolson method to obtain a numerical solution. This method is a well-known reliable method for solving heat conduction problems.

2 The Model Equations

The coupled partial and ordinary differential equations which describe heat transfer through a shed wall between the air outside the shed and the air inside the shed are as follows :

$$\frac{\partial}{\partial t}v(x,t) = \kappa\frac{\partial^2}{\partial x^2}v(x,t) \quad , \quad 0 < x < \ell \tag{2.1}$$

$$K\frac{\partial}{\partial x}v(0,t) + H_{in}[s(t) - v(0,t)] = 0 \tag{2.2}$$

$$-K\frac{\partial}{\partial x}v(\ell,t) + H_{out}[v_{out}(t) - v(\ell,t)] = 0 \tag{2.3}$$

$$v(x,0) = f(x) \tag{2.4}$$

$$Mc\frac{d}{dt}s(t) + AH_{in}[s(t) - v(0,t)] = 0 \tag{2.5}$$

Here, κ, ℓ , K and A represent diffusivity (m^2/s) , width (m^2) , heat conductivity $(W/^\circ Cm)$ and area (m^2) of the wall, respectively. $v(x,t)$ is wall temperature at dept into the wall at time $t(^\circ C)$, H_{in}, H_{out} are the combined radiative-convective heat transfer coefficients $(W/^\circ Cm^2)$ at the inside and outside wall surfaces, respectively. $s(t)$ is the air temperature inside the shed $(^\circ C)$, and M and c are the mass(kg) and specific heat $kJ/kg^\circ C)$ of the air inside the shed. $v_{out}(t)$ is the air temperature outside the shed $(^\circ C)$.

3 The Numerical Method

3.1 Crank-Nicolson Method for Partial Differential Equations

Crank and Nicolson [1] suggested a modified implicit finite-difference method for solving parabolic partial differential equations. To illustrate this method we consider the heat conduction equation (3.6).

$$\frac{\partial}{\partial t}v(x,t) = \kappa\frac{\partial^2}{\partial x^2}v(x,t) \quad ; \quad 0 < x < \ell, \ t > 0 \tag{3.1}$$

The first step is to replace the partial derivatives in equation (3.1) by finite difference approximations. The x and t domains are divided into small steps Δx and Δt, as illustrated in figure 1, so that the values of x and t are :

$$\begin{aligned} x &= i\Delta x, \quad i = 0,\ 1,\ 2,\ ...\ n \\ t &= j\Delta t, \quad j = 0,\ 1,\ 2,\ ... \\ \textit{where} \qquad n &= \frac{\ell}{\Delta x} \end{aligned}$$

Then the temperature $v(x,t)$ at a location x_i and time t_j is denoted by the symbol v_i^j that is

$$v(x_i, t_j) \quad = \quad v(i\Delta x, j\Delta t) \quad = \quad v_i^j \tag{3.2}$$

The Crank-Nicolson method is based on numerical approximations for solutions of equation (3.1) at points $(x_i, t_j + \frac{\Delta t}{2})$ that lie between the rows in the grid, as illustrated in figure 2. Specifically, the approximation used for $\frac{\partial}{\partial t}v(x, t + \frac{\Delta t}{2})$ is obtained from the central-difference formula

$$\frac{\partial}{\partial t}v(x, t + \frac{\Delta t}{2}) \quad = \quad \frac{v(x, t + \Delta t) - v(x, t)}{\Delta t} + O[(\Delta t)^2] \tag{3.3}$$

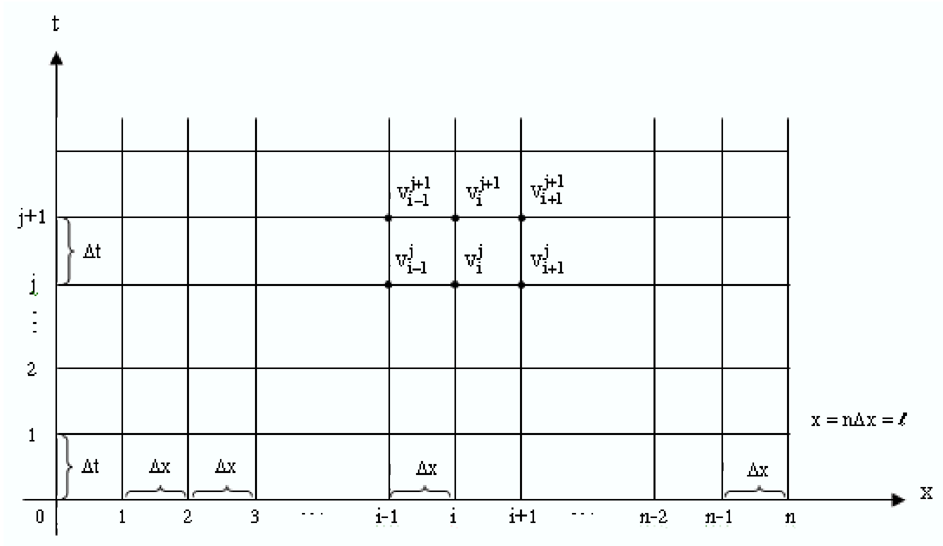


Figure 1 : Subdivision of the domain into intervals of Δx and Δt for finite-difference representation of the one-dimensional, time-dependent heat conduction equation.

The approximation used for $\frac{\partial^2}{\partial x^2}v(x, t + \frac{\Delta t}{2})$ is the average of the approximations for $\frac{\partial^2}{\partial x^2}v(x,t)$ and $\frac{\partial^2}{\partial x^2}v(x, t + \Delta t)$. These approximations, which have an accuracy of the order $O[(\Delta x)^2]$, are

$$\frac{\partial^2}{\partial x^2}v(x,t) \quad = \quad \frac{v(x - \Delta x, t) - 2v(x,t) + v(x + \Delta x, t)}{(\Delta x)^2} + O[(\Delta x)^2] \tag{3.4}$$

$$\frac{\partial^2}{\partial x^2}v(x, t + \Delta t) \quad = \quad \frac{v(x - \Delta x, t + \Delta t) - 2v(x, t + \Delta t) + v(x + \Delta x, t + \Delta t)}{(\Delta x)^2} + O[(\Delta x)^2] \tag{3.5}$$

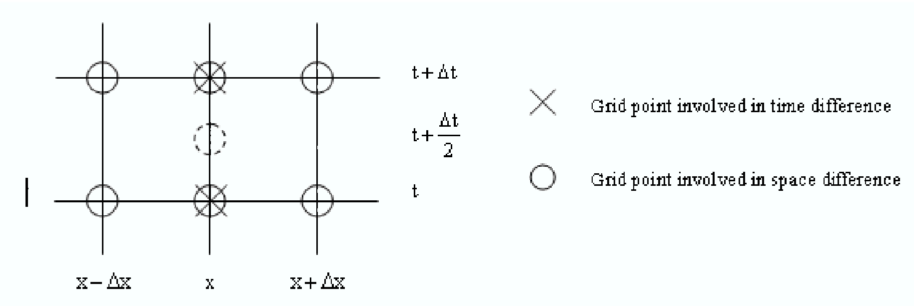


Figure 2 : A computational molecule for the Crank-Nicolson method.

Therefore the approximation for $\frac{\partial^2}{\partial x^2}v(x\,t+\frac{\Delta t}{2})$ becomes

$$\begin{aligned}\frac{\partial^2}{\partial x^2}v(x,t+\frac{\Delta t}{2}) &= \frac{1}{2}\left[\frac{v(x-\Delta x,t)-2v(x,t)+v(x+\Delta x,t)}{(\Delta x)^2}\right.\\ &\quad \left.+\frac{v(x-\Delta x,t+\Delta t)-2v(x,t+\Delta t)+v(x+\Delta x,t+\Delta t)}{(\Delta x)^2}\right]\\ &\quad +O[(\Delta x)^2] \qquad (3.6)\end{aligned}$$

Then using the notation $v(x_i,t_j)=v_i^j$ and $v(x_i,t_{j+1})=v_i^{j+1}$ in equation (3.3) we obtain the equation

$$\left.\frac{\partial v}{\partial t}\right|_{i,j+\frac{1}{2}} = \frac{v_i^{j+1}-v_i^j}{\Delta t}+O[(\Delta t)^2] \qquad (3.7)$$

Also introducing the notation

$$\begin{array}{lll} v(x,t)=v_i^j & ;\quad v(x-\Delta x,t)=v_{i-1}^j & ;\quad v(x+\Delta x,t)=v_{i+1}^j \\ v(x,t+\Delta t)=v_i^{j+1} & ;v(x-\Delta x,t+\Delta t)=v_{i-1}^{j+1} & ;v(x+\Delta x,t+\Delta t)=v_{i+1}^{j+1} \end{array}$$

into equation (3.6) we obtain

$$\left.\frac{\partial^2 v}{\partial x^2}\right|_{i,j+\frac{1}{2}} = \frac{1}{2}\left[\frac{v_{i-1}^j-2v_i^j+v_{i+1}^j}{(\Delta x)^2}+\frac{v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1}}{(\Delta x)^2}\right]+O[(\Delta x)^2] \quad (3.8)$$

Then, introducing equations (3.7) and equation (3.8) into equation (3.1) and neglecting the error terms $O[(\Delta x)^2]$ and $O[(\Delta t)^2]$, we obtain the difference equation

$$\frac{v_i^{j+1}-v_i^j}{\Delta t} = \frac{\kappa}{2}\left[\frac{v_{i-1}^j-2v_i^j+v_{i+1}^j}{(\Delta x)^2}+\frac{v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1}}{(\Delta x)^2}\right] \qquad (3.9)$$

This equation can be rearranged in the form

$$-\lambda v_{i-1}^{j+1} + (2+2\lambda)v_i^{j+1} - \lambda v_{i+1}^{j+1} = \lambda v_{i-1}^{j} + (2-2\lambda)v_i^{j} + \lambda v_{i+1}^{j}; \quad (3.10)$$

for $i = 1, 2, ..., n,$ where $\lambda = \frac{\kappa \Delta t}{(\Delta x)^2}$
The left side of equation (3.10) contains the unknown v at the nodes $i-1,\ i$ and $i+1$ at the time step $j+1$, that is $v_{i-1}^{j+1},\ v_i^{j+1}$ and v_{i+1}^{j+1} . The temperature, v , on the right-hand side of equation at these three nodes are known at the time step j, that is $v_{i-1}^{j},\ v_i^{j}$ and v_{i+1}^{j} are known. The computational molecule corresponding to equation (3.10) is shown in figure 3. Equation (3.10) correspond to a matrix equation $AV = B$,
where A represents the coefficient square matrix of order $n+1$
V represents an unknown vector of order $n+1$ at step $j+1$
And B represents a known vector of order $n+1$, that is the value of V at step j.
An important feature of the matrix A that makes the system easier to solve is that A is close to a tridiagonal matrix.

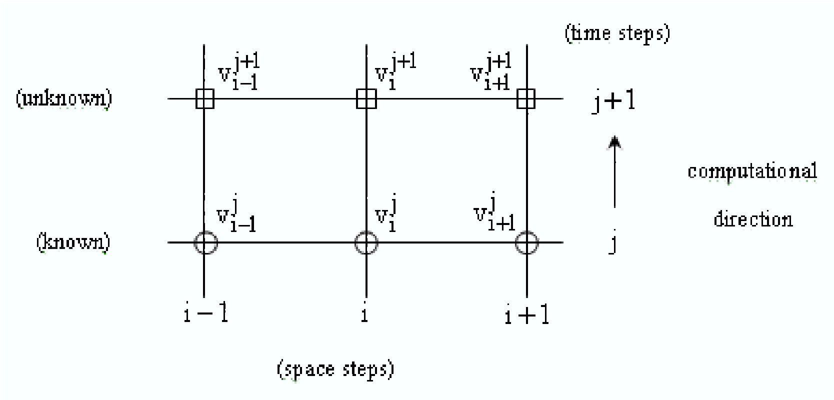


Figure 3 : Schematic form of the Crank-Nicolson method.

3.2 Central Difference Approximations for the Boundary Conditions

The boundary conditions at the outer and inner surfaces of the shed wall are :

$$A_1 v(0,t) + B_1 \frac{\partial}{\partial x} v(0,t) = F_1 \quad (3.11)$$

$$A_2 v(\ell,t) + B_2 \frac{\partial}{\partial x} v(\ell,t) = F_2 \quad (3.12)$$

The finite-difference representation of boundary conditions (16) and (17) using central-differences are given, respectively, as

$$A_1 v_0^j + B_1 \left(\frac{v_1^j - v_{-1}^j}{2\Delta x} \right) = F_1 \tag{3.13}$$

$$A_2 v_n^j + B_2 \left(\frac{v_{n+1}^j - v_{n-1}^j}{2\Delta x} \right) = F_2 \tag{3.14}$$

where v_{-1}^j and v_{n+1}^j are fictitious temperatures outside the xt domain.

3.3 Using the Crank-Nicolson method to Calculate a Numerical Solution

The following is the steps in finding the numerical solution of equations (2.1)-(2.5). The (x, t) domain is divided into intervals $\Delta x,\ \Delta t$ such that

$$\begin{aligned} x_i &= i\Delta x \quad ; \qquad i = 0,\ 1,\ 2,\ ...,\ n \\ t_j &= j\Delta t \quad ; \qquad i = 0,\ 1,\ 2,\ ... \end{aligned}$$

where $n = \frac{l}{\Delta x}$. We also consider fictitious nodal points $i = -1$ and $i = n + 1$ outside the region as illustrated in figure 4. The differential equation (2.1) is represented in finite differences using the Crank-Nicolson method, according to equation (3.10). The initial condition (2.4) is written as

$$v_i^0 = f(i\Delta x) \equiv f_i \ ; \ i = 0, 1, 2, ..., n$$

Using a central difference for the boundary condition (2) we have

$$K \left(\frac{v_1^j - v_{-1}^j}{2\Delta x} \right) + H_{in}(s^j - v_0^j) = 0 \tag{3.15}$$

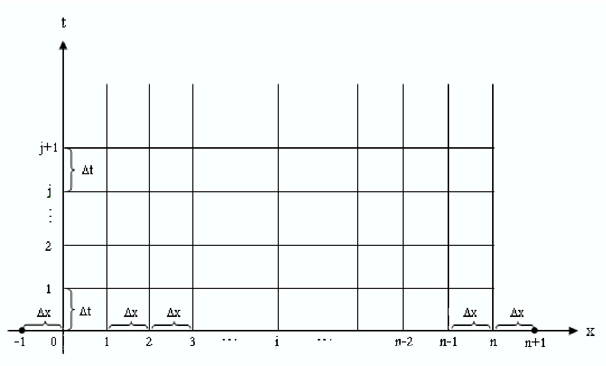


Figure 4 : Fictitious nodes $i = -1$ and $i = n + 1$.

Solving equation (3.15) for v_{-1}^j we obtain

$$v_{-1}^j = v_1^j + \frac{2\Delta x H_{in}}{K}(s^j - v_0^j) \tag{3.16}$$

and

$$v_{-1}^{j+1} = v_1^{j+1} + \frac{2\Delta x H_{in}}{K}(s^{j+1} - v_0^{j+1}) \tag{3.17}$$

For $i = 0$ we introduce equations (3.16) and equation (3.17) into equation (3.10) and obtain

$$\begin{aligned}(2 + 2\lambda + \frac{2\lambda\Delta x H_{in}}{K})v_0^{j+1} - 2\lambda v_1^{j+1} - \frac{2\lambda\Delta x H_{in}}{K}s^{j+1} &= (2 - 2\lambda - \frac{2\lambda\Delta x H_{in}}{K})v_0^j \\ &+2\lambda v_1^j + \frac{2\lambda\Delta x H_{in}}{K}s^j\end{aligned} \tag{3.18}$$

The finite-difference representation of the boundary condition (2.3) using a central difference is given as

$$-K\left(\frac{v_{n+1}^j - v_{n-1}^j}{2\Delta x}\right) + H_{out}(v_{out}^j - v_n^j) = 0 \tag{3.19}$$

Equation (3.19) can be solved for v_{n+1}^j to obtain

$$v_{n+1}^j = v_{n-1}^j + \frac{2\Delta x H_{out}}{K}(v_{out}^j - v_n^j) \tag{3.20}$$

and then

$$v_{n+1}^{j+1} = v_{n-1}^{j+1} + \frac{2\Delta x H_{out}}{K}(v_{out}^{j+1} - v_n^{j+1}) \tag{3.21}$$

For $i = n$ we introduce equations (3.20) and (3.21) into equation (3.10) to obtain

$$\begin{aligned}-2\lambda v_{n-1}^{j+1} + (2 + 2\lambda + \frac{2\lambda\Delta x H_{out}}{K})v_n^{j+1} &= 2\lambda v_{n-1}^j + (2 - 2\lambda - \frac{2\lambda\Delta x H_{out}}{K})\, v_n^j \\ &+ \frac{2\lambda\Delta x H_{out}}{K}(v_{out}^j + v_{out}^{j+1})\end{aligned} \tag{3.22}$$

Using the Crank-Nicolson method for equation (3.10), we find

$$Mc\left(\frac{s^{j+1} - s^j}{\Delta t}\right) + \frac{1}{2}AH_{in}[(s^{j+1} - v_0^{j+1}) + (s^j - v_0^j)] = 0 \tag{3.23}$$

This equation can be rearranged in the form

$$-\frac{1}{2}AH_{in}\, v_0^{j+1} + (\frac{Mc}{\Delta t} + \frac{1}{2}AH_{in})\, s^{j+1} = \frac{1}{2}AH_{in}\, v_0^j + (\frac{Mc}{\Delta t} - \frac{1}{2}AH_{in})\, s^j \tag{3.24}$$

Values of $v_0^{j+1}, v_1^{j+1}, ...v_n^{j+1}$ and s^{j+1} can be calculated from equations (3.18), (3.24), (3.10) and (3.22), respectively. We obtain
for $i = 0$;

$$(2 + 2\lambda\beta_1)v_0^{j+1} - 2\lambda v_1^{j+1} - 2\lambda\gamma_1 s^{j+1} = (2 - 2\lambda\beta_1)v_0^j + 2\lambda v_1^j + 2\lambda\gamma_1 s^j \quad (3.25)$$

for $i = 1, 2, 3, ..., n - 1$;

$$-\eta_1 v_0^{j+1} + (\eta_2 + \eta_1)s^{j+1} = \eta_1 v_0^j + (\eta_2 - \eta_1)s^j \quad (3.26)$$

for $i = n$;

$$-\lambda v_{i-1}^{j+1} + (2 + 2\lambda)v_i^{j+1} - \lambda v_{i+1}^{j+1} = \lambda v_{i-1}^j + (2 - 2\lambda)v_i^j + \lambda v_{i+1}^j \quad (3.27)$$

$$-2\lambda v_{n-1}^{j+1} + (2+2\lambda\beta_2)v_n^{j+1} = 2\lambda v_{n-1}^j + (2-2\lambda\beta_2)v_n^j + 2\lambda\gamma_2(v_{out}^j + v_{out}^{j+1}) \quad (3.28)$$

where

$$\begin{array}{lllllll} \beta_1 & = & 1 + \frac{\Delta x H_{in}}{K} & ; & \beta_2 & = & 1 + \frac{\Delta x H_{out}}{K} \\ \gamma_1 & = & \frac{\Delta x H_{in}}{K} & ; & \gamma_2 & = & \frac{\Delta x H_{out}}{K} \\ \eta_1 & = & \frac{1}{2} A H_{in} & ; & \eta_2 & = & \frac{Mc}{\Delta t} \end{array}$$

Equations (3.25) to (3.28) are $(n+2)$ simultaneous algebraic equations for $(n+2)$ unknown nodal point temperatures v_0^{j+1}, v_1^{j+1}, $...v_n^{j+1}$ and s^{j+1} at the time level $(j+1)$ in term of the $(n+2)$ known temperatures v_0^j, v_1^j, ... v_n^j and s^j of the previous time level j. Values of v_{out}^j and v_{out}^{j+1} can be calculated directly. Equations (3.25) to (3.28) can be written in the matrix form as

$$\begin{bmatrix} 2+2\lambda\beta_1 & -2\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & -2\lambda\gamma_1 \\ -\eta_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \eta_2+\eta_1 \\ -\lambda & 2+2\lambda & -\lambda & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 2+2\lambda & -\lambda & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & 2+2\lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2\lambda & 2+2\lambda\beta_2 & 0 \end{bmatrix} \begin{bmatrix} v_0^{j+1} \\ v_1^{j+1} \\ v_2^{j+1} \\ v_3^{j+1} \\ \vdots \\ v_n^{j+1} \\ s^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} 2-2\lambda\beta_1 & 2\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & 2\lambda\gamma_1 \\ \eta_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \eta_2-\eta_1 \\ \lambda & 2-2\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \lambda & 2-2\lambda & \lambda & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 2-2\lambda & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\lambda & 2-2\lambda\beta_2 & 0 \end{bmatrix} \begin{bmatrix} v_0^j \\ v_1^j \\ v_2^j \\ v_3^j \\ \vdots \\ v_n^j \\ s^j \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 2\lambda\gamma_2(v_{out}^{j} + v_{out}^{j+1}) \end{bmatrix} \tag{3.29}$$

where $j = 0, 1, 2, ...$

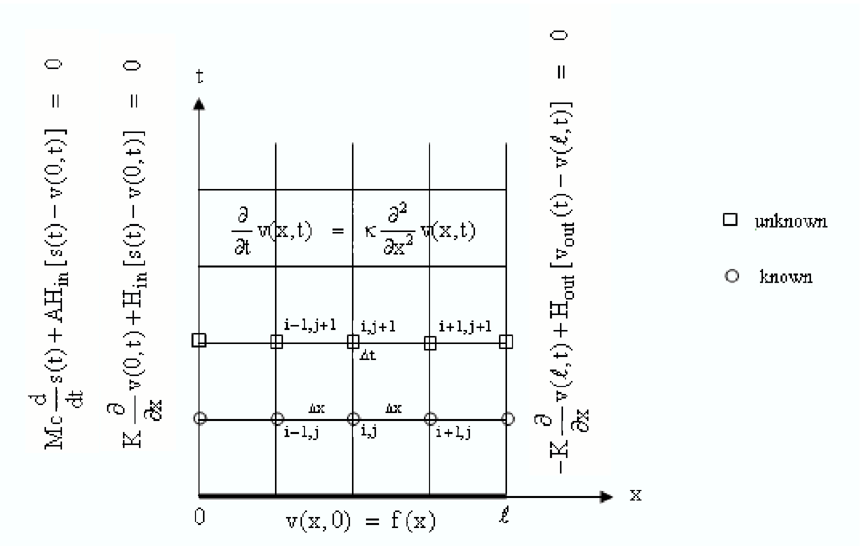


Figure 5 : Boundary condition and values needed for computation at time $j + 1$.

The values $v_0^{j+1}, v_1^{j+1}, ...v_n^{j+1}$ and s^{j+1} can be found from this system and can be solved using MATLAB.

The truncation error associated with the finite difference representation of the heat-conduction problem using the Crank-Nicolson method for the differential equation (2.1) and the boundary condition (2.5) is of the order of $(\Delta t)^2 + (\Delta x)^2$ and $\Delta t)^2$, respectively. Also, using the central-differences for the boundary conditions (2.2) and (2.3) gives a truncation error of the order $(\Delta x)^2$. Therefore, the Crank-Nicolson is called a $'second - orderaccurate'$ method.

3.4 Numerical solution to heat-conduction problems

We have used the Crank-Nicolson method to obtain solutions for the coupled set of ordinary and partial differential equations in a model of heat flow through the walls of a poultry shed. We have obtained solutions for both a constant and a time varying outside air temperature. We have also obtained numerical solutions for walls with values for thermal conductivity in the range 0.11-1.28 $W/^{\circ}Cm$ (see Chanthaweeroj [9]).

An example of the solutions obtained is given in figure 7 for parameter values: $\kappa = 0.66{\times}10^{-6}\, m^2/s$, $\ell = 0.1\, m$, $K = 1.28\, W/^{\circ}Cm, H_{in} = 8.29\, W/^{\circ}Cm^2$ $H_{out} = 22.7\, W/^{\circ}Cm^2$, $A = 32\, m^2$, $M = 226.0608\, kg$, $c = 1.0057\, kJ/kg^{\circ}C$, and for outside air temperature given by the periodic function $v_{out}(t) = 27 + 4\sin(\frac{t}{14400})$ shown in figure 6

3.5 Conclusions

The Crank-Nicolson method is a useful method for solving the set of coupled ordinary and partial differential equations in a model for heat flow through the walls of a poultry shed. The results can be used to assist in the design of sheds suitable for housing poultry or animals.

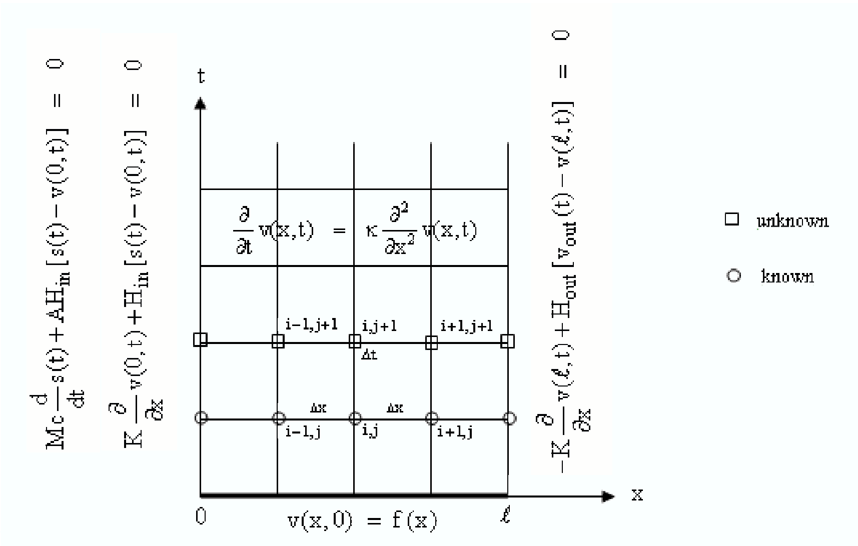
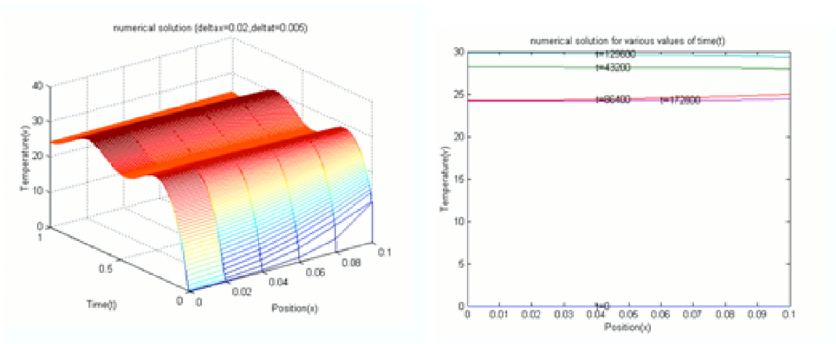
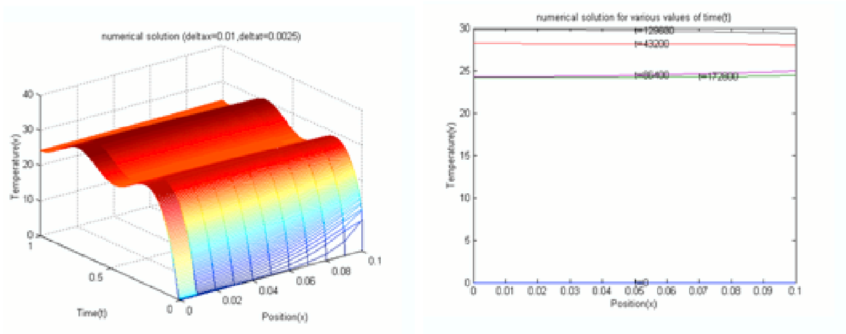


Figure 6 : Graph of function $v_{out}(t) = 27 + 4\sin(\frac{t}{14400})$ when $0 \leqslant t \leqslant 172800\, s$.

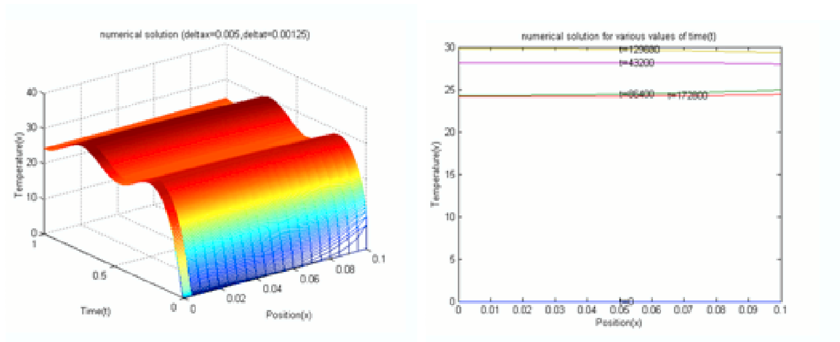
The numerical solution for $0 \leqslant t \leqslant 172800$ s. is shown in figure 7.



$(a)\quad \Delta x \;=\; 0.02\;,\;\; \Delta t \;=\; 0.005$



$(a)\quad \Delta x \;=\; 0.01\;,\;\; \Delta t \;=\; 0.0025$



$(a)\quad \Delta x \;=\; 0.005\;,\;\; \Delta t \;=\; 0.000125$

Figure 7 : Temperature in wall in three-dimension (left) and two-dimension for some (right).

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