Thai Journal of Mathematics
Volume 18 Number 2 (2020)
Pages 715–732

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Fractional Differential Operators and Generalized Oscillatory Dynamics

Rami Ahmad El-Nabulsi

Athens Institute for Education and Research, Mathematics and Physics Divisions, 8 Valaoritou Street, Kolonaki, 10671, Athens, Greece
e-mail : nabulsiahmadrami@yahoo.fr

Abstract In this paper, a new generalized fractional derivative is introduced holding many important properties. By implementing this new definition inside the Lagrangian $L : \mathcal{TQ} \to \mathbb{R}$, where $\mathcal{Q}$ is an n-dimensional manifold and $\mathcal{TQ}$ its tangent bundle, the new definition was used to discuss many interesting and general properties of the Lagrangian and Hamiltonian formalisms starting from a fractional actionlike variational approach. Applications of the new formalism for solving some dynamical oscillatory models of fractional order are given. Additional attractive features are explored in some details.

MSC: 26A33; 26A18

Keywords: fractional dynamics; fractional derivative; fractional variational approach

Submission date: 29.06.2017 / Acceptance date: 02.02.2019

1. Introduction

Fractional integral and derivative is a subject of mathematics which dated back to the late part of seventeenth century. It has been the subject of study for many researchers in pure and applied mathematics. It is considered as a powerful mathematical tool and has been applied successfully in different fields of science [1–3]. For an excellent list of historical survey in this area, the reader in refereed to the bibliography prepared by Ross and reprinted in the monograph by Oldham and Spanier [2]. The fractional calculus of variations (CoV) and correspondent fractional Euler-Lagrange equations (FELE) is one of the most and important topics encountered in fractional calculus [4–28]. The FELE has been studied in recent years and its applications in treating various problems have been extensively addressed. However, the fractional problem of the CoV still needs more amplification as the problem is remarkably related to the fractional quantization process and to the occurrence of non-local fractional differential operators. However, recent studies confirm the importances of fractional derivatives which have been used to describe more precisely the non-trivial behavior of complex physical systems whose dynamics are distant from equilibrium and the evolution of physical systems with loss. Furthermore, the derived FELE depends explicitly on the type of the fractional derivative

Published by The Mathematical Association of Thailand.

used inside the action and hence the subjects of applications are limited. Our intention in this paper is to introduce a new definition of the fractional derivative that is suitable to use while dealing with more generalized applications.

The paper is organized as follows: in Section 2, we introduce the definition of the generalized fractional complexified derivative (GFCD) that we will apply through this work and we summarize some of its important properties. In Section 3, we discuss the fractional Lagrangian formalism. For this, we introduce the new fractional action integral in which the Lagrangian is a function of the GFCD. We derive the corresponding fractional Euler-Lagrange equations and we discuss some of its properties. Section 4 deals with the complex Lagrangian fractional formalism in one, two and n-dimensional manifold. Some nice and interesting particular cases of the new fractional approaches are discussed in Section 5 along with some motivating applications illustrated in Section 6. Conclusions and perspectives are given in the last section.

2. Generalized Fractional Complexified Derivative

We start by introducing the following new definition:

Definition 2.1. The fractional derivatives of order (α, β), $0 < \alpha, \beta < 1$, $a, b, t \in \mathbb{R}, a < t$ that is defined in the paper is as follows:

$$D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} f(t) = \frac{1}{2}\left(1-\varepsilon+i\gamma\right) D^{\alpha}_{t;+} f(t) - \frac{1}{2}\left(1-\varepsilon-i\gamma\right) D^{\beta}_{t;-} f(t) \tag{2.1}$$

$i = \sqrt{-1}$, (ε, γ) are free parameters in the theory which could be real or complex and where

$$\begin{aligned} D^{\alpha}_{t;+} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(\tau)}{(t-\tau)^{\alpha+1}} d\tau \right], \end{aligned} \tag{2.2}$$

$$\begin{aligned} D^{\beta}_{t;-} f(t) &= \frac{1}{\Gamma(1-\beta)} \left(-\frac{d}{dt} \int_t^b f(\tau)(\tau-t)^{-\beta} d\tau \right) \\ &= \frac{(-1)^{\beta}}{\Gamma(1-\beta)} \left[\frac{f(t)}{(b-t)^{\beta}} + \beta \int_t^b \frac{f(t)-f(\tau)}{(\tau-t)^{\beta+1}} d\tau \right], \end{aligned} \tag{2.3}$$

are the left and right Riemann-Liouville fractional derivative of order (α, β), augmented by the following properties:

$$\int_a^b D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} f(t) g(t) dt = -\int_a^b f(t) D^{(\beta,\alpha)}_{-(\gamma),(\varepsilon)} g(t) dt, \qquad a \le t \le b, \tag{2.4}$$

$$\int_a^b D^{(\alpha)}_{(a_+)} f(t) g(t) dt = (-1)^{\alpha} \int_a^b f(t) D^{(\alpha)}_{(b_-)} g(t) dt, \tag{2.5}$$

$$D^{(\alpha)}_{(a)} D^{(\beta)}_{(a)} f(t) = D^{(\alpha+\beta)}_{(a)} f(t) - \sum_{i=1}^{k} D^{(\sigma-i)}_{(a)} f(t)\Big|_{t=a} \frac{(t-a)^{-\alpha-i}}{\Gamma(1-\alpha-i)}, 0 \le k-1 \le q \le k,$$

(2.6)

provided that $f(a) = f(b) = 0$ and $g(a) = g(b) = 0$. k in equation (2.6) is a whole number.

The following particular properties hold as well:

$$\varepsilon = 1 : D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = \frac{1}{2} i\gamma \left(D^{\alpha}_{t;+} + D^{\beta}_{t;-} \right) \begin{cases} \gamma = i : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = -\frac{1}{2} \left(D^{\alpha}_{t;+} + D^{\beta}_{t;-} \right) \\ \gamma = -i : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = \frac{1}{2} \left(D^{\alpha}_{t;+} + D^{\beta}_{t;-} \right) \end{cases}, \tag{2.7}$$

$$\varepsilon = 0 : D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = \frac{1}{2} \left(1 + i\gamma \right) D^{\alpha}_{t;+} - \frac{1}{2} \left(1 - i\gamma \right) D^{\beta}_{t;-} \begin{cases} \gamma = i : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = -D^{\beta}_{t;-} \\ \gamma = 0 : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = \frac{1}{2} \left(D^{\alpha}_{t;+} - D^{\beta}_{t;-} \right), \\ \gamma = -i : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = D^{\alpha}_{t;+} \end{cases} \tag{2.8}$$

$$\gamma = i : D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = \frac{1}{2} \left(-\varepsilon \right) D^{\alpha}_{t;+} - \frac{1}{2} \left(2 - \varepsilon \right) D^{\beta}_{t;-} \begin{cases} \varepsilon = 2 : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = -D^{\alpha}_{t;+} \\ \varepsilon = 1 : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = -\frac{1}{2} \left(D^{\alpha}_{t;+} + D^{\beta}_{t;-} \right), \\ \varepsilon = 0 : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = -D^{\beta}_{t;-} \end{cases} \tag{2.9}$$

$$\gamma = -i : D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = \frac{1}{2} \left(2 - \varepsilon \right) D^{\alpha}_{t;+} - \frac{1}{2} \left(-\varepsilon \right) D^{\beta}_{t;-} \begin{cases} \varepsilon = 2 : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = D^{\beta}_{t;-} \\ \varepsilon = 1 : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = \frac{1}{2} \left(D^{\alpha}_{t;+} + D^{\beta}_{t;-} \right). \\ \varepsilon = 0 : & D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = D^{\alpha}_{t;+} \end{cases} \tag{2.10}$$

Finally, for $\alpha \to 1$, $\beta \to 1$ and $\varepsilon \to 0$, $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} = d/dt$. $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t) dt$ is the Euler gamma function.

It is remarkable that if $f(t)$ is an analytical function, then

(1) $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} f(t)$ must be an analytic function as well.

(2) The operation $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}$ must produce the same result as ordinary differentiation when the parameters (α, β) are positive integers (n^+, m^+) respectively. Whereas when the parameters (α, β) are negative integers (n^-, m^-), $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}$ must produce the same result as ordinary (n^-, m^-)-fold integration.

(3) The fractional derivative (2.1) must also be linear, in other words, the following requirement must be satisfied $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} \left[Af(t) + Bg(t) \right] = AD^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} f(t) + BD^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} g(t), (A, B) \in \mathbb{R}$.

(4) The operation of order zero must leave the function unchanged.

(5) The law of exponents must hold for integration of arbitrary order.

However, as the Riemann-Liouville fractional derivatives $D^{\alpha}_{t;+}$ and $D^{\beta}_{t;-}$ fulfill these criteria, $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}$ must fulfill them as well.

We recognize in equation (2.8) the Cresson fractional derivatives [11] and for $\gamma = 0$, the Luchko-Martinez-Trujillo fractional derivative [29]. In equation (2.8) and for the particular case $\gamma = 0$, we recognize also the Riesz fractional derivative [1]. It should be mentioned that Definition 2.1 will be supported by arguments or examples in the subsequent sections. Our main aim by introducing such a kind of fractional derivative is to extend to extend the problem of classical mechanics. As stated previously, there exists a difference between Definition 2.1 and Cresson derivative. As the parameter εin our arguments may be set as real, the fractional derivative introduced in equation (2.1) is then decomplexified if, for instance, $\gamma = \pm i$. This fact is motivating as it may lead to a decomplexified fractional dynamical system in the complex plane. Furthermore, the fact that ε in our arguments may also be set as complex will lead to a more generalized fractional dynamics that will be characterized by a certain classification of the ensuing fractional Euler-Lagrange equations. Such a classification is appealing and vital for any variational problem. All these imitate the exclusivity of the research reported herein.

Remarks 1:

1-*For* $\varepsilon = 1 + i\gamma$, $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} f(t) = i\gamma D^{\beta}_{t;-} f(t)$ *is complexified unless* $\gamma = \pm i$.

2-*For* $\varepsilon = 1 - i\gamma$, $D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} f(t) = i\gamma D^{\alpha}_{t;+} f(t)$ *is as well complexified unless* $\gamma = \pm i$.

3. Fractional Lagrangian Formalism

Our central aim is to discuss the Lagrangian formalism. We introduce the following definition:

Definition 3.1. Let $\mathcal{Q}$ be an n-dimensional manifold and $\mathcal{TQ}$ its tangent bundle. Coordinates $q^i, i = 1, 2, ..., n$ on $\mathcal{Q}$ induces tangent coordinates $(q^i, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} q^i)$ on $\mathcal{TQ}$. The mapping $L : \mathcal{TQ} \to \mathbb{R}$ is the Lagrangian of the theory assumed to be a C^2-function with respect to all its arguments. The fractional variational Hamilton's approach singles out particular curves $q(\tau) \in \mathcal{Q}$ by the condition $\delta S = 0$ where the fractional action is defined by:

$$S[L] = \frac{1}{\Gamma(\alpha)} \int_a^b L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)} q(\tau), q(\tau), \tau\right) (t-\tau)^{\alpha-1} d\tau, 0 < \alpha < 1. \tag{3.1}$$

The variation is over smooth curves in $\mathcal{Q}$ with fixed endpoints. We choose variations $\delta q(\tau)$ of the curve $q(\tau)$ that satisfy $\delta q(\tau) \in \mathcal{D}_q(\tau)$ for each τ. Here $\mathcal{D}$ is a collection of linear subspaces denoted $\mathcal{D}_q(\tau) \in \mathcal{T}_q\mathcal{Q}$ for each $q \in \mathcal{Q}$ under the initial condition $q(a) = q_a$, τ is the intrinsic time, t is the observer time, $t \neq \tau$.

We may now start with the following theorem:

Theorem 3.2 ([16]). *The fractional Hamilton's variational approach for a curve* $q(\tau)$ *is equivalent to the condition that this curve satisfies the fractional Euler-Lagrange equations*

It should be pointed here that the tangent bundle $\mathcal{TQ}$ is a vectorial space and we already know that the fractional derivative that we introduce in the coordinate system may not provide a basis for this vectorial space. Besides, there is also a problem with respect to the dependence of this coordinates with respect to change between charts on the manifold. This is important as we define the Lagrangian not only locally but globally on the tangent bundle. We are aware of this point but work in this direction is in progress. To the best of our knowledge, this is a completely open question in the complexified fractional setting.

(FELE):

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q(\tau),q(\tau),\tau\right)}{\partial q}-D^{(\beta,\alpha)}_{-(\gamma),(\varepsilon)}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}\right)$$

$$=\frac{1-\alpha}{t-\tau}\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}, \tag{3.2}$$

$\forall\tau\in(a,t)$.

Proof. We execute a small perturbation of the generalized coordinates $x(\bullet)\rightarrow x(\bullet)+\delta h(\bullet), \delta<<1$, for which

$$D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}(x(\bullet)+\delta h(\bullet))\approx D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}x(\bullet)+\delta D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}h(\bullet).$$

It should be mentioned at this stage that we are performing a perturbation of the curve itself that we can look inside a particular chart on the manifold as we have not the analogue of the linear tangent map to know what the behaviour of our mapping, at least we do not stay in the tangent bundle a priori. Consequently:

$$S\left[x+\delta h\right]=\frac{1}{\Gamma(\alpha)}\int_a^b L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}x+\delta D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}h,x+\delta h,\tau\right)(t-\tau)^{\alpha-1}d\tau.$$

Performing a Taylor expansion of $L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}x+\delta D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}h,x+\delta h,\tau\right)$ in ε around zero and integrating by parts:

$$S\left[x+\varepsilon h\right]=S[x]-\frac{1}{\Gamma(\alpha)}\int_a^b\left[-\frac{\partial L}{\partial x}h(\tau)(t-\tau)^{\alpha-1}d\tau\right.$$

$$\left.+(t-\tau)^{\alpha-1}\frac{\partial L}{\partial y}D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}h(t)+(t-\tau)^{\alpha-1}\ \frac{\partial L}{\partial y}D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}h(t)\right]dt+\mathrm{O}(\delta).$$

Making use of the least action principle we arrive to equation (3.2) after simple algebra. ■

<u>Remarks 2</u>: *A variety of FELE may at the moment arise from equation* (3.2):

1-*For* $\varepsilon=\gamma=0$ *and* $\alpha=1$ *the fractional Euler-Lagrange equation* (3.2) *is reduced, in the sense of Riesz to*:

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(0),(0)}q(\tau),q(\tau),\tau\right)}{\partial q}-D^{(\beta,\alpha)}_{-(0),(0)}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(0),(0)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}\right)=0. \tag{3.3}$$

2-*For* $(\varepsilon,\gamma)=(0,-i)$, *equation* (3.2) *is reduced, in the sense FALVA to*:

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(-i),(0)}q(\tau),q(\tau),\tau\right)}{\partial q}-D^{\alpha}_{\tau;+}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(-i),(0)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}\right)$$

$$=\frac{1-\alpha}{t-\tau}\frac{\partial L\left(D^{(\alpha,\beta)}_{(-i),(0)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}. \tag{3.4}$$

3-*For* $(\varepsilon,\gamma)=(0,i)$, *equation* (3.2) *is reduced to the special form*:

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(i),(0)}q(\tau),q(\tau),\tau\right)}{\partial q}+D^{\beta}_{\tau;-}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(i),(0)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}\right)$$
$$=\frac{1-\alpha}{t-\tau}\frac{\partial L\left(D^{(\alpha,\beta)}_{(i),(0)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}. \tag{3.5}$$

4-*For* $(\varepsilon,\gamma)=(1,\pm i)$, *equation* (3.2) *is reduced to the particular form*:

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(\pm i),(0)}q(\tau),q(\tau),\tau\right)}{\partial q}\mp\frac{1}{2}\left(D^{\alpha}_{\tau;+}+D^{\beta}_{\tau;-}\right)\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(\pm i),(0)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}\right)$$
$$=\frac{1-\alpha}{t-\tau}\frac{\partial L\left(D^{(\alpha,\beta)}_{(\pm i),(0)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}. \tag{3.6}$$

5-*For* $\varepsilon=1,\gamma\in\mathbb{R}$, *equation* (3.2) *is reduced furthermore to the particular complexified form*:

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(1)}q(\tau),q(\tau),\tau\right)}{\partial q}-\frac{1}{2}i\gamma\left(D^{\alpha}_{\tau;+}+D^{\beta}_{\tau;-}\right)\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(1)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}\right)$$
$$=\frac{1-\alpha}{t-\tau}\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(1)}q(\tau),q(\tau),\tau\right)}{\partial \dot{q}}. \tag{3.7}$$

For $\varepsilon=1,\gamma\in\mathbb{R}$, the Lagrangian inside equation (3.7) is complexified as follows:

$$L\left(D^{(\alpha,\beta)}_{(\gamma),(1)}q(\tau),q(\tau),\tau\right)\to L\left(\frac{1}{2}i\gamma\left(D^{\alpha}_{\tau;+}q(\tau)+D^{\beta}_{\tau;-}q(\tau)\right),q(\tau),\tau\right),$$

and hence for the particular case $\gamma=2$,

$$L\left(D^{(\alpha,\beta)}_{(2),(1)}q(\tau),q(\tau),\tau\right)\to L\left(i\left(D^{\alpha}_{\tau;+}q(\tau)+D^{\beta}_{\tau;-}q(\tau)\right),q(\tau),\tau\right),$$

and accordingly, one may conjecture that the action and the FELE are in their turns complexified. It was, however, observed in more recent studies that a complexified action may have interesting consequences in different branches of theoretical physics, i.e. classical electrodynamics [30], complex Lagrangian mechanics [31] and applied mathematics [32]

In the next section, we will discuss the complexified question in some details.

4. Complexified Fractional Lagrangian Formalism

More generally, observe that the Lagrangian inside equation (3.7) is complexified as follows:

$$L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q(\tau),q(\tau),\tau\right)\ \to\ L\Bigg(\frac{1}{2}(1-\varepsilon)\left(D^{\alpha}_{\tau;+}q(\tau)-D^{\beta}_{\tau;-}q(\tau)\right)$$
$$+\frac{1}{2}i\gamma\left(D^{\alpha}_{\tau;+}q(\tau)+D^{\beta}_{\tau;-}q(\tau)\right),q(\tau),\tau\Bigg). \tag{4.1}$$

One can deal with complex Lagrangian if, for instance, we let $\mathcal{M}$ be an n-dimensional manifold and $\mathcal{TM}$ its almost complex manifold with fixed almost complex structure J such that $J^2 = -I$. It is noteworthy that a tensor field J on $\mathcal{TM}$ is called an almost complex structure on $\mathcal{TM}$ if at every point p of $\mathcal{TM}$, J is endomorphism of the tangent space $\mathcal{T}_p\mathcal{M}$ such that $J^2 = -I$. For this, we can assume $q = q_1 + iq_2$ and $\dot{q} = \dot{q}_1 + i\dot{q}_2$ with

$$J\left(\frac{\partial}{\partial q_1^i}\right) = \frac{\partial}{\partial q_2^i}, \quad J\left(\frac{\partial}{\partial q_2^i}\right) = -\frac{\partial}{\partial q_1^i}, \tag{4.2}$$

and equation (3.2) is now splitted into:

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial q_1} - D^{(\beta,\alpha)}_{-(\gamma),(\varepsilon)}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_1}\right)$$

$$= \frac{1-\alpha}{t-\tau}\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_1}, \tag{4.3}$$

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial q_2} - D^{(\beta,\alpha)}_{-(\gamma),(\varepsilon)}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_2}\right)$$

$$= \frac{1-\alpha}{t-\tau}\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_2}. \tag{4.4}$$

Since the vector fields are defined by:

$$\left.\frac{\partial}{\partial q}\right|_p = \frac{1}{2}\left(\frac{\partial}{\partial q_1} - i\frac{\partial}{\partial q_2}\right), \tag{4.5}$$

$$\left.\frac{\partial}{\partial q^*}\right|_p = \frac{1}{2}\left(\frac{\partial}{\partial q_1} + i\frac{\partial}{\partial q_2}\right), \tag{4.6}$$

and using the complex notation $q = q_1 + iq_2$, we can write:

$$\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial q_1} + i\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial q_2}$$

$$-D^{(\beta,\alpha)}_{-(\gamma),(\varepsilon)}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_1} + i\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_1}\right)$$

$$= \frac{1-\alpha}{t-\tau}\left(\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_1} + i\frac{\partial L\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_1, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q_2, q_1, q_2, \tau\right)}{\partial \dot{q}_2}\right), \tag{4.7}$$

that reads using

$$\tilde{L}\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q^{*}, q, q^{*}, \tau\right)$$
$$\equiv L\left(\frac{D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q + D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q^{*}}{2}, \frac{D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q - D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q^{*}}{2i}, \frac{q+q^{*}}{2}, \frac{q-q^{*}}{2}, \tau\right),$$

as:

$$\frac{\partial\tilde{L}\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q^{*}, q, q^{*}, \tau\right)}{\partial q^{*}} - D^{(\beta,\alpha)}_{-(\gamma),(\varepsilon)}\left(\frac{\partial\tilde{L}\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q^{*}, q, q^{*}, \tau\right)}{\partial\dot{q}^{*}}\right)$$
$$= \frac{1-\alpha}{t-\tau}\frac{\partial\tilde{L}\left(D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q, D^{(\alpha,\beta)}_{(\gamma),(\varepsilon)}q^{*}, q, q^{*}, \tau\right)}{\partial\dot{q}^{*}}, \tag{4.8}$$

which is the complex Euler-Lagrange equation on the complex manifold.

In two dimensions, the previous argument may be generalized as follows:

Definition 4.1. Let $\mathcal{M}$ be an 2-dimensional manifold and $\mathcal{TM}$ its almost complex manifold with fixed almost complex structure J such that $J^2 = -I$. The two-dimensional fractional action integral is defined by

$$S_{2D}[L] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\iint_{\Omega(\xi,\lambda)}\tilde{L}\Big(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q(x,y), D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^{*}(x,y), D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q(x,y), D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^{*}(x,y), q(x,y), q^{*}(x,y), x, y\Big)$$
$$\times(\xi-x)^{\alpha-1}(\lambda-y)^{\beta-1}\,dxdy, \tag{4.9}$$

where ξ and λ are the observer times, $(\xi,\lambda)\in\Omega$, x and y are the intrinsic times, $(x,y)\in\Omega(\xi,\lambda)\subseteq\Omega$, $x\neq\xi$ and $y\neq\lambda$; $q=q(x,y)$; $D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}$ and $D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}$ are the fractional derivative operators (2.1) respectively of orders (δ,α) and (χ,β) with respect to x and y. The Lagrangian is supposed to be a sufficiently smooth function of its arguments.

Theorem 4.2. *Given a smooth Lagrangian*

$$\tilde{L}\left(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q(x,y), D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^{*}(x,y), D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q(x,y), D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^{*}(x,y), q(x,y), q^{*}(x,y), x, y\right).$$

If $q=q(x,y)$ and $q^{*}=q^{*}(x,y)$ make the fractional action stationary $\forall(\xi,\lambda)\in\Omega$, then the following double-weighted Euler-Lagrange equation holds $\forall(x,y)\in\Omega(\xi,\lambda)$:

$$\frac{\partial\tilde{L}\left(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q(x,y), D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^{*}(x,y), D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q(x,y), D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^{*}(x,y), q(x,y), q^{*}(x,y), x, y\right)}{\partial q^{*}}$$
$$-D^{(\delta,\alpha)}_{-(\gamma),(\varepsilon)}\left(2\frac{\partial\tilde{L}\left(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^{*}(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^{*}(x,y),q(x,y),q^{*}(x,y),x,y\right)}{\partial\dot{q}^{*}_{x}}\right)$$
$$-D^{(\chi,\beta)}_{-(\gamma),(\varepsilon)}\left(2\frac{\partial\tilde{L}\left(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^{*}(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^{*}(x,y),q(x,y),q^{*}(x,y),x,y\right)}{\partial\dot{q}^{*}_{y}}\right)$$
$$=\frac{1-\alpha}{\xi-x}\frac{\partial\tilde{L}\left(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^{*}(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^{*}(x,y),q(x,y),q^{*}(x,y),x,y\right)}{\partial\dot{q}^{*}_{x}}$$

$$+\frac{1-\beta}{\lambda-y}\frac{\partial\tilde{L}\left(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^*(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q(x,y),D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^*(x,y),q(x,y),q^*(x,y),x,y\right)}{\partial\dot{q}^*_y}. \tag{4.10}$$

Proof. Following the same arguments in [15], we let q be a stationary solution, $h << 1$ a real small parameter and $w(x,y)$ be an arbitrary smooth complex function satisfying $w(x,y)=0$ for all $(x,y)\in\partial\Omega$ so that q^*+hw^* satisfies the given Dirichlet boundary conditions $\forall h$. Now, we can write the fractional action as:

$$S_{2D}[L]=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\times$$

$$\iint_{\Omega(\xi,\lambda)}\tilde{L}\Big(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q+hD^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}w,D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}q^*+hD^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}w^*,D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q+hD^{(\beta,\chi)}_{(\gamma),(\varepsilon)}w,$$
$$D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}q^*+hD^{(\beta,\chi)}_{(\gamma),(\varepsilon)}w^*,q+hw,q^*+hw^*,x,y\Big)$$
$$\times(\xi-x)^{\alpha-1}(\lambda-y)^{\beta-1}\,dxdy.$$

Thus the stationary condition gives

$$\frac{d}{dh}S_{2D}\left[q^*+hw^*\right]\Big|_{h=0}=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\iint_{\Omega(\xi,\lambda)}\tilde{L}\left(w\frac{\partial\tilde{L}}{\partial q^*}+D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}w\frac{\partial\tilde{L}}{\partial q^*_x}+D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}w\frac{\partial\tilde{L}}{\partial q^*_y}\right)(\xi-x)^{\alpha-1}(\lambda-y)^{\beta-1}\,dxdy.$$

Integration by parts and making use of the Green's theorems:

$$\iint_{\Omega(\xi,\lambda)}\left(\frac{\partial P}{\partial\xi}G_1+\frac{\partial P}{\partial\lambda}G_2\right)d\bar{\xi}d\bar{\lambda}$$
$$=\oint_{\partial\Omega}P\left(-G_2d\bar{\xi}+G_1d\bar{\lambda}\right)-\iint_{\Omega(\xi,\lambda)}\left(P\text{-}\left(\frac{\partial G_1}{\partial\xi}+\frac{\partial G_2}{\partial\lambda}\right)\right)d\bar{\xi}d\bar{\lambda}$$

where

$$\Gamma(1+\alpha)\,\bar{\xi}=\xi^{\alpha}-(\xi-x)^{\alpha},$$
$$\Gamma(1+\alpha)\,\bar{\lambda}=\lambda^{\beta}-(\lambda-y)^{\beta},$$

G_1,G_2 are any smooth functions, we find:

$$\frac{d}{dh}S_{2D}\left[q^*+hw^*\right]\Big|_{h=0}=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\iint_{\Omega(\mathrm{x},\mathrm{Y})}w\Big[(\xi-x)^{\alpha-1}(\lambda-y)^{\beta-1}$$
$$\times\left(D^{(\alpha,\delta)}_{(\gamma),(\varepsilon)}\left(\frac{\partial\tilde{L}}{\partial q^*_x}\right)+D^{(\beta,\chi)}_{(\gamma),(\varepsilon)}\left(\frac{\partial\tilde{L}}{\partial q^*_y}\right)+(\alpha-1)(\xi-x)^{\alpha-2}(\lambda-y)^{\beta-1}\left(\frac{\partial\tilde{L}}{\partial q^*_x}\right)\right.$$
$$\left.+(\beta-1)(\xi-x)^{\alpha-1}(\lambda-y)^{\beta-2}\left(\frac{\partial\tilde{L}}{\partial q^*_y}\right)-(\xi-x)^{\alpha-1}(\lambda-y)^{\beta-1}\frac{\partial\tilde{L}}{\partial q^*}\right)dxdy=0,$$

where the boundary integral vanishes owing to the boundary conditions. Due to the arbitrariness of $w(x,y)$ inside $\Omega(\xi,\lambda)$, it follows then equation (4.10). ∎

We may generalize the previous arguments to N-dimensions:

Definition 4.3. Let $\mathcal{M}$ be an n-dimensional manifold and $\mathcal{TM}$ its almost complex manifold with fixed almost complex structure J such that $J^2 = -I$. The n-dimensional fractional action integral is defined by

$$S_{2D}[L] = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int ... \int_{\Omega(\xi)} \tilde{L}\left(\nabla^{(\alpha,\delta)}_{(\gamma),(\varepsilon)} q(x), \nabla^{(\alpha,\delta)}_{(\gamma),(\varepsilon)} q^*(x), q(x), q^*(x), x\right) \prod_{i=1}^{n} (\xi_i - x_i)^{\alpha_i - 1} dx, \tag{4.11}$$

where $\xi = (\xi_1, ..., \xi_n)$ is the observer time vector, $x = (x_1, ..., x_n)$ the intrinsic time vector, $dx = dx_1...dx_n$, $x \neq \xi$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\delta = (\delta_1, ..., \delta_n)$, $0 < \alpha_i < 1, i = 1, ..., n$ and $\nabla^{(\alpha,\delta)}_{(\gamma),(\varepsilon)} = (\nabla^{(\alpha_1,\delta_1)}_{(\gamma),(\varepsilon)}, .., \nabla^{(\alpha_n,\delta_n)}_{(\gamma),(\varepsilon)})$.

Theorem 4.4. *Given a smooth Lagrangian*

$$\tilde{L}\left(\nabla^{(\alpha,\delta)}_{(\gamma),(\varepsilon)} q(x), \nabla^{(\alpha,\delta)}_{(\gamma),(\varepsilon)} q^*(x), q(x), q^*(x), x\right),$$

If $q = q(x)$ and $q^* = q^*(x)$, $x = (x_1, ..., x_n)$, make the fractional action stationary $\forall \xi \in \Omega$, $\xi = (\xi_1, ..., \xi_n)$, then the following fractional complexified Euler-Lagrange equation holds:

$$\sum_{i=1}^{N} \left[D^{(\delta_i,\alpha_i)}_{-(\gamma);\varepsilon_i} \left(\frac{\partial \tilde{L}}{\partial q^*_{x_i}} \right) + \frac{1-\alpha_i}{\xi_i - x_i} \left(\frac{\partial \tilde{L}}{\partial q^*_{x_i}} \right) \right] - \frac{\partial \tilde{L}}{\partial q^*} = 0, \tag{4.12}$$

where all partial derivatives of the Lagrangian are evaluated at $(\nabla^{(\alpha,\delta)}_{(\gamma),(\varepsilon)} q(x), q(x), x), x \in \Omega(\xi)$.

5. Special and Particular Cases of the New Fractional Formalsim

All the previous arguments discussed in the previous section could be extended *mutadis mutandis* for a given Lagrangian given by

$$L\left(\frac{1}{2}(1-\varepsilon+i\gamma) D^{\alpha}_{\tau;+} q(\tau), \frac{1}{2}(1-\varepsilon-i\gamma) D^{\alpha}_{\tau;-} q(\tau), q(\tau), \tau\right).$$

The associated one-dimensional fractional Euler-Lagrange equation derived from the fractional functional

$$S[L] = \frac{1}{\Gamma(\alpha)} \int_a^b L\left(\frac{1}{2}(1-\varepsilon+i\gamma) D^{\alpha}_{\tau;+} q(\tau), \frac{1}{2}(1-\varepsilon-i\gamma) D^{\alpha}_{\tau;-} q(\tau), q(\tau), \tau\right) (t-\tau)^{\alpha-1} d\tau, \tag{5.1}$$

is given by [10]

$$\frac{\partial L}{\partial q^i} - \frac{1-\varepsilon+i\gamma}{1-\varepsilon-i\gamma} D^{\alpha}_{\tau;+} \left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-} q^i} \right)$$
$$- \frac{1-\varepsilon-i\gamma}{1-\varepsilon+i\gamma} D^{\alpha}_{\tau;-} \left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+} q^i} \right) = \frac{1-\alpha}{t-\tau} \left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-} q^i} + \frac{\partial L}{\partial D^{\alpha}_{\tau;+} q^i} \right), \tag{5.2}$$

or more explicitly

$$\frac{\partial L}{\partial q} - \frac{(1-\varepsilon)^2 - \gamma^2}{(1-\varepsilon)^2 + \gamma^2} \left(D^{\alpha}_{\tau;+} \left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-} q} \right) + D^{\alpha}_{\tau;-} \left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+} q} \right) \right)$$

$$-i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}\left(D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}\right)-D^{\alpha}_{\tau;-}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right)\right)=\frac{1-\alpha}{t-\tau}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}+\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right). \tag{5.3}$$

This is a complexified FELE.

Remarks 3:

1-*For* $\gamma=0$, *equation* (5.3) *is reduced to*:

$$\frac{\partial L}{\partial q}-\underbrace{\left(D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}\right)+D^{\alpha}_{\tau;-}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right)\right)}_{{}^{\alpha}_{\tau}}=\frac{1-\alpha}{t-\tau}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}+\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right), \tag{5.4}$$

which obviously is independent of ε *and thus is logically expected.*

2-*For* $\varepsilon=1$, *equation* (5.3) *is reduced to*:

$$\frac{\partial L}{\partial q}+\underbrace{\left(D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}\right)+D^{\alpha}_{\tau;-}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right)\right)}_{-{}^{\alpha}_{\tau}}=\frac{1-\alpha}{t-\tau}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}+\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right). \tag{5.5}$$

which apparently is independent of γ *and thus is rationally expected. This equation differs from its preceding equation with the negative sign of the fractional gradient* ${}^{\alpha}_{\tau}$. *This indicates the presence of a potential with positive sign of the gradient or negative momentum, e.g. harmonic oscillator with complex frequency* $(\omega^2<0)$ *or complex energy. This later makes further investigation meaningful. It is noteworthy that momentums with negative signs may play a crucial role in different aspects of theoretical physics as their presence may indicate the presence of some kind of negative fluctuations.*

3-*For* $\varepsilon=0$, *equation* (5.3) *is complexified in particular for* $\gamma\in\mathbb{R}$ *and takes now the special form:*

$$\frac{\partial L}{\partial q}-\frac{1-\gamma^2}{1+\gamma^2}\left(D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}\right)+D^{\alpha}_{\tau;-}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right)\right)$$

$$-i\frac{2\gamma}{1+\gamma^2}\left(D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}\right)-D^{\alpha}_{\tau;-}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right)\right)=\frac{1-\alpha}{t-\tau}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}+\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right). \tag{5.6}$$

4-*For a given Lagrangian given by*

$$L\left(\frac{1}{2}\left(1-\varepsilon+i\gamma\right)D^{\alpha}_{\tau;+}q(\tau),\frac{1}{2}\left(1-\varepsilon-i\gamma\right)D^{\alpha}_{\tau;+}q(\tau),q(\tau),\tau\right),$$

the associated one-dimensional fractional Euler-Lagrange equation derived from the fractional functional

$$S[L]=\frac{1}{\Gamma(\alpha)}\int_a^b L\left(\frac{1}{2}\left(1-\varepsilon+i\gamma\right)D^{\alpha}_{\tau;+}q(\tau),\frac{1}{2}\left(1-\varepsilon-i\gamma\right)D^{\alpha}_{\tau;+}q(\tau),q(\tau),\tau\right)(t-\tau)^{\alpha-1}d\tau, \tag{5.7}$$

is given by:

$$\frac{\partial L}{\partial q}-2\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}\right)=2\frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}. \tag{5.8}$$

Amazingly, for $(\gamma,\varepsilon)\in\mathbb{R}$*, equation* (5.8) *is not complexified although the fractional action is complexified.*

6. Illustrations and Applications

To illustrate, we consider the following form of equation (5.3):

$$\frac{\partial L}{\partial q}-\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}\right)-i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}D^{\alpha}_{\tau;+}\left(\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}\right)=\frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}. \tag{6.1}$$

Definition 6.1. The fractional generalized momenta associated to the FELE (6.1) is defined by

$$p_\alpha=\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}, \tag{6.2}$$

and the fractional Hamiltonian by:

$$H=p_\alpha D^{\alpha}_{\tau;-}q-L. \tag{6.3}$$

Theorem 6.2. *The associated fractional canonical equations for the fractional Hamiltonian* (6.3) *are given by:*

$$D^{\alpha}_{\tau;+}p_\alpha=-\frac{1-\varepsilon-i\gamma}{1-\varepsilon+i\gamma}\left(\frac{\partial H}{\partial q}+\frac{1-\alpha}{t-\tau}p_\alpha\right), \tag{6.4}$$

$$D^{\alpha}_{\tau;-}q=\frac{\partial H}{\partial p_\alpha}. \tag{6.5}$$

Proof. We let $F=p_\alpha D^{\alpha}_{\tau;-}q-H$. Requiring that:

$$\delta\left\{\frac{1}{\Gamma(\alpha)}\int_a^b F\,(t-\tau)^{\alpha-1}\,d\tau\right\}=0,$$

and varying the variables q and p independently, we obtain the fractional Euler-Lagrange equations $\delta F/\delta q=0$, $\delta F/\delta p=0$. When written out, we find easily:

$$\frac{\partial F}{\partial q}-\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}D^{\alpha}_{\tau;+}\left(\frac{\partial F}{\partial D^{\alpha}_{\tau;-}q}\right)-i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}D^{\alpha}_{\tau;+}\left(\frac{\partial F}{\partial D^{\alpha}_{\tau;-}q}\right)=\frac{1-\alpha}{t-\tau}\frac{\partial F}{\partial D^{\alpha}_{\tau;-}q},$$

$$\frac{\partial F}{\partial p_\alpha}-\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}D^{\alpha}_{\tau;+}\left(\frac{\partial F}{\partial D^{\alpha}_{\tau;-}p_\alpha}\right)-i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}D^{\alpha}_{\tau;+}\left(\frac{\partial F}{\partial D^{\alpha}_{\tau;-}p_\alpha}\right)=\frac{1-\alpha}{t-\tau}\frac{\partial F}{\partial D^{\alpha}_{\tau;-}p_\alpha},$$

from which we deduce equations (6.4) and (6.5). ∎

Examples:

A-To illustrate, we consider the fractional generalization of the harmonic oscillator Lagrangian $L=\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2$ like:

$$L=\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2\;\rightarrow\;L=\frac{1}{2}D^{\alpha}_{\tau;-}qD^{\alpha}_{\tau;+}q-\frac{1}{2}q^2.$$

The fractional equation of motion:

$$\left(\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}+i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}\right)D^{\alpha}_{\tau;+}(D^{\alpha}_{\tau;+}q)+\frac{1-\alpha}{t-\tau}D^{\alpha}_{\tau;+}q+2q=0, \tag{6.6}$$

and the fractional generalized momenta is

$$p_\alpha=\frac{\partial L}{\partial D^{\alpha}_{\tau;-}q}=\frac{1}{2}D^{\alpha}_{\tau;+}q, \tag{6.7}$$

and the fractional Hamiltonian by:

$$H=p_\alpha D^{\alpha}_{\tau;-}q-L=\frac{1}{2}q^2. \tag{6.8}$$

Accordingly, the fractional canonical equations (6.4) and (6.5) are reduced to:

$$D^{\alpha}_{\tau;+}p_\alpha=-\frac{1-\varepsilon-i\gamma}{1-\varepsilon+i\gamma}\left(q+\frac{1-\alpha}{t-\tau}p_\alpha\right), \tag{6.9}$$

$$D^{\alpha}_{\tau;-}q=\frac{\partial H}{\partial p_\alpha}=0. \tag{6.10}$$

We may conclude from equations (6.9), (6.10) and (6.7) that the fractional Lagrangian and the fractional Hamiltonian formalism are equivalent. At very long time, equation (6.6) is approximated by:

$$\left(\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}+i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}\right)D^{\alpha}_{\tau;+}(D^{\alpha}_{\tau;+}q)+2q=0. \tag{6.11}$$

Let us assume the boundary conditions:

$$D^{\alpha-1}_{\tau;+}\left(D^{\alpha}_{\tau;+}q(\tau)\right)\big|_{\tau=0}=Q_1, \tag{6.12}$$

$$D^{\alpha-1}_{\tau;+}q(\tau)\big|_{\tau=0}=Q_2, \tag{6.13}$$

where $0<\alpha<1$ and (Q_1,Q_2) are real constants. Making use of the Laplace transform of the sequential fractional derivative [33]:

$$\int_0^\infty \exp(-p\tau)D^{\sigma_m}_{\tau}f(\tau)d\tau=p^{\sigma_m}F(p)-\sum_{k=0}^{m}p^{\sigma_m-\sigma_{m-k}}\;D^{\sigma_{m-k}-1}_{\tau}f(\tau)\big|_{\tau=0}, \tag{6.14}$$

$$D^{\sigma_m}_{\tau}\equiv D^{\alpha_k}_{\tau}D^{\alpha_{k-1}}_{\tau}...D^{\alpha_1}_{\tau}, \tag{6.15}$$

$$D^{\sigma_{k-1}}_{\tau}\equiv D^{\alpha_{k-1}}_{\tau}D^{\alpha_{k-2}}_{\tau}...D^{\alpha_1}_{\tau}, k=1,2,...,n, \tag{6.16}$$

$$\sigma_k=\sum_{j=1}^{k}\alpha_j, 0\le\alpha_j\le 1, j=1,2,...n, \tag{6.17}$$

by taking $\alpha_1=\alpha_2=\alpha$ and $m=2$, i.e. $\sigma_1=\alpha$, $\sigma_2=2\alpha$ and therefore:

$$\left(\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}+i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}\right)\left(p^{2\alpha}+2\right)Y(p)=Q_2p^\alpha+Q_1, \tag{6.18}$$

or

$$Y(p)=\frac{Q_2}{\Omega}\frac{p^{\alpha}}{p^{2\alpha}+2}+\frac{Q_1}{\Omega}\frac{1}{p^{2\alpha}+2}, \tag{6.19}$$

where

$$\Omega=\frac{(1-\varepsilon)^2-\gamma^2}{(1-\varepsilon)^2+\gamma^2}+i\frac{2\gamma(1-\varepsilon)}{(1-\varepsilon)^2+\gamma^2}. \tag{6.20}$$

With the help of:

$$\int_0^{\infty}\exp(-p\tau)\tau^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}\left(\pm at^{\alpha}\right)dt=\frac{k!p^{\alpha-\beta}}{\left(p^{\alpha}\mp a\right)^{k+1}},\quad \Re(p)>|a|^{1/\alpha}, \tag{6.21}$$

where

$$E_{\alpha,\beta}^{(k)}(y)\equiv\frac{d^k}{dy^k}E_{\alpha,\beta}(y), \tag{6.22}$$

$E_{\alpha,\beta}(y)$ being the Mittag-Leffler two-parameter function, we find the complexified solution

$$y(\tau)=\frac{Q_2}{\Omega}\tau^{\alpha-1}E_{2\alpha,\alpha}(-2\tau^{2\alpha})+\frac{Q_1}{\Omega}\tau^{2\alpha-1}E_{2\alpha,2\alpha}(-2\tau^{2\alpha}). \tag{6.23}$$

B-Another illustration concerns the following fractional generalization of the harmonic oscillator Lagrangian like:

$$L=\frac{1}{2}D_{\tau;-}^{\alpha}q(\tau)D_{\tau;+}^{\alpha}q(\tau)+gq(\tau)D_{\tau;+}^{\alpha}q(\tau)+q(\tau)f(\tau),\ g\in\mathbb{R},$$

$f(\tau)$ is any real and differentiable function. As $\partial L/\partial D_{\tau;-}^{\alpha}q=\frac{1}{2}D_{\tau;+}^{\alpha}q$, we find at very long time:

$$D_{\tau;+}^{\alpha}\left(D_{\tau;+}^{\alpha}q\right)-\frac{2g}{\Omega}D_{\tau;+}^{\alpha}q=\frac{2}{\Omega}f(\tau)\equiv F(\tau). \tag{6.24}$$

Assuming the boundary conditions:

$$D_{\tau;+}^{\alpha-1}\left(D_{\tau;+}^{\alpha}q(\tau)\right)\Big|_{\tau=0}+D_{\tau;+}^{\alpha-1}q(\tau)\Big|_{\tau=0}=P_1, \tag{6.25}$$

$$D_{\tau;+}^{\alpha-1}q(\tau)\Big|_{\tau=0}=P_2, \tag{6.26}$$

we obtain after simple algebraic manipulation, making use of equations (6.14)-(6.17) and equation (6.21):

$$y(\tau)=\frac{Q_2}{\Omega}\tau^{\alpha-1}E_{\alpha,\alpha}(-\tau^{\alpha})+\frac{Q_1}{\Omega}\tau^{\alpha}E_{\alpha,2\alpha}(-\tau^{\alpha})+\frac{2}{\Omega}\int_0^t(t-\tau)^{2\alpha-1}E_{\alpha,2\alpha}(-(t-\tau)^{\alpha})f(\tau)d\tau. \tag{6.27}$$

Both equations (6.23) and (6.27) are the sum of a real part and an imaginary part, i.e. the classical trajectories are typically complex. However, it was observed more recently that complexified harmonic oscillators and their fractional counterparts are one of the features of fractional actionlike variational approaches [15]. It is amazing to obtain at the end a complexified harmonic oscillators emerging from non-complexified Lagrangians like those introduced in examples A and B. It is worth-mentioning that complexified harmonic oscillators are not excluded in literature as they play an important role in many aspects of theoretical physics related to PT-symmetry, e.g. Lotka-Volterra predictor-prey model

[34, 35]. In fact, there exist quantum mechanical models with explicit complex terms in the Hamiltonian that admit real spectra and unitary evolution. In a general context, it was observed that the real part of the Hamiltonian is responsible of the dynamics in a real phase space whereas the imaginary part, treated as a constraint, is responsible of the symmetry transformation [36]. The ideas discussed here might be viewed as a fuzzy alternative fractional version of a hidden variable formulation of quantum mechanics.

If, in contrast, we choose equation (5.8), the associated fractional generalized momenta is

$$p_\alpha = \frac{\partial L}{\partial D^{\alpha}_{\tau;+}q}, \tag{6.28}$$

and the fractional Hamiltonian by:

$$H = p_\alpha D^{\alpha}_{\tau;+}q - L. \tag{6.29}$$

The corresponding fractional canonical equations for the fractional Hamiltonian (6.29) are given by:

$$D^{\alpha}_{\tau;+}p_\alpha = -\frac{1}{2}\frac{(1-\varepsilon)^2+\gamma^2}{(1-\varepsilon)^2-\gamma^2}\left(\frac{\partial H}{\partial q} + \frac{1-\alpha}{t-\tau}p_\alpha\right), \tag{6.30}$$

$$D^{\alpha}_{\tau;+}q = \frac{\partial H}{\partial p_\alpha}. \tag{6.31}$$

For $\gamma = 0$ and equations (6.2) and (6.3) are reduced to:

$$D^{\alpha}_{\tau;+}p_\alpha = -\frac{1}{2}\left(\frac{\partial H}{\partial q} + \frac{1-\alpha}{t-\tau}p_\alpha\right), \tag{6.32}$$

$$D^{\alpha}_{\tau;-}q = \frac{\partial H}{\partial p_\alpha}, \tag{6.33}$$

whereas for $\varepsilon = 0$, we get:

$$D^{\alpha}_{\tau;+}p_\alpha = \frac{1}{2}\left(\frac{\partial H}{\partial q} + \frac{1-\alpha}{t-\tau}p_\alpha\right), \tag{6.34}$$

$$D^{\alpha}_{\tau;-}q = \frac{\partial H}{\partial p_\alpha}, \tag{6.35}$$

The particular case where $\gamma^2 = -(1-\varepsilon)^2, \gamma \in \mathbb{C}$ is interesting as it yields $D^{\alpha}_{\tau;+}p_\alpha = 0$, i.e. p_α is a fractional constant of motion. To illustrate we consider a fractional generalization of a free Lagrangian of one degree of freedom like:

$$L = \frac{1}{2}(D^{\alpha}_{\tau;+}q)^2 + CD^{\alpha}_{\tau;+}q - \frac{1}{2}q^2, C \in \mathbb{R}^+. \tag{6.36}$$

The fractional equation of motion (5.8) gives:

$$\begin{aligned} &D^{\alpha}_{\tau;+}D^{\alpha}_{\tau;+}q + \frac{1-\alpha}{t-\tau}\frac{(1-\varepsilon)^2+\gamma^2}{(1-\varepsilon)^2-\gamma^2}D^{\alpha}_{\tau;+}q + \frac{C}{\Gamma(1-\alpha)(b-\tau)^\alpha} \\ &+C\frac{1-\alpha}{t-\tau}\frac{(1-\varepsilon)^2+\gamma^2}{(1-\varepsilon)^2-\gamma^2} + \frac{1}{2}\frac{(1-\varepsilon)^2+\gamma^2}{(1-\varepsilon)^2-\gamma^2}q = 0, \end{aligned} \tag{6.37}$$

which for very large time is approximated by:

$$D^{\alpha}_{\tau;+}D^{\alpha}_{\tau;+}q+\frac{1}{2}\frac{(1-\varepsilon)^2+\gamma^2}{(1-\varepsilon)^2-\gamma^2}q=0. \tag{6.38}$$

The solution of this equation making use of the boundary conditions

$$D^{\alpha-1}_{\tau;+}\left(D^{\alpha}_{\tau;+}q(\tau)\right)\big|_{\tau=0}=M_1, \tag{6.39}$$

$$D^{\alpha-1}_{\tau;+}q(\tau)\big|_{\tau=0}=M_2, \tag{6.40}$$

where $0<\alpha<1$ and (M_1,M_2) are real constants and equations (6.14)-(6.17) and (6.21) is:

$$y(\tau)=M_2\tau^{\alpha-1}E_{2\alpha,\alpha}\left(-\frac{1}{2}\frac{(1-\varepsilon)^2+\gamma^2}{(1-\varepsilon)^2-\gamma^2}\tau^{2\alpha}\right)+M_2\tau^{2\alpha-1}E_{2\alpha,2\alpha}\left(-\frac{1}{2}\frac{(1-\varepsilon)^2+\gamma^2}{(1-\varepsilon)^2-\gamma^2}\tau^{2\alpha}\right). \tag{6.41}$$

7. Conclusions and Perspectives

The new fractional derivative introduced in this work and its implementation inside the Lagrangian of a dynamical system has revealed many interesting consequences. The new fractional actionlike variational approach and its corresponding fractional Euler-Lagrange equations are highly prosperous and universal than the ones discussed in literature. The complexified fractional Lagrangian and Hamiltonian formalisms modeled in this work may have interesting consequences in many aspects of theoretical physics, i.e. gauge field theories as well as in many aspects of applied mathematics, i.e. chaotic dynamics and gauge theory. We believe it will open up in the future a new exciting research area and supply us by a potent tool to comprehend many fundamental problems in the area of mathematical physics. The next step, to acquire an improved understanding of the generic features of the new fractional formalism, would be to examine some more significant examples within the framework of complexification of gauge theories and complex Lagrangian mechanics with constraints on the Kählerian manifold. Our main aim in a future work is to make contribution to the modern development of classical and quantum Lagrangian dynamics in terms of some new fractional differentiable geometric methods on differentiable manifold.

References

[1] K.S. Miller, B. Ross, An Introduction to the Fractional Integrals and Derivatives-Theory and Application. Wiley, New York, 1993.

[2] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic, New York, 1974.

[3] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, Advance in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, 2007.

[4] R. Almeida, D.F.M. Torres, Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives, Commun. Nonlinear Sci. Numer. Simulat. 16 (3) (2011) 1490–1500.

[5] R. Almeida, D.F.M. Torres, Fractional variational calculus for nondifferentiable functions, Comput. Math. Appl. 61 (10) (2011) 3097–3104.

[6] R. Almeida, A.B. Malinowska, D.F.M. Torres, Fractional Euler-Lagrange differential equations via Caputo derivatives, Fractional Dynamics and Control, Springer New York (2012) Part 2, 109–118.

[7] R. Almeida, S. Pooseh, D.F.M. Torres, Fractional variational problems depending on indefinite integrals, Nonlinear Anal. 75 (3) (2012) 1009–1025.

[8] R. Almeida, A.B. Malinowska, D.F.M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, J. Math. Phys. 51 (3) (2010) 033503.

[9] O.P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl. 272 (2002) 368–379.

[10] D. Baleanu, S.I. Muslih, E.M. Rabei, On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative, arXiv:0708.1690.

[11] J. Cresson, Fractional embedding of differential operators and Lagrangian systems, J. Math. Phys. 48 (2007) 033504.

[12] R.A. El-Nabulsi, Fractional variational problems from extended exponentially fractional integral, Applied Mathematics and Computation 217 (22) (2011) 9492–9496.

[13] R.A. El-Nabulsi, A periodic functional approach to the calculus of variations and the problem of time dependent damped harmonic oscillators, Applied Mathematics Letters 24 (10) (2011) 1647–1653.

[14] R.A.El-Nabulsi, D.F.M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (alfa,beta), Math. Methods Appl. Sci. 30 (15) (2007) 1931–1939.

[15] R.A. El-Nabulsi, D.F.M. Torres, Fractional actionlike variational problems, J. Math. Phys. 49 (5) (2008) 053521.

[16] R.A. El-Nabulsi, A fractional approach to nonconservative Lagrangian dynamical systems, Fizika A14 (4) (2005) 289–298.

[17] R.A. El-Nabulsi, The fractional calculus of variations from extended Erdelyi-Kober operator, Int. J. Mod. Phys. B23 (16) (2009) 3349–3361.

[18] R.A. El-Nabulsi, Universal fractional Euler-Lagrange equation from a generalized fractional derivate operator, Central Europ. J. Phys. 9 (1) (2010) 250–256.

[19] R.A. El-Nabulsi, G.-c. Wu, Fractional complexified field theory from Saxena-Kumbhat fraction integral, fractional derivative of order alfa-beta and dynamical fractional integral exponent, African Dias. J. Math. 13 (2) (2012) 45–61.

[20] R.A. El-Nabulsi, M. Jamil, G.-c. Wu, Complexified Lie algebroids from a generalized Stieltjes action approach to the calculus of variations, Sarajevo J. Math. 80 (2) (2012) 1–16.

[21] R.A. El-Nabulsi, The fractional Boltzmann transport equation, Comp. Math. Appl. 62 (3) (2011) 1568–1575.

[22] A.B. Malinowska, M.R.S. Ammi, D.F.M. Torres, Composition functionals in fractional calculus of variations, Commun. Frac. Calc. 1 (2010) 32–40.

[23] A.B. Malinowska, D.F.M. Torres, Euler-Lagrange equations for composition functionals in calculus of variations on time scales, Discrete Contin. Dyn. Syst. 29 (2) (2011) 577–593.

[24] A.B. Malinowska, D.F.M. Torres, A general backwards calculus of variations via duality, Optim. Lett. 5 (4) (2011) 587–599.

[25] A.B. Malinowska, D.F.M. Torres, Fractional calculus of variations for a combined Caputo derivative, Fract. Calc. Appl. Anal. 14 (4) (2011) 523–537.

[26] A.B. Malinowska, D.F.M. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative, Comput. Math. Appl. 59 (9) (2010) 3110–3116.

[27] A.B. Malinowska, D.F.M. Torres, Multiobjective fractional variational calculus in terms of a combined Caputo derivative, Appl. Math. Comput. 218 (9) (2012) 5099–5111.

[28] N. Martins, D.F.M. Torres, Noether's symmetry theorem for nabla problems of the calculus of variations, Appl. Math. Lett. 23 (12) (2010) 1432–1438.

[29] Y.F. Luchko, H. Martinez and J.J. Trujillo, Fractional Fourier transform and some of its applications, Frac. Cal. Appl. Anal. 11 (4) (2008) 457–470.

[30] P. Weiss, Born's Electrodynamics in complex form, Proc. Camb. Philo. Soc., 33 (1937) 79–93.

[31] P. Polesello, P. Schapira, Stacks of quantization-deformation modules on complex symplectic manifolds, Int. Math. Res. Notices 49 (2004) 2637–2664.

[32] M. Tekkoyum, A. Gorgulu, Complex Lagrangian mechanics with constraints, arXiv:0902.4128.

[33] I. Podlubny, The Laplace transform method for linear differential equations of the fractional order, In: I. Podlubny, Fractional Differential Equations, Academic Press, San Diego (1999), Chapters 4–5.

[34] C.M. Bender, D.D. Holm, D.W. Hook, Complexified dynamical systems, J. Phys. A40 (2007) F793–F804.

[35] S. Ghosh, S.K. Modak, Classical oscillator with position-dependent mass in a complex domain, Phys. Lett. A373 (2009) 1212–1232.

[36] K. Fujikawa, H. Suzuki, Path Integrals and Quantum Anomalies, Oxford University Press-USA, 2004.