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G-Frames, Fusion Frames and the Restricted Isometry Property

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Abstract In this paper we study the restricted isometry property for g-frames, we will show how to use tight g-frames that have the restricted isometry property to construct fusion frames. We also study the conditions which under removing some element from a g-frame, again we obtain another g-frame so that Theorem 4.3 obtained in [M.S. Asgari, On the stability of fusion frames (frames of subspaces), Acta Math. Sci. (Ser. B) 31 (4) (2011) 1633–1642] is a special case of it.

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1. Introduction

The g-frame theory is an emerging mathematical theory that provides a natural framework for performing hierarchical data processing. A g-frame is a frame-like collection of operators on a Hilbert space, thereby generalizing the concept of a frame for signal representation. Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [1], reintroduced in 1986 by Daubechies, Grossmann, and Meyer [2]. Frames are a generalization of the orthonormal bases and g-frames are a generalization of frames in Hilbert spaces which were introduced by W. Sun in [3]. Related approaches with a different focus were undertaken by Casazza and Kutyniok in [4]. G-Frames and fusion frames play important roles in many applications in mathematics, science, and engineering, including coding theory [5], compressed sensing [6], filter bank theory [7], applications to sensor networks [8], construction methods [9–11], and many other areas. The restricted isometry property is one of the cornerstones of compressed sensing. Today, compressed sensing is one of the most active areas of research in applied analysis and so we refer the reader to the tutorials [12] and their references for a background in the area. Our goal here is to use tools from

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compressed sensing, namely operators with the restricted isometry property, to construct fusion frames with very strong properties.

The paper is organized as follows: Section 2, contains an extension of the restricted isometry property to the g-frame situation. In this section, we will show how to use tight g-frames which have the restricted isometry property to construct fusion frames. In Section 3, we study the conditions which under removing some element from a g-frame, again we obtain another g-frame.

Throughout this paper, $\mathcal{H}, \mathcal{K}$ are separable Hilbert spaces and $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. I, J, J_i denote the countable (or finite) index sets and $\{W_i\}_{i\in I}$ is a sequence of closed subspaces of $\mathcal{K}$. Also $\mathcal{B}(\mathcal{H}, W_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ into W_i.

We start by recalling the definition of frames for $\mathcal{H}$. A frame for $\mathcal{H}$ is an indexed set of vectors $\{f_i : i \in I\} \subseteq \mathcal{H}$ for which there exist positive constants $0 < A \leq B < \infty$ so that for all $f \in \mathcal{H}$ we have

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

The numbers A, B are called lower (respectively, upper) frame bounds for the frame. If we only have the right-hand inequality of (1.1), then $\mathcal{F}$ is called a Bessel sequence. If $A = B$ it is an A-tight frame, and if $A = B = 1$, it is a Parseval frame. If $\|f_i\| = c$ for all $i \in I$ this is an equal norm frame, and if $c = 1$ it is a unit norm frame. The *synthesis* operator for $\mathcal{F}$ is the bounded linear operator $T_{\mathcal{F}} : \ell^2(I) \to \mathcal{H}$, given by $T_{\mathcal{F}}(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$. The *analysis* operator for $\mathcal{F}$ is $T^*_{\mathcal{F}}$ and is given by $T^*_{\mathcal{F}} f = \{\langle f, f_i\rangle\}_{i\in I}$. The frame operator is the positive self-adjoint invertible operator $S_{\mathcal{F}} = T_{\mathcal{F}} T^*_{\mathcal{F}}$ and satisfies $S_{\mathcal{F}} f = \sum_{i\in I}\langle f, f_i\rangle f_i$. Reconstruction is given by

$$f = \sum_{i\in I}\langle f, f_i\rangle S_{\mathcal{F}}^{-1} f_i = \sum_{i\in I}\langle f, S_{\mathcal{F}}^{-\frac{1}{2}} f_i\rangle S_{\mathcal{F}}^{-\frac{1}{2}} f_i \qquad \forall f \in \mathcal{H}.$$

In particular, $\{S_{\mathcal{F}}^{-\frac{1}{2}} f_i\}_{i\in I}$ is a Parseval frame for $\mathcal{H}$.

Definition 1.1. Let $\Lambda_i \in \mathcal{B}(\mathcal{H}, W_i)$ for all $i \in I$. Then a family of operators $\Lambda = \{\Lambda_i\}_{i\in i}$ is called a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq D\|f\|^2 \qquad \forall f \in \mathcal{H}. \tag{1.2}$$

The constants C and D are called g-frame bounds and $\sup_{i\in I} \Lambda_i$ is the multiplicity of Λ. A g-frame is called tight if C and D can be chosen to be equal, Parseval if $C = D = 1$ and ε-g-frame if $C = \frac{1}{1+\varepsilon}$ and $D = 1 + \varepsilon$ for some $\varepsilon > 0$. If the right-hand side of (1.2) holds, then Λ is said a g-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Moreover if $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\Lambda_i = \pi_{W_i}$ be the orthogonal projection of $\mathcal{H}$ onto W_i for all $i \in I$. Then $\{W_i\}_{i\in I}$ is said a fusion frame for $\mathcal{H}$. The representation space associated with a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ is defined by

$$\Big(\sum_{i\in I} \oplus W_i\Big)_{\ell^2} = \Big\{\{g_i\}_{i\in I} | g_i \in W_i \ \text{ and } \sum_{i\in I} \|g_i\|^2 < \infty\Big\}. \tag{1.3}$$

The synthesis operator of Λ given by

$$T_\Lambda : \Big(\sum_{i\in I} \oplus W_i\Big)_{\ell^2} \to \mathcal{H} \qquad T_\Lambda(\{g_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* g_i.$$

The adjoint operator of T_Λ, which is called the analysis operator also obtain as follows

$$T_\Lambda^* : \mathcal{H} \to \Big(\sum_{i\in I} \oplus W_i\Big)_{\ell^2} \qquad T_\Lambda^* f = \{\Lambda_i f\}_{i\in I}.$$

By composing T_Λ with its adjoint T_Λ^*, we obtain the fusion frame operator

$$S_\Lambda : \mathcal{H} \to \mathcal{H} \qquad S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f,$$

which is a bounded, self-adjoint, positive and invertible operator and $CI_{\mathcal{H}} \le S_\Lambda \le DI_{\mathcal{H}}$. The canonical dual g-frame for $\{\Lambda_i\}_{i\in I}$ is defined by $\{\widetilde{\Lambda}_i\}_{i\in I}$ where $\widetilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, which is also a g-frame for $\mathcal{H}$ with g-frame bounds $\frac{1}{D}$ and $\frac{1}{C}$, respectively. Also we have

$$f = \sum_{i\in I} \Lambda_i^* \widetilde{\Lambda}_i f = \sum_{i\in I} \widetilde{\Lambda}_i^* \Lambda_i f \qquad \forall f \in \mathcal{H}.$$

For more details about the theory and applications of frames we refer the readers to [1, 2, 13, 14] and for fusion frames to [4, 11], about g-frames to [3, 15].

2. G-Frames with the Restricted Isometry Property

In this section we generalize the restricted isometry property for g-frames, we will show how to use tight g-frames which have the ε-restricted isometry property to construct fusion frames. We denote $\mathcal{H}_N$ for a Hilbert space with dimension N and $\{e_j\}_{j=1}^N$ an orthonormal basis for $\mathcal{H}_N$. Moreover, the Hilbert-Schmidt norm of operator $T \in \mathcal{B}(\mathcal{H}_N, \mathcal{K})$ is defined by $\|T\|_{HS}^2 = \sum_{j=1}^N \|Te_j\|^2$.

Proposition 2.1. *Let $\{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then*

(i) If $\mathcal{H}$ is finite-dimensional, then $\{\|\Lambda_i\|_{HS}^2\}_{i\in I}$ is summable.

(ii) If $\mathcal{H}$ is finite-dimensional and $\alpha = \|\Lambda_i\| = \|\Lambda_j\|$ for all $i, j \in I$. Then $\{\Lambda_i\}_{i\in I}$ is a finite sequence.

(iii) If $\mathcal{H}$ is finite-dimensional and $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with g-frame bounds A and B. Then

$$A \le \frac{\sum_{i\in I} \|\Lambda_i\|_{HS}^2}{\dim \mathcal{H}} \le B.$$

Proof. (i) Let B be the g-Bessel bound for $\{\Lambda_i\}_{i\in I}$ and let $\{e_j\}_{j=1}^N$ be an orthonormal basis for $\mathcal{H}$. Then we have

$$\sum_{i\in I} \|\Lambda_i\|_{HS}^2 = \sum_{i\in I}\sum_{j=1}^N \|\Lambda_i e_j\|^2 = \sum_{j=1}^N \sum_{i\in I} \|\Lambda_i e_j\|^2$$
$$\le B \sum_{j=1}^N \|e_j\|^2 = BN.$$

(ii) Since $\|\Lambda_i\|_{HS} \geq \|\Lambda_i\| = \alpha$ hence from the part (i) follows that $\{\Lambda_i\}_{i\in I}$ is finite.
(iii) Since $\sum_{i\in I} \|\Lambda_i\|_{HS}^2 = \sum_{j=1}^{N} \langle S_\Lambda e_j, e_j\rangle$ and $AI_{\mathcal{H}} \leq S_\Lambda \leq BI_{\mathcal{H}}$ thus we obtain

$$A \dim \mathcal{H} = A\sum_{j=1}^{N} \|e_j\|^2 \leq \sum_{j=1}^{N} \langle S_\Lambda e_j, e_j\rangle \leq B\sum_{j=1}^{N} \|e_j\|^2 = B \dim \mathcal{H}.$$

This yields

$$A \dim \mathcal{H} \leq \sum_{i\in I} \|\Lambda_i\|_{HS}^2 \leq B \dim \mathcal{H}.$$

From this the claim follows immediately. ∎

Theorem 2.2. *Let $\Lambda = \{\Lambda_i\}_{i=1}^{M}$ be a g-frame for $\mathcal{H}_N$ with respect to $\{W_i\}_{i=1}^{M}$. Then*

- (i) *The optimal bounds of Λ are the smallest and biggest eigenvalues of g-frame operator S_Λ.*
- (ii) *If $\{\lambda_i\}_{i=1}^{N}$ is a representation of eigenvalues of S_Λ. Then*

$$\sum_{j=1}^{N} \lambda_j = \sum_{i=1}^{M} \|\Lambda_i\|_{HS}^2 \quad \text{and} \quad \lambda_j = \sum_{i=1}^{M} \|\Lambda_i e_j\|^2,$$

where $\{e_j\}_{j=1}^{N}$ is the orthonormal basis consisting of eigenvectors of S_Λ.

Proof. To prove (i) see that since S_Λ is a self-adjoint operator on $\mathcal{H}_N$, thus $\mathcal{H}_N$ has an orthonormal basis include eigenvectors of S_Λ. Let $\{e_j\}_{j=1}^{N}$ be an orthogonal basis of $\mathcal{H}_N$ include of eigenvectors of S_Λ. Let $\{\lambda_j\}_{j=1}^{N}$ be eigenvalues of $\{e_j\}_{j=1}^{N}$. Then for any $f \in \mathcal{H}_N$ we have

$$\begin{aligned}\sum_{i=1}^{M} \|\Lambda_i f\|^2 = \langle S_\Lambda f, f\rangle &= \Big\langle \sum_{j=1}^{N} \langle f, e_j\rangle S_\Lambda e_j, f\Big\rangle \\ &= \sum_{j=1}^{N} \langle f, e_j\rangle \langle S_\Lambda e_j, f\rangle = \sum_{j=1}^{N} \langle f, e_j\rangle \langle \lambda_j e_j, f\rangle \\ &= \sum_{j=1}^{N} \lambda_j |\langle f, e_j\rangle|^2.\end{aligned}$$

Since for any $1 \leq i \leq N$ we have $\lambda_{\min} \leq \lambda_i \leq \lambda_{\max}$, thus

$$\lambda_{\min}\|f\|^2 \leq \sum_{i=1}^{M} \|\Lambda_i f\|^2 \leq \lambda_{\max}\|f\|^2.$$

To prove (ii) we have:

$$\begin{aligned}\sum_{j=1}^{N}\lambda_j &= \sum_{j=1}^{N}\langle\lambda_j e_j, e_j\rangle = \sum_{j=1}^{N}\langle S_\Lambda e_j, e_j\rangle \\ &= \sum_{j=1}^{N}\sum_{i=1}^{M}\|\Lambda_i e_j\|^2 = \sum_{i=1}^{M}\sum_{j=1}^{N}\|\Lambda_i e_j\|^2 \\ &= \sum_{i=1}^{M}\|\Lambda_i\|_{HS}^2.\end{aligned}$$

Also we obtain

$$\sum_{i=1}^{M}\|\Lambda_i e_j\|^2 = \langle S_\Lambda e_j, e_j\rangle = \langle\lambda_j e_j, e_j\rangle = \lambda_j. \qquad \blacksquare$$

Corollary 2.3. *Let $\{\Lambda_i\}_{i=1}^{M}$ be a A-tight g-frame with unit Hilbert-Schmidt norm for $\mathcal{H}_N$ with respect to $\{W_i\}_{i=1}^{M}$, then $A = \frac{M}{N}$.*

Proof. This is a direct result from section (iii) in Proposition 2.1. $\blacksquare$

Definition 2.4. Let $\Lambda_i \in \mathcal{B}(\mathcal{H}, W_i)$ for all $i \in I$. Then

(i) $\{\Lambda_i\}_{i\in I}$ is called an orthonormal g-system for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, if $\Lambda_i\Lambda_j^* g_j = \delta_{ij} g_j$ for all $i, j \in I, g_j \in W_j$.

(ii) If $\mathcal{H} = \overline{\text{span}}\{\Lambda_i^*(W_i)\}_{i\in I}$, then we say that $\{\Lambda_i\}_{i\in I}$ is g-complete.

(iii) We say that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, if it is a g-orthonormal g-complete system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

(iv) If $\{\Lambda_i\}_{i\in I}$ is g-complete and there are positive constants A and B such that for any finite subset $J \subset I$ and $g_j \in W_j$,

$$A\sum_{j\in J}\|g_j\|^2 \le \Big\|\sum_{j\in J}\Lambda_j^* g_j\Big\|^2 \le B\sum_{j\in J}||g_j||^2.$$

Then $\{\Lambda_i\}_{i\in I}$ is called a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Moreover, $\{\Lambda_i\}_{i\in I}$ is called a ε-g-Riesz basis for $\mathcal{H}$ if $A = \frac{1}{1+\varepsilon}$ and $B = 1+\varepsilon$ for some $\varepsilon > 0$. Also $\{\Lambda_i\}_{i\in I}$ is a ε-g-Riesz sequence if $\{\Lambda_i\}_{i\in I}$ is a ε-g-Riesz basis for $\overline{\text{span}}\{\Lambda_i^*(W_i)\}_{i\in I}$.

The next proposition is similar to a result of Bodmann, Cahill and Casazza [16] to the situation of g-frames.

Proposition 2.5. *Let $\{\Lambda_i\}_{i\in I}$ be a ε-g-Riesz sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and let $\{I_j\}_{j=1}^{L}$ be a partition of I. Then for every $1 \le j \le L$ and for any sequence $\{g_{jk}\}_{k\in I_j} \in \big(\sum_{k\in I_j}\oplus W_k\big)_{\ell^2}$*

$$\frac{1}{1+\varepsilon}\sum_{j=1}^{L}\Big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2 \le \sum_{j=1}^{L}\sum_{k\in I_j}\|g_{jk}\|^2 \le (1+\varepsilon)\sum_{j=1}^{L}\Big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2.$$

Also,

$$\frac{1}{(1+\varepsilon)^2}\sum_{j=1}^{L}\Big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2 \le \Big\|\sum_{j=1}^{L}\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2 \le (1+\varepsilon)^2\sum_{j=1}^{L}\Big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2.$$

Proof. For each $1 \leq j \leq L$ and any sequence $\{g_{jk}\}_{k\in I_j} \in \left(\sum_{k\in I_j} \oplus W_k\right)_{\ell^2}$ we have

$$\frac{1}{1+\varepsilon}\sum_{j=1}^{L}\big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\big\|^2 \leq \frac{1}{1+\varepsilon}\sum_{j=1}^{L}(1+\varepsilon)\sum_{k\in I_j}\|g_{jk}\|^2 = \sum_{j=1}^{L}\sum_{k\in I_j}\|g_{jk}\|^2$$
$$\leq \sum_{j=1}^{L}(1+\varepsilon)\big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\big\|^2 = (1+\varepsilon)\sum_{j=1}^{L}\big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\big\|^2.$$

This yields

$$\frac{1}{(1+\varepsilon)^2}\sum_{j=1}^{L}\big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\big\|^2 \leq \frac{1}{1+\varepsilon}\sum_{j=1}^{L}\sum_{k\in I_j}\|g_{jk}\|^2 \leq \Big\|\sum_{j=1}^{L}\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2$$
$$\leq (1+\varepsilon)\sum_{j=1}^{L}\sum_{k\in I_j}\|g_{jk}\|^2 \leq (1+\varepsilon)^2\sum_{j=1}^{L}\big\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\big\|^2. \qquad \blacksquare$$

It is known that if $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with g-Riesz constants A and B, then $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with same bounds A and B. The next lemma is analogous to Lemma 3.3 in [16] to the situation of g-frames.

Lemma 2.6. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a ε-g-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then for all $n \in \mathbb{N}$*

$$\frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq S_\Lambda^n \leq (1+\epsilon)^n I_{\mathcal{H}} \quad \text{and} \quad \frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq S_\Lambda^{-n} \leq (1+\epsilon)^n I_{\mathcal{H}}.$$

Proof. Since $\{\Lambda_i\}_{i\in I}$ is a ε-g-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, so this family is a g-frame for $\mathcal{H}$ with bounds $\frac{1}{1+\varepsilon}, 1+\varepsilon$ respectivelt. Hence $\frac{1}{1+\varepsilon} \leq \|S_\Lambda\| \leq (1+\varepsilon)$ and $\frac{1}{1+\varepsilon} \leq \|S_\Lambda^{-1}\| \leq (1+\varepsilon)$. On the other hand for any $f \in \mathcal{H}$ and $n \in \mathbb{N}$ we have $\|S_\Lambda^{-1}\|^{-n}\|f\| \leq \|S_\Lambda^n f\| \leq \|S_\Lambda\|^n\|f\|$ which implies that $\|S_\Lambda^{-1}\|^{-n}I_{\mathcal{H}} \leq S_\Lambda^n \leq \|S_\Lambda\|^n I_{\mathcal{H}}$. Consequently

$$\frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq \|S_\Lambda^{-1}\|^{-n}I_{\mathcal{H}} \leq S_\Lambda^n \leq \|S_\Lambda\|^n I_{\mathcal{H}} \leq (1+\varepsilon)^n I_{\mathcal{H}}.$$

This shows that $\frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq S_\Lambda^n \leq (1+\epsilon)^n I_{\mathcal{H}}$ and so $\frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq S_\Lambda^{-n} \leq (1+\epsilon)^n I_{\mathcal{H}}$. $\blacksquare$

Proposition 2.7. *Let $\{\Lambda_i\}_{i\in I}$ be a ε-g-Riesz sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then for all partition $\{I_1, I_2\}$ of I and $f \in \overline{\operatorname{span}}\{\Lambda_i^*(W_i)\}_{i\in I_1}, g \in \overline{\operatorname{span}}\{\Lambda_i^*(W_i)\}_{i\in I_2}$ with $\|f\| = \|g\| = 1$, $|\langle f, g\rangle| \leq 2\varepsilon + \varepsilon^2$.*

Proof. For all finite subsets $F_1 \subseteq I_1, F_2 \subseteq I_2$ and arbitrary vectors $g_i \in W_i (i \in F_1 \bigcup F_2)$, suppose that $\varphi = \sum_{i\in F_1}\Lambda_i^* g_i$ and $\psi = \sum_{i\in F_2}\Lambda_i^* g_i$ with conditions $||\varphi|| = ||\psi|| = 1$. Then for any $|\lambda| = 1$ we have

$$(\langle\varphi,\lambda\psi\rangle) = \frac{2(\langle\varphi,\lambda\psi\rangle)+2}{2} - 1 = \frac{\|\varphi+\lambda\psi\|^2}{2} - 1$$
$$\leq \frac{(1+\varepsilon)}{2}\sum_{i\in F_1\cup F_2}\|g_i\|^2 - 1 = \frac{(1+\varepsilon)}{2}\Big(\sum_{i\in F_1}\|g_i\|^2 + \sum_{i\in F_2}\|g_i\|^2\Big) - 1$$
$$\leq \frac{(1+\varepsilon)^2}{2}(\|\varphi\|^2 + \|\psi\|^2) - 1 = 2\varepsilon + \varepsilon^2.$$

This yields

$$|\langle \varphi, \psi\rangle| = \max_{|\lambda|=1}\langle \varphi, \lambda\psi\rangle \leq 2\epsilon + \epsilon^2,$$

which implies that $|\langle f, g\rangle| \leq 2\epsilon + \epsilon^2$. ■

Definition 2.8. For every $1 \leq i \leq M$, let $\Lambda_i \in \mathcal{B}(\mathcal{H}_N, W_i)$. Then we say that the family $\{\Lambda_i\}_{i=1}^M$ has the restricted isometry property with constant $0 < \varepsilon < 1$ for sets of size $s \leq N$, if for every $I \subseteq \{1, 2, ..., M\}$ with $|I| \leq s$, the family $\{\Lambda_i\}_{i\in I}$ is a ε-g-Riesz sequence for $\mathcal{H}_N$ with respect to $\{W_i\}_{i\in I}$.

The next theorem is a generalization of Theorem 4.2 in [16] to the g-frames situation.

Theorem 2.9. *Let $\{\Lambda_i\}_{i=1}^M$ be a tight g-frame for $\mathcal{H}_N$ with respect to $\{W_i\}_{i=1}^M$ with the restricted isometry property with constant $0 < \varepsilon < 1$ for sets of size $s \leq N$. Suppose that $\{I_j\}_{j=1}^L$ is an arbitrary partition of $\{1, 2, ..., M\}$ with $|I_j| \leq s$. Define $V_j = \operatorname{span}\{\Lambda_i^*(W_i)\}_{i\in I_j}$ for all $1 \leq j \leq L$, then $\{V_j\}_{j=1}^L$ is a fusion frame for $\mathcal{H}_N$ with fusion frame bounds $\frac{\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{(1+\varepsilon)N}$, $\frac{(1+\varepsilon)\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N}$ and*

$$\frac{1}{1+\varepsilon}\sum_{i\in I_j}\|\Lambda_i f\|^2 \leq \|\pi_{V_j} f\|^2 \leq (1+\varepsilon)\sum_{i\in Ij}\|\Lambda_i f\|^2.$$

Proof. By the hypothesis $\{\Lambda_i\}_{i\in I_j}$ is a g-frame for V_j with respect to $\{W_i\}_{i\in I_j}$ with g-frame bounds $\frac{1}{1+\varepsilon}, 1+\varepsilon$ respectively, for all $1 \leq j \leq L$. Let S_j be g-frame operator of $\{\Lambda_i\}_{i\in I_j}$, which is a self-adjoint operator on $\mathcal{H}_N$. Suppose that $\{e_i\}_{i=1}^N$ is an orthonormal basis of eigenvectors for S_j with eigenvalues $\{\lambda_i\}_{i=1}^N$, then $\lambda_i = 0$ for all $|I_j| < i \leq N$ and $\frac{1}{1+\varepsilon} \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|I_j|} \leq 1+\varepsilon$. Since $\{e_i\}_{i=1}^{|I_j|}$ is an orthonormal basis for V_j, hence $\pi_{V_j} f = \sum_{i=1}^{|I_j|}\langle f, e_i\rangle e_i$, for any $f \in \mathcal{H}_N$. We also have

$$S_j f = S_j\big(\sum_{i=1}^N \langle f, e_i\rangle e_i\big) = \sum_{i=1}^N \langle f, e_i\rangle S_j e_i = \sum_{i=1}^{|I_j|}\langle f, e_i\rangle \lambda_i e_i,$$

which implies that $\langle S_j f, f\rangle = \sum_{i=1}^{|I_j|}\lambda_i |\langle f, e_i\rangle|^2$. Thus we have

$$\begin{aligned}\frac{1}{1+\varepsilon}\sum_{i\in I_j}\|\Lambda_i f\|^2 &= \frac{1}{1+\varepsilon}\langle S_j f, f\rangle = \sum_{i\in I_j}\frac{\lambda_i}{1+\varepsilon}|\langle f, e_i\rangle|^2\\ &\leq \|\pi_{V_j}\|^2 \leq \sum_{i\in I_j}\lambda_i(1+\varepsilon)|\langle f, e_i\rangle|^2\\ &= (1+\varepsilon)\langle S_j f, f\rangle = (1+\varepsilon)\sum_{i\in I_j}\|\Lambda_i f\|^2.\end{aligned}$$

It follows that

$$\frac{1}{1+\varepsilon}\sum_{j=1}^L\sum_{i\in I_j}\|\Lambda_i f\|^2 \leq \sum_{j=1}^L \|\pi_{V_j} f\|^2 \leq (1+\varepsilon)\sum_{j=1}^L\sum_{i\in I_j}\|\Lambda_i f\|^2.$$

Now by Proposition 2.1 we have

$$\frac{\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{(1+\varepsilon)N}\|f\|^2 \leq \sum_{i=1}^L \|\pi_{V_j} f\|^2 \leq \frac{(1+\varepsilon)\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N}\|f\|^2.$$ ■

Corollary 2.10. *Under the assumptions of Theorem 2.9 if* $\{1,2,\cdots,L\}\subseteq\{1,2,\cdots,M\}$ *and there exists a family* $\{J_j\}_{j=1}^L$ *such that* $\sum_{j=1}^L |J_j|\leq s$ *and* $J_j\subseteq I_j$ *for all* $1\leq j\leq L$. *Then*

$$\frac{1}{(1+\varepsilon)^2}\sum_{j=1}^L\|\sum_{i\in J_j}\Lambda_i^* g_i\|^2\leq\Big\|\sum_{j=1}^L\sum_{i\in J_j}\Lambda_i^* g_i\Big\|^2\leq(1+\varepsilon)^2\sum_{j=1}^L\|\sum_{i\in J_j}\Lambda_i^* g_i\|^2.$$

Proof. This follows from the Proposition 2.5. ∎

The following theorem will give another method for obtaining a fusion frame from an unit norm tight frame for $\mathcal{H}_N$ without having the restricted isometry property. Another form of this result can be found in [16] Theorem 4.2.

Theorem 2.11. *Let* $\{f_i\}_{i=1}^M$ *be an unit norm tight frame of vectors for* $\mathcal{H}_N$ *and let* $\{I_j\}_{j=1}^L$ *be a partition of* $\{1,2,\cdots,M\}$. *Define* $W_j=\operatorname{span}\{f_i\}_{i\in I_j}$, *then the family* $\{W_j\}_{j=1}^L$ *is a fusion frame for* $\mathcal{H}_N$ *with fusion frame bounds* $\frac{AM}{N}$ *and* $\frac{BM}{N}$ *where*

$$A=\min_{1\leq j\leq L}\min_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}},\qquad B=\max_{1\leq j\leq L}\max_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}},$$

and $\{\lambda_{jk}\}_{k=1}^{\dim W_j}$ *is the family of eigenvalues of frame operator associated to* $\{f_i\}_{i\in I_j}$.

Proof. Let S_j be the frame operator associated to $\{f_i\}_{i\in I_j}$ and let $\{e_{jk}\}_{k=1}^N$ be the orthonormal basis for $\mathcal{H}_N$ of eigenvectors of S_j with eigenvalues $\{\lambda_{jk}\}_{k=1}^N$. Then $\lambda_{jk}=0$ for any $\dim W_j<k\leq N$ and $\{e_{jk}\}_{k=1}^{\dim W_j}$ is a orthonormal basis for W_j, which implies that $\langle S_j f,f\rangle=\sum_{k=1}^{\dim W_j}\lambda_{jk}|\langle f,e_k\rangle|^2$. Now for any $f\in\mathcal{H}_N$ we have

$$\begin{aligned}\min_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}\sum_{i\in I_j}|\langle f,f_i\rangle|^2&=\min_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}\langle S_j f,f\rangle\\&=\sum_{k=1}^{\dim W_j}\frac{\lambda_{jk}}{\max_{1\leq k\leq\dim W_j}\lambda_{jk}}|\langle f,e_{jk}\rangle|^2\\&\leq\|\pi_{W_j}\|^2\leq\sum_{k=1}^{\dim W_j}\frac{\lambda_{jk}}{\min_{1\leq k\leq\dim W_j}\lambda_{jk}}|\langle f,e_{jk}\rangle|^2\\&=\max_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}\langle S_j f,f\rangle\\&=\max_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}\sum_{i\in I_j}|\langle f,f_i\rangle|^2.\end{aligned}$$

This yields

$$\sum_{j=1}^L\sum_{i\in I_j}\min_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}|\langle f,f_i\rangle|^2\leq\sum_{j=1}^L\|\pi_{W_j}f\|^2\leq\sum_{j=1}^L\sum_{i\in I_j}\max_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}|\langle f,f_i\rangle|^2.$$

Put $A=\min_{1\leq j\leq L}\min_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}$, $B=\max_{1\leq j\leq L}\max_{1\leq k\leq\dim W_j}\frac{1}{\lambda_{jk}}$. Then

$$\frac{AM}{N}\|f\|^2\leq\sum_{j=1}^L\|\pi_{W_j}f\|^2\leq\frac{BM}{N}\|f\|^2.$$

∎

The next corollary generalizes Theorem 2.11 to the g-frames situation which the proof leave to interested readers.

Corollary 2.12. *Let* $\{\Lambda_i\}_{i=1}^M$ *be a tight g-frame for* $\mathcal{H}_N$ *with respect to* $\{W_i\}_{i=1}^M$ *and let* $\{I_j\}_{j=1}^L$ *be a partition of* $\{1, 2, \cdots, M\}$. *Define* $V_j = \text{span}\{\Lambda_i^*(W_i)\}_{i\in I_j}$, *then the family* $\{V_j\}_{j=1}^L$ *is a fusion frame for* $\mathcal{H}_N$ *with fusion frame bounds* $\frac{A\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N}$ *and* $\frac{B\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N}$ *where*

$$A = \min_{1\le j\le L}\min_{1\le k\le \dim W_j}\frac{1}{\lambda_{jk}}, \qquad B = \max_{1\le j\le L}\max_{1\le k\le \dim W_j}\frac{1}{\lambda_{jk}},$$

and $\{\lambda_{jk}\}_{k=1}^{\dim W_j}$ *is the family of eigenvalues of g-frame operator associated to* $\{\Lambda_i\}_{i\in I_j}$.

3. Excess of G-Frames

Our purpose of this section is to study the conditions which under removing some element from a g-frame, again we obtain another g-frame. The next theorem gives a erasure result of g-frames so that Theorem 4.3 obtained in [11] is a special case of it.

Theorem 3.1. *Let* $\Lambda = \{\Lambda_i\}_{i\in I}$ *be a g-frame for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$ *with g-frame bounds* C *and* D *and let* $J \subset I$. *Then* $\{\Lambda_i\}_{i\in I-J}$ *is a g-frame for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I-J}$ *with bounds* $\frac{C^2}{D}\big\|(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i)^{-1}\big\|^{-2}$ *and* D *if and only if the operator* $I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i$ *is bounded and invertible on* $\mathcal{H}$.

Proof. Since $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, for any $f \in \mathcal{H}$ we have

$$f = \sum_{i\in I} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f = \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f + \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f$$

Therefore, $I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i = \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i$. Moreover we have

$$\begin{aligned}
\big\|(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i)f\big\| &= \Big\|\sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f\Big\| = \sup_{\|g\|=1}\big|\langle \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f, g\rangle\big| \\
&= \sup_{\|g\|=1}\big|\sum_{i\in I-J}\langle \Lambda_i f, \Lambda_i S_\Lambda^{-1} g\rangle\big| \\
&\le \sup_{\|g\|=1}\sum_{i\in I-J}\|\Lambda_i f\|\|\Lambda_i S_\Lambda^{-1} g\| \\
&\le \sup_{\|g\|=1}\big(\sum_{i\in I-J}\|\Lambda_i f\|^2\big)^{\frac{1}{2}}\big(\sum_{i\in I-J}\|\Lambda_i S_\Lambda^{-1} g\|^2\big)^{\frac{1}{2}} \\
&\le \sup_{\|g\|=1}\sqrt{D}\|S_\Lambda^{-1} g\|\big(\sum_{i\in I-J}\|\Lambda_i f\|^2\big)^{\frac{1}{2}} \\
&\le \frac{\sqrt{D}}{C}\big(\sum_{i\in I-J}\|\Lambda_i f\|^2\big)^{\frac{1}{2}}.
\end{aligned}$$

Also, if $I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i$ is invertible on $\mathcal{H}$, then

$$\frac{C^2}{D}\Big\|\big(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i\big)^{-1}\Big\|^{-2}\|f\|^2 \leq \frac{C^2}{D}\Big\|\big(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i\big)f\Big\|^2$$
$$\leq \sum_{i\in I-J}\|\Lambda_i f\|^2 \leq \sum_{i\in I}\|\Lambda_i f\|^2 \leq D\|f\|^2.$$

From this the conclusion follows. To Prove the opposite direction, we first show that the operator $I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i$ is injective. Let $(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i)f = 0$. Then $\sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f = 0$ hence $\sum_{i\in I-J}\Lambda_i^*\Lambda_i f = 0$. It follows that

$$C\|f\|^2 \leq \sum_{i\in I-J}\|\Lambda_i f\|^2 = \sum_{i\in I-J}\langle\Lambda_i f, \Lambda_i f\rangle = \Big\langle \sum_{i\in I-J}\Lambda_i^*\Lambda_i f, f\Big\rangle = 0,$$

which implies that $f = 0$. Moreover, if

$$\big(I_{\mathcal{H}} - \sum_{i\in J}\Lambda_i^*\Lambda_i S_\Lambda^{-1}\big)f = \big(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i\big)^* f = 0,$$

then $\sum_{i\in I-J}\Lambda_i^*\Lambda_i S_\Lambda^{-1} f = 0$ and therefore $S_\Lambda^{-1} f = 0$, it follows that $f = 0$. This finishes the proof. ■

Corollary 3.2. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and let $J \subset I$. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0 = f_0$, then $\{\Lambda_i\}_{i\in I-J}$ is not a g-frame for $\mathcal{H}$.*

Proof. Assume that there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0 = f_0$. Then $\sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0 = 0$, hence $\sum_{i\in I-J}\Lambda_i^*\Lambda_i f_0 = 0$. It follows that

$$\sum_{i\in I-J}\|\Lambda_i f_0\|^2 = \sum_{i\in I-J}\langle\Lambda_i f_0, \Lambda_i f_0\rangle = \Big\langle \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0, f_0\Big\rangle = 0$$

Therefore $\{\Lambda_i\}_{i\in I-J}$ is not a g-frame. ■

Corollary 3.3. *Let $\{\Lambda_i\}_{i\in I}$ be a A-tight g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and let $J \subset I$. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i\in J}\Lambda_i^*\Lambda_i f_0 = Af_0$, then $\{\Lambda_i\}_{i\in I-J}$ is not a g-frame for $\mathcal{H}$.*

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