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Optimal Approximate Solution for $\mathcal{S}$-Weakly Contraction and Generalized Suzuki-Contraction by Using P_p-Property and Weak-P-Property in Complete Metric Spaces

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Abstract In this article, we obtained the best proximity point theorems for $\mathcal{S}$-weakly contraction and generalized Geraghty-Suzuki contractions in the setting of a complete metric spaces by using P_p-property and weak P-property. Also, established the conditions for the existence of uniqueness of optimal approximate solutions. We also gave simple examples to show the validity of our results.

MSC: 55M20

Keywords: best proximity point; P-best proximity point; P_p-property; w_s-distance; generalized α-Geraghty Suzuki contraction; $\mathcal{S}$-weakly contraction

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1. Introduction

The Banach contraction principle [1], which is a useful tool in the study of many branches of mathematics and mathematical sciences, is one of the earlier and fundamental result in fixed point theory. Because of its importance in nonlinear analysis, a number of mathematicians have improved, generalized and extended this basic result either by defining a new contractive mappings in the context of a complete metric space or by investigating the existing contractive mappings in various abstract spaces; see, e.g., [2–8] and references therein. When a mapping from a metric space into itself has no fixed points, it could be interesting to study the existence and uniqueness of some points that minimize the distance between the origin and its corresponding image. These points are known as best proximity points and were introduced by [9] and modified by Sadiq Basha

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in [10]. Best proximity point theorems for several types of non-self mappings have been derived in [10–18].

Recently, Geraghty [3] obtained a generalization of the Banach contraction principle in the setting of complete metric spaces by considering an auxiliary function. Later, Amini-Harandi and Emami [19] characterized the result of Geraghty in the context of a partially ordered complete metric space. This result is of particular interest since many real world problems can be identified in a partially ordered complete metric space. Cabellero et al. [20] discussed the existence of a best proximity point of Geraghty contraction. After the introduction of new notion about Generalized Geraghty-Suzuki contraction the existence and uniqueness of best proximity point has also been proved by using different conditions, see references [1]-[31]. In this paper, we obtained the best proximity point theorems and fixed point theorems for generalized Geraghty-Suzuki contractions in the setting of a complete metric space by replacing the P-property of [25] with another suitable property which is weaker than P-property. Also, motivated by [32] results, we generalized some results and contractions. We present example to prove the validity of our main result. Our results extended and unify many existing results in the literature.

2. Preliminaries

In this section, we collect some notions and notations which will be used throughout the rest of this work.

Definition 2.1 ([23])**.** Let X be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } d(a,b) = d(A,B)\}, \\ B_0 &= \{b \in B : \text{there exists some } a \in A \text{ such that } d(a,b) = d(A,B)\}. \end{aligned}$$

In [26], the authors present sufficient conditions which determine when the sets A_0 and B_0 are nonempty. We denote by $\mathcal{F}$ the class of all functions $\beta : [0,\infty) \to [0,1)$ satisfying $\beta(t_n) \to 1$, implies $t_n \to 0$ as $n \to \infty$.

Definition 2.2 ([3])**.** Let (X,d) be a metric space. A map $f : X \to X$ is called *Geraghty contraction* if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$d(fx, fy) \leq \beta(d(x,y))d(x,y).$$

Theorem 2.3 ([3])**.** *Let (X,d) be a complete metric space. Mapping $f : X \to X$ is Geraghty contraction. Then f has a fixed point $x \in X$, and $\{f^n x_1\}$ converges to x.*

Cho et al. [27] generalized the concept of Geraghty contraction to α-Geraghty contraction and prove the fixed point theorem for such contraction.

Definition 2.4 ([27])**.** Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ a function. A map $f : X \to X$ is called *α-Geraghty contraction* if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x,y)d(fx,fy) \leq \beta(d(x,y))d(x,y).$$

Definition 2.5 ([26])**.** Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if for any $x_1, x_2, x_3, x_4 \in A_0$,

$$\left.\begin{array}{ll} d(x_1, fx_3) & = d(A,B) \\ d(x_2, fx_4) & = d(A,B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(fx_3, fx_4).$$

Definition 2.6 ([21])**.** Let (A,B) be a part of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$

$$\left.\begin{array}{ll} d(x_1, y_1) & = d(A,B) \\ d(x_2, y_2) & = d(A,B) \end{array}\right\} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Also [26] showed that any pair of nonempty closed convex subset of real Hilbert space satisfies the P-property. Also one can see that the pair (A, A) has also P-property.

Theorem 2.7 ([20])**.** *Let A, B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty, and $\alpha : A \times A \to \mathbb{R}$ a function. Define a map $f : A \to B$ satisfying the following conditions:*

(1) f is continuous Geraghty contraction with $f(A_0) \subseteq B_0$;
(2) the pair (A, B) has the weak P-property.

Then there exists a unique x^* in A such that $d(x^*, fx^*) = d(A, B)$.

Definition 2.8 ([25])**.** Let (X, d) be a metric space. A mapping $f : A \to B$ is called *generalized Geraghty-Suzuki contraction* (GS-contraction) if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in A$,

$$\frac{1}{2} d^*(x, fx) \leq d(x, y) \Rightarrow d(fx, fy) \leq \beta(M(x, y))[M(x, y) - d(A, B)], \tag{2.1}$$

where $A, B \subseteq X$, $d^*(x, y) = d(x, y) - d(A, B)$ and $M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy)\}$.

Definition 2.9 ([17])**.** Given a non-self mapping $f : A \to B$, then an element x^* is called *best proximity point* of the mappings if this condition satisfied:

$$d(x^*, fx^*) = d(A, B),$$

where $BPP(f)$ denotes the set of best proximity points of f.

Definition 2.10. Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called *w-distance* on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
(2) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semi continuous;
(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ $d(x, y) \leq \epsilon$.

Definition 2.11 ([32])**.** Let (X, d) be a metric space. A set valued mapping $T : X \to X$ is called *weakly contractive* if there exists a w-distance p on X and $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $p(y_1, y_2) \leq rp(x_1, x_2)$.

In this paper, motivated by [21, 25, 32] and Suzuki [30] in which he proved fixed point theorem that is generalization of the Banach contraction principle and characterized the metric completeness, by using such type of Suzuki mapping, we ensures the existence and uniqueness of the best proximity points and fixed points.

3. Best Proximity Point Theorems for S-Weakly Contraction

In this part of research paper we introduced some new notions and contractions by using [32] results. Further more, find out best proximity point for such contractions.

Definition 3.1. Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called *w_s-distance* on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
(2) $p(x, y) \geq 0$, for any $x, y \in X$;
(3) if $\{x_m\}$ and $\{y_m\}$ be any sequences in X such that $x_n \to x$, $y_n \to y$ as $n \to \infty$, then $p(x_n, y_n) \to p(x, y)$ as $x \to \infty$;
(4) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ $d(x, y) \leq \epsilon$.

Example 3.2. Let $X = \mathbb{R}$ endowed with Euclidean metric $d = |.|$ and s a positive constant. Define $p : X \times X \to \mathbb{R}$ by $p(x, y) = y^s$, for all $x, y \in X$.

Proof. To prove triangular inequality, let us take $x, y, z \in X$. then $p(x, z) = z^s \leq y^s + z^s = p(x, y) + p(y, z)$. So, 1st axiom of Definition 3.1 holds and 2nd, 3rd are proved easily. For 4th axiom of Definition 3.1. Let us take $\epsilon > 0$ and put $\delta = \epsilon^s$. Suppose that $p(x, y) \leq \delta$ and $p(z, y) \leq \delta$. It follows that $d(x, y) = |x - y| \leq \max\{x, y\} \leq \{\delta^{\frac{1}{s}}, \delta^{\frac{1}{s}}\} = \epsilon$. ∎

Definition 3.3. Let X be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} p(A, B) &= \inf\{p(a, b) : a \in A, b \in B\}, \\ A_{0,p} &= \{a \in A : \text{there exists some } b \in B \text{ such that } p(a, b) = p(A, B)\}, \\ B_{0,p} &= \{b \in B : \text{there exists some } a \in A \text{ such that } p(a, b) = p(A, B)\}. \end{aligned}$$

Example 3.4. Let $X = \mathbb{N}$ and $A, B \subset X$ where

$$A = \{(1, 0), (4, 5), (5, 4)\},$$
$$B = \{(2, 0), (0, 4), (4, 0)\}.$$

Then, $p(A, B) = 1$ with $A_{0,p} = \{(1, 0)\}$ and $B_{0,p} = \{(2, 0)\}$.

Definition 3.5. Let (X, d) be a metric space, $A, B \subseteq X$ and $A_{0,p} \neq \phi$. A set valued mapping $T : A \to B$ with $T(A_{0,p}) \subseteq B_{0,p}$ is called *S-weakly contractive* or *P_p-contractive* if there exists a w_s-distance p on A and $r \in [0, 1)$ such that for any $x_1, x_2 \in A$ and $y_1 \in Tx_1$ in B there is $y_2 \in Tx_2$ in B with $p(y_1, y_2) \leq rp(x_1, x_2)$.

Definition 3.6. Let (A, B) be a part of nonempty subsets of a metric space (X, d) with $A_{0,p} \neq \emptyset$. Then the pair (A, B) is said to have P_p-property if and only if for any $x_1, x_2 \in A_{0,p}$ and $y_1, y_2 \in B_{0,p}$

$$\left.\begin{aligned} p(x_1, y_1) &= p(A, B) \\ p(x_2, y_2) &= p(A, B) \end{aligned}\right\} \Rightarrow p(x_1, x_2) = p(y_1, y_2).$$

Definition 3.7. Given a non-self mapping $f : A \to B$, then an element x^* is called *p-best proximity point* of the mappings if this condition satisfied:

$$p(x^*, fx^*) = p(A, B),$$

Now, we are defining next example to show the existence of p-best proximity point with the help of P_p-property.

Example 3.8. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$. Assume that $A, B \subset X$ where $A = \{0, \frac{1}{3}, \frac{1}{5}, ...\}$. Then, $p(A, B) = 0$ and $A_{0,p} = \{0\}$ and $B_{0,p} = \{0\}$. Define $T : A \to B$ as $Tx = \frac{x}{1+x}$, for all x and $d, p : X \times X \to X$ by $d, p(x, y) = 0$ for $x = y$ and $d, p(x, y) = \max\{x, y\}$ if $x \neq y$. Here $p(A, B) = 0$ and $p(x, Tx) = 0 = p(A, B)$. So, $p(0, T0) = p(A, B) = 0$, thus it has a unique p-best proximity point and that is 0, also P_p-property satisfies.

Theorem 3.9. *Let (X, d) be a complete metric space, A and B nonempty closed subsets of X and $T : A \to B$ a continuous set valued S-weakly contractive or p_p-contractive mapping with (A, B) satisfying the P_p-property where p is the w_s-distance. Then T has unique p-best proximity point.*

Proof. Since T is S-weakly-contractive mapping, so $A_{0,p}$ is nonempty and $T(A_{0,p}) \subseteq B_{0,p}$, we take $x_0 \in A_{0,p}$, there exists $x_1 \in A_{0,p}$ such that

$$p(x_1, Tx_0) = p(A, B). \tag{3.1}$$

Again, since $T(A_{0,p}) \subseteq B_{0,p}$, there exists $x_2 \in A_{0,p}$ such that

$$p(x_2, Tx_1) = p(A, B). \tag{3.2}$$

Repeating this process, we get a sequence $\{x_n\}$ in $A_{0,p}$ satisfying

$$p(x_{n+1}, Tx_n) = p(A, B),$$

for any $n \in \mathbb{N}$.
Since (A, B) has P_p-property, we have that

$$p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n),$$

for any $n \in \mathbb{N}$.
Note that T is S-weakly-contractive mapping and (A, B) has P_p-property. So for any $n \in \mathbb{N}$, we have that

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq rp(x_{n-1}, x_n) \\ &< p(x_{n-1}, x_n), \end{aligned}$$

where $0 \leq r < 1$. This means

$$p(x_n, x_{n+1}) < p(x_{n-1}, x_n).$$

So, $\{p(x_n, x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers.
Suppose that there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0}, x_{n_0+1}) = 0$. In this case,

$$0 = p(x_{n_0}, x_{n_0+1}) = p(Tx_{n_0-1}, Tx_{n_0}),$$

and consequently

$$Tx_{n_0-1} = Tx_{n_0}.$$

Therefore,

$$p(A, B) = p(x_{n_0}, Tx_{n_0-1}) = p(x_{n_0}, Tx_{n_0}).$$

Note that $x_{n_0} \in A_0$, $Tx_{n_0-1} \in B_0$, and $x_{n_0} = Tx_{n_0-1}$ for any $n_0 \in \mathbb{N}$. So, $A \bigcap B$ is nonempty. Then $p(A, B) = 0$. Thus in this case, there exists unique p-best proximity point, i.e., there exists unique x^* in A such that $p(x^*, Tx^*) = p(A, B)$.

In the contrary case, suppose that $p(Tx_{n_0}, Tx_{n_0-1}) > 0$. This implies that $p(x_n, x_{n+1}) > 0$, for any $n \in N$. Since $\{p(x_n, x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers, there exists $k \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = k.$$

We have to show that k=0. Let $k \neq 0$ and $k > 0$. From

$$p(x, y) = \lim_{n\to\infty} p(x_n, x_{n+1})$$

and

$$p(x, y) \leq \liminf_{n\to\infty} p(x, x_{n+1}) \leq 0,$$

we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0,$$

for any $n \in \mathbb{N}$. This yields that

$$\lim_{n\to\infty} p(x_{n-1}, x_n) = 0.$$

Hence $k = 0$ and this contradicts our assumption that $k > 0$. Therefore,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Since $p(x_{n+1}, Tx_n) = p(A, B)$, for any $n \in \mathbb{N}$ and for fixed $p, q \in \mathbb{N}$, we have

$$p(x_p, Tx_{p-1}) = p(x_q, Tx_{q-1}) = p(A, B).$$

Since (A, B) satisfies weak P_p-property, so

$$p(x_p, x_q) \leq p(Tx_{p-1}, Tx_{q-1}).$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence.
On contrary, we suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that for all $k > 0$, there exists $m(k) > n(k) > k$ with (the smallest number satisfying the condition below)

$$p(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } p(x_{m(k)-1}, x_{n(k)}) < \epsilon.$$

Then, we have

$$\begin{aligned}
\epsilon &\leq p(x_{m(k)}, x_{n(k)}) \\
&\leq p(x_{m(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{n(k)}) \\
&< p(x_{m(k}, x_{m(k)-1}) + \epsilon.
\end{aligned}$$

This implies that

$$\epsilon \leq p(x_{m(k)}, x_{n(k)}) < p(x_{m(k)}, x_{m(k)-1}) + \epsilon. \tag{3.3}$$

Let $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{3.4}$$

Now by using Triangular inequality, we have

$$p(x_{m(k)}, x_{n(k)}) \leq p(x_{m(k)}, x_{m(k)-1}) \; + \; p(x_{m(k)-1}, x_{n(k)-1}) \; + \; p(x_{n(k)-1}, n_k).$$

Take limit on both sides, we get

$$\begin{aligned}\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) &\geq \lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) - \lim_{k\to\infty} p(x_{m(k)}, x_{m(k)-1}) \\ &\quad - \lim_{k\to\infty} p(x_{n(k)-1}, x_{n(k)}).\end{aligned}$$

We obtain

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \epsilon.$$

We have

$$\begin{aligned}p(x_{m(k)}, x_{n(k)}) &= p(Tx_{m(k)-1}, Tx_{n(k)-1}) \\ &\leq rp(x_{n(k)-1}, x_{m(k)-1}).\end{aligned}$$

Thus,

$$\lim_{n\to\infty} p(x_{n(k)-1}, x_{m(k)-1}) = 0.$$

Hence $\epsilon = 0$, which contradicts our supposition that $\epsilon > 0$. So we conclude that $\{x_n\}$ is a Cauchy sequence in A. Since $\{x_n\} \subseteq A$ and A is closed subset of a complete metric space (X, d). There is $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. Since T is continuous, we have

$$Tx_n \to Tx^*.$$

So, $p(x_{n+1}, Tx_n) \to p(x^*, Tx^*)$. Taking into account that $\{p(x_{n+1}, Tx_n)\}$ is a constant sequence with a value $p(A, B)$, we deduce

$$p(x^*, Tx^*) = p(A, B),$$

i.e., x^* is best proximity point of T.
For uniqueness of p-best proximity point.
Since p is a w-distance, also T is P_p-contractive, then $p(Tx, Ty) \leq rp(x, y)$, for every $x, y \in A$ of X. We suppose that given mapping T has two distinct p-best proximity points $x_0, x_1 \in A$, that is $p(x_0, Tx_0) = p(x_1, Tx_1) = p(A, B)$, Since T has P_p-property, then

$$\begin{aligned}p(x_0, x_1) &= p(Tx_0, Tx_1) \\ &\leq rp(x_0, x_1),\end{aligned}$$

which shows

$$p(x_0, y_0) \leq rp(x_0, y_0).$$

It contradicts towards our assumption and so we get $x_0 = y_0$.
Therefore, T has unique p-best proximity point. ∎

4. Some Results about Generalized α-Geraghty Suzuki Contraction

In this section, we show the existence and uniqueness of best proximity point in our main result for generalized α-Geraghty-Suzuki contraction by using weak P-property in the field of a complete metric space.

Definition 4.1. Let (X,d) be a metric space. A mapping $f : A \to B$ is called *generalized α-Geraghty Suzuki contraction ($\alpha - GS$-contraction)* if there exists $\beta \in \mathcal{F}$ and a function $\alpha : A \times A \to \mathbb{R}^+$ such that $\alpha(x,y) \geq 1$, for all $x, y \in A$,

$$\frac{1}{2}d^*(x,fx) \leq d(x,y) \Rightarrow \alpha(x,y)d(fx,fy) \leq \beta(M(x,y))[M(x,y) - d(A,B)], \quad (4.1)$$

where $A, B \subseteq X$, $d^*(x,y) = d(x,y) - d(A,B)$ and $M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy)\}$.

Now, we are in a position to prove our main result.

Theorem 4.2. *Let (A,B) be the pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Define a mapping $f : A \to B$ satisfying the following conditions:*

(1) f is generalized α-Geraghty Suzuki contraction with $f(A_0) \subseteq B_0$;
(2) for each $x, y \in A_0$ satisfying $d(x, f(y)) = d(A,B)$ implies $\alpha(x,y) \geq 1$;
(3) the pair (A,B) has the weak P-property.

Then there exists x^* in A such that $d(x^*, fx^*) = d(A,B)$.

Proof. Since A_0 is nonempty, we take $x_0 \in A_0$. Since $f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$d(x_1, fx_0) = d(A,B) \text{ with } \alpha(x_0, x_1) \geq 1. \quad (4.2)$$

Again, since $f(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, fx_1) = d(A,B) \text{ implies } \alpha(x_1, x_2) \geq 1. \quad (4.3)$$

Repeating this process, we get a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, fx_n) = d(A,B) \text{ for any } n \in \mathbb{N} \cup \{0\}, \quad (4.4)$$

with $\alpha(x_n, x_{n+1}) \geq 1$, for any $n \in \mathbb{N}$.
Since (A,B) has the weak P-property, we have that

$$d(x_n, x_{n+1}) \leq d(fx_{n-1}, fx_n), \text{ for any } n \in \mathbb{N}. \quad (4.5)$$

Now by (4.4), we get

$$d(x_{n-1}, fx_{n-1}) \leq d(x_{n-1}, x_n) + d(x_n, fx_{n-1}) = d(x_{n-1}, x_n) + d(A,B). \quad (4.6)$$

Now from (4.4) and (4.6), we obtain

$$\begin{aligned} d(x_n, fx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, fx_n) \\ &= d(A,B) + d(x_n, x_{n+1}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + d(A,B). \end{aligned} \quad (4.7)$$

Clearly, if there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then we have nothing to prove. The conclusion is immediate. So

$$0 = d(x_{n_0}, x_{n_0+1}) \leq d(fx_{n_0-1}, fx_{n_0}),$$

this implies $0 = d(fx_{n_0-1}, fx_{n_0})$ and consequently, $fx_{n_0-1} = fx_{n_0}$. Thus, we conclude that

$$d(A,B) = d(x_{n_0}, fx_{n_0-1}) = d(x_{n_0}, fx_{n_0}).$$

For the rest of proof, we suppose that $0 < d(fx_{n-1}, fx_n)$. It shows that $d(x_n, x_{n+1}) > 0$, for any $n \in \mathbb{N} \cup \{0\}$. Now from (4.1), we deduce that

$$\frac{1}{2} d^*(x_{n-1}, fx_{n-1}) \le d^*(x_{n-1}, fx_{n-1}) \le d(x_n, x_{n-1})$$

and by (4.5), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(fx_{n-1}, fx_n) \\ &\le \alpha(x_n, x_{n-1}) d(fx_{n-1}, fx_n) \\ &\le \beta(M(x_{n-1}, x_n))[M(x_{n-1}, x_n) - d(A, B)] \\ &< M(x_{n-1}, x_n) - d(A, B). \end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we obtain

$$d(x_n, x_{n+1}) < M(x_{n-1}, x_n) - d(A, B) \le \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Now, if $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

which is a contradiction. Thus

$$M(x_{n-1}, x_n) \le \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + d(A, B) = d(x_{n-1}, x_n) + d(A, B).$$

Therefore, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(fx_{n-1}, fx_n) \\ &\le \alpha(x_n, x_{n-1}) d(fx_{n-1}, fx_n) \\ &\le \beta(M(x_{n-1}, x_n)) d(x_{n-1}, x_n) \\ &< d(x_{n-1}, x_n), \end{aligned} \tag{4.9}$$

for all $n \in \mathbb{N}$. Consequently, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence and bounded below and so

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = L.$$

Suppose $L > 0$. From (4.9), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \le \beta(M(x_n, x_{n+1})) \le 1,$$

for any $n \ge 0$, which implies that

$$\lim_{n \to \infty} \beta(M(x_n, x_{n+1})) = 1.$$

On the other hand, since $\beta \in \mathcal{F}$, we get

$$\lim_{n \to \infty} M(x_n, x_{n+1}) = 0.$$

That is,

$$L = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Since $d(x_n, fx_{n-1}) = d(A, B)$ holds, for all $n \in \mathbb{N}$, and the pair (A, B) satisfies the weak P-property, then for all $m, n \in \mathbb{N}$, we can write $d(x_m, x_n) \le d(fx_{m-1}, fx_{n-1})$.
Using the fact that

$$d(x_l, fx_l) \le d(x_l, x_{l+1}) + d(x_{l+1}, fx_l) = d(x_l, x_{l+1}) + d(A, B),$$

for all $l \in \mathbb{N}$, we deduce easily

$$\begin{aligned} M(x_m, x_n) &\leq \max\{d(x_m, x_n), d(x_m, fx_m), d(x_n, fx_n)\} \\ &\leq \max\{d(x_m, x_n), d(x_m, x_{m+1}), d(x_n, x_{n+1})\} + d(A, B). \end{aligned}$$

Since, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, then we have

$$\lim_{m,n\to\infty} M(x_m, x_n) \leq \lim_{m,n\to\infty} d(x_m, x_n) + d(A, B). \tag{4.10}$$

We shall show that $\{x_n\}$ is a Cauchy sequence. If not, then we get

$$\lim_{m,n\to\infty} d(x_m, x_n) > 0.$$

Thus, without loss of generality, we can assume that

$$\epsilon = \lim_{m,n\to\infty} d(x_m, x_n) > 0. \tag{4.11}$$

By using the triangular inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m). \tag{4.12}$$

Now, since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, then

$$\begin{aligned} d(A, B) &\leq \lim_{m\to\infty} d(x_m, fx_m) \\ &\leq \lim_{m\to\infty} [d(x_m, x_{m+1}) + d(x_{m+1}, fx_m) \\ &= \lim_{m\to\infty} [d(x_m, x_{m+1}) + d(A, B)] = d(A, B), \end{aligned}$$

which implies $\lim_{m\to\infty} d(x_m, fx_m) = d(A, B)$, that is

$$\lim_{m\to\infty} \frac{1}{2} d^*(x_m, fx_m) = \lim_{m\to\infty} \frac{1}{2}[d(x_m, fx_m) - d(A, B)] = 0.$$

On the other hand, from (4.1), it follows that there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$,we have

$$\frac{1}{2} d^*(x_m, fx_m) \leq d(x_n, x_m).$$

Now from (4.1) and (4.8), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(fx_n, fx_m) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + \alpha(x_n, x_m) d(fx_n, fx_m) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + \beta(M(x_n, x_m))[M(x_n, x_m) - d(A, B)] + d(x_{m+1}, x_m). \end{aligned} \tag{4.13}$$

Then by taking limits as $n \to \infty$ and from $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we have

$$\begin{aligned} \lim_{m,n\to\infty} d(x_m, x_n) &\leq \lim_{m,n\to\infty} \beta(M(x_n, x_m)) \lim_{m,n\to\infty} [M(x_n, x_m) - d(A, B)] \\ &\leq \lim_{m,n\to\infty} \beta(M(x_n, x_m)) \lim_{m,n\to\infty} d(x_m, x_n). \end{aligned}$$

So, we get

$$1 \leq \lim_{m,n\to\infty} \beta(M(x_n, x_m)),$$

that is, $\lim_{m,n\to\infty} \beta(M(x_n, x_m)) = 1$. Therefore, $\lim_{m,n\to\infty} M(x_n, x_m) = 0$ and consequently,

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subset A$ and A is closed subset of complete metric space (X, d), we can find $x^* \in A$ such that $x_n \to x^*$, as

$n \to \infty$.
We shall now show that $d(x^*, fx^*) = d(A, B)$. Suppose on the contrary that $d(x^*, fx^*) > d(A, B)$. At first, we have

$$\begin{aligned} d(x^*, fx^*) &\leq d(x^*, fx_n) + d(fx_n, fx^*) \\ &\leq d(x^*, x_{n+1}) + d(x_{n+1}, fx_n) + d(fx_n, fx^*) \\ &= d(x^*, x_{n+1}) + d(A, B) + d(fx_n, fx^*), \end{aligned}$$

and taking limit $n \to \infty$, we get

$$d(x^*, fx^*) - d(A, B) \leq \lim_{n\to\infty} d(fx_n, fx^*). \tag{4.14}$$

Also, we have

$$\begin{aligned} d(x_n, fx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, fx_n) \\ &= d(x_n, x_{n+1}) + d(A, B). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(x_n, fx_n) \leq d(A, B),$$

that is,

$$\lim_{n\to\infty} d(x_n, fx_n) = d(A, B).$$

Then, we get

$$\begin{aligned} \lim_{n\to\infty} M(x_n, x^*) &= \max\{\lim_{n\to\infty} d(x_n, x^*), \lim_{n\to\infty} d(x_n, fx_n), \lim_{n\to\infty} d(x^*, fx^*)\} \\ &= d(x^*, fx^*) \end{aligned}$$

and hence

$$M(x_n, x^*) - d(A, B) = d(x^*, fx^*) - d(A, B). \tag{4.15}$$

Next, we have

$$\begin{aligned} d^*(x_n, fx_n) &= d(x_n, fx_n) - d(A, B) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, fx_n) - d(A, B) \\ &= d(x_n, x_{n+1}) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} d^*(x_{n+1}, fx_{n+1}) &= d(x_{n+1}, fx_{n+1}) - d(A, B) \\ &\leq d(x_{n+2}, fx_{n+1}) + d(x_{n+1}, x_{n+2}) - d(A, B) \\ &= d(x_{n+1}, x_{n+2}) \\ &< d(x_n, x_{n+1}), \end{aligned} \tag{4.17}$$

and so by adding (4.16) and (4.17), we get

$$\frac{1}{2}[d^*(x_n, fx_n) + d^*(x_{n+1}, fx_{n+1})] \leq d(x_n, x_{n+1}). \tag{4.18}$$

Now we suppose that following inequalities hold,

$$\frac{1}{2} d^*(x_n, fx_n) > d(x_n, x^*)$$

and

$$\frac{1}{2} d^*(x_{n+1}, fx_{n+1}) > d(x_{n+1}, x^*),$$

for some $n \in \mathbb{N} \cup \{0\}$. Hence, we can write

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x^*) + d(x_{n+1}, x^*) \\ &< \frac{1}{2}[d^*(x_n, fx_n) + d^*(x_{n+1}, fx_{n+1}) \\ &\leq d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Then for any $n \in \mathbb{N} \cup \{0\}$, either

$$\frac{1}{2}d^*(x_n, fx_n) > d(x_n, x^*)$$

or

$$\frac{1}{2}d^*(x_{n+1}, fx_{n+1}) > d(x_{n+1}, x^*)$$

holds. Therefore, we deduce that

$$\begin{aligned} d(x^*, fx^*) - d(A,B) &\leq \lim_{n\to\infty} d(fx_n, fx^*) \\ &\leq \lim_{n\to\infty} \alpha(x_n, x^*) d(fx_n, fx^*) \\ &\leq \lim_{n\to\infty} \beta(M(x_n, x^*)) \lim_{n\to\infty} [M(x_n, x^*) - d(A,B)] \\ &= \lim_{n\to\infty} \beta(M(x_n, x^*))[d(x^*, fx^*) - d(A,B)]. \end{aligned} \tag{4.19}$$

Then, we get

$$1 \leq \lim_{n\to\infty} \beta(M(x_n, x^*)),$$

that is,

$$\lim_{n\to\infty} \beta(M(x_n, x^*)) = 1,$$

which implies

$$\lim_{n\to\infty} M(x_n, x^*) = d(x_n, fx^*) = 0$$

and so $d(x^*, fx^*) = 0 > d(A,B)$, a contradiction. Therefore, $d(x^*, fx^*) \leq d(A,B)$, that is, $d(x^*, fx^*) = d(A,B)$. This means that x^* is a best proximity point of f and so existence of a best proximity point has been proved.
We shall show the uniqueness of the best proximity point of f. Suppose that x^* and y^* are two distinct best proximity points of f, that is, $x^* \neq y^*$. This implies that

$$d(x^*, fx^*) = d(A,B) = d(y^*, fy^*).$$

Using the weak P-property, we have

$$d(x^*, y^*) \leq d(fx^*, fy^*)$$

and so

$$\begin{aligned} M(x^*, y^*) &= \max\{d(x^*, y^*), d(x^*, fx^*), d(y^*, fy^*)\} \\ &= \max\{d(x^*, y^*), d(A,B)\}. \end{aligned}$$

Also, we have $\frac{1}{2}d^*(x^*, fx^*) = \frac{1}{2}[d(x^*, fx^*) - d(A,B)] = 0 \leq d(x^*, y^*))$.
Since $M(x^*, y^*) - d(A,B) \leq d(x^*, y^*)$, we have

$$\begin{aligned} d(x^*, y^*) &\leq d(fx^*, fy^*) \\ &\leq \alpha(x^*, y^*) d(fx^*, fy^*) \\ &\leq \beta(M(x^*, y^*))[M(x^*, y^*) - d(A,B)] \\ &< d(x^*, y^*) \end{aligned}$$

which is a contradiction. Thus there exist best proximity point, that is, $x^* = y^*$. This completes the proof. ■

To show the independence of our main result, we give the following example:

Example 4.3. Consider $X = \mathbb{R}^2$, with the usual metric d. Define the sets $A = \{(1,1),(0,2),(2,0)\}$ and $B = \{(0,1),(1,0)\}$, so that $d(A,B) = 1$. Let $A_0 = \{(2,0),(0,2)\}$ and $B_0 = \{(1,0),(0,1)\}$. Then the pair (A,B) has the weak P-property.
Let $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ defined as:

$$\alpha((x_1,y_1),(x_2,y_2)) = \begin{cases} 1 & \text{if } 0 \le x_1, x_2, y_1, y_2 \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Also define $f : A \to B$ by:

$$f(x_1,x_2) = \begin{cases} (x_1, x_2 - 1) & \text{if } x_1 \le x_2, \\ (x_1 - 1, x_2) & \text{if } x_1 > x_2. \end{cases}$$

Notice that $f(A_0) \subseteq B_0$. Now, consider the function $\beta : [0,\infty) \to [0,1)$ given by

$$\beta(t) = \frac{t-1}{t}; 0 < t \le 1,$$

otherwise 0. Note that $\beta \in \mathcal{F}$. Assume that $\frac{1}{2} d^*(x, fx) \le d(x,y)$, for some $x, y \in A$. Then,

$$x = (2,0), y = (0,2)$$

or

$$y = (2,0), x = (0,2).$$

Since $d(fy, fx) = d(fx, fy)$ and $M(x,y) = M(y,x)$, for all $x, y \in A$, hence without loss of generality, we can assume that $(x,y) = ((0,2),(2,0))$.
Now, we distinguish as follows:

$$\begin{aligned} \alpha((0,2),(2,0)) d(f(0,2), f(2,0)) &= d((0,1),(1,0)) = 2 \\ &\le \frac{4-1}{4}.(4-1) \\ &= \beta(M((0,2),(2,0)))[M((0,2),(2,0)) - 1]. \end{aligned}$$

Consequently, we have

$$\frac{1}{2} d^*(x, fx) \le d(x,y) \;\Rightarrow\; \alpha(x,y) d(fx, fy) \le \beta(M(x,y))[M(x,y) - d(A,B)]$$

and hence all conditions of Theorem 4.2 hold and f has the best proximity point. Here, $x = (2,0)$ and $(0,2)$ are best proximity points of f.

Now for uniqueness, next theorem established as follows:

Theorem 4.4. *Under the same hypothesis of Theorem 4.2, suppose that f is α-regular. Then for all best proximity points x and y of f in A_0, we get that $x = y$; In particular, f has unique best proximity point.*

Proof. Following Theorem 4.2, we obtain $\lim_{m,n\to\infty} M(x_n, x_m) \leq \lim_{m,n\to\infty} d(x_n, x_m) + d(A, B)$. Let $x, y \in A_0$ be two best proximity

$$d(x, y) \leq d(fx, fy).$$

We consider two cases:
Case-I: If $\alpha(x, y) \geq 1$,

$$d(x, fx) = d(A, B) = d(y, fy).$$

By using weak P-property, we have

$$d(x, y) \leq d(fx, fy).$$

Using the fact that f is generalized α-Geraghty Suzuki contraction, we have

$$\begin{aligned} d(x, y) \leq d(fx, fy) &\leq \alpha(x, y)d(fx, fy) \\ &\leq \beta(d(x, y))M[d(x, y) - d(A, B)] \\ &< d(x, y). \end{aligned}$$

Thus, $d(x, y) < d(x, y)$, which is contradiction. So $x = y$.
Case-II: If $\alpha(x, y) < 1$, then by the α-regularity of f, there exists $z_0 \in A_0$ such that $\alpha(x, z_0) \geq 1$ and $\alpha(y, z_0) \geq 1$. Based on z_0, we define a sequence $\{z_n\}$ and suppose that z_n converges to x and y, which proves the uniqueness. First, we shall prove that $\{z_n\}$ converges to x.

Indeed, $fz_0 \in fA_0 \subseteq B_0$ implies that $z_1 \in A_0$ such that $d(z_1, fz_0) = d(A, B)$. Follow the similar arguments, there exists a sequence $\{z_n\} \subseteq A_0$ such that $d(z_{n+1}, fz_n) = d(A, B)$, for all $n \geq 0$. In particular, $z_{n+1} \in A_0$ and $fz_n \in B_0$. We claim that

$$\alpha(x, z_n) \geq 1, \tag{4.20}$$

for all $n \geq 0$. If $n = 0$, $\alpha(x, z_0) \geq 1$ by the choice of z_0. Suppose that $\alpha(x, z_n) \geq 1$, for some $n \geq 0$. As triangular α-admissibility of f, so we have for $x, z_n, z_{n+1} \in A_0$, $\alpha(x, z_n) \geq 1$, $\alpha(z_n, z_{n+1}) \geq 1$ implies $\alpha(x, z_{n+1}) \geq 1$. Hence (4.20) holds for all $n \geq 0$. We have by weak P-property, $x, z_n, z_{n+1} \in A_0$, $d(x, fx) = d(A, B), d(z_{n+1}, fz_n) = d(A, B)$ imply that $d(x, z_{n+1}) \leq d(fx, fz_n)$. From Theorem 4.2, we have $M(x_n, x_{n-1}) \leq d(x_{n-1}, x_n) + d(A, B)$. So for all $n \geq 0$, we have

$$\begin{aligned} d(x, z_{n+1}) &\leq d(fx, fz_n) \\ &\leq \alpha(x, z_n)d(fx, fz_n) \\ &\leq \beta(d(x, z_n))M[d(x, z_n) - d(A, B)] \\ &\leq \beta(d(x, z_n))d(x, z_n) \\ &< d(x, z_n), \end{aligned}$$

which shows that $\{d(x, z_{n+1})\}$ is a decreasing sequence of nonnegative real numbers, and there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x, z_{n+1}) = r$. Assume $r > 0$, then we have

$$0 < \frac{d(x, z_{n+1})}{d(x, z_n)} \leq \beta(d(x, z_n)) < 1,$$

for any $n \in \mathbb{N}$.
The last inequality implies that $\lim_{n\to\infty} \beta(d(x, z_n)) = 1$. Since $\beta \in F$, so $r = 0$ and this contradicts our assumption.
Therefore $\lim_{n\to\infty} d(x, z_{n+1}) = 0$, that is $z_{n+1} \to x$ as $n \to \infty$.
Repeat this argument,we have that $z_n \to x$, as $n \to \infty$, which proves that $\{z_n\}$ is a

sequence converging to x. Similarly z_n converges to y. By uniqueness of limit we have $x = y$. ∎

If we take $\alpha(x, y) = 1$, in Theorem 4.2, then we obtain the following corollary:

Corollary 4.5. *Let (A, B) be the pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Define a mapping $f : A \to B$ satisfying the following conditions:*

(1) f is generalized Geraghty-Suzuki contraction with $f(A_0) \subseteq B_0$;
(2) the pair (A, B) has the weak P-property.

Then there unique exists x^* in A such that $d(x^*, fx^*) = d(A, B)$.

Example 4.6. Consider $X = \mathbb{R}^2$, with the usual metric d. Define the sets $A = \{(1, 0), (4, 5), (5, 4)\}$ and $B = \{(2, 0), (0, 4), (4, 0)\}$, so that $d(A, B) = 1$, Let $A_0 = \{(1, 0)\}$ and $B_0 = \{(2, 0)\}$ and the pair (A, B) has the weak P-property. Also define $f : A \to B$ as:

$$f(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \le x_2, \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases}$$

Notice that $f(A_0) \subseteq B_0$.
Now, consider the function $\beta : [0, \infty) \to [0, 1)$ given by

$$\beta(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{\ln(1+t)}{t} & \text{if } 0 < t \le 1, \\ \frac{(t)}{1+t} & \text{if } 1 < t \le 10, \\ \frac{(10)}{11} & \text{if } t > 10, \end{cases}$$

and note that $\beta \in \mathcal{F}$. Assume that $\frac{1}{2} d^*(x, fx) \le d(x, y)$, for some $x, y \in A$. Then,

$$x = (1, 0), y = (4, 5)$$

or

$$x = (1, 0), y = (5, 4)$$

or

$$y = (1, 0), x = (4, 5)$$

or

$$y = (1, 0), x = (5, 4).$$

Since $d(fy, fx) = d(fx, fy)$ and $M(x, y) = M(y, x)$, for all $x, y \in A$, hence without loss of generality, we can assume that $(x, y) = ((1, 0), (4, 5))$ or $(x, y) = ((1, 0), (5, 4))$.
Now, we distinguish the following cases:
if $(x, y) = ((1, 0), (4, 5))$, then

$$\begin{aligned} d(f(1,0), f(4,5)) &= d((2,0),(4,0)) = 2 \\ &\le \frac{8}{1+8}.(8-1) \\ &= \beta(M((1,0),(4,5)))[M((1,0),(4,5)) - 1], \end{aligned}$$

if $(x, y) = ((1, 0), (5, 4))$, then

$$\begin{aligned} d(f(1,0), f(5,4)) &= d((2,0),(2,4)) = 4 \\ &\leq \frac{8}{1+8}.(8-1) \\ &= \beta(M((1,0),(5,4)))[M((1,0),(5,4)) - 1]. \end{aligned}$$

Consequently, we have

$$\frac{1}{2}d^*(x, fx) \leq d(x, y) \Rightarrow d(fx, fy) \leq \beta(M(x, y))[M(x, y) - d(A, B)]$$

and hence all conditions of Theorem 4.2 hold and f has the unique best proximity point. Here, $x = (1, 0)$ is unique best proximity point of f.

Corollary 4.7. *Let (A, B) be the pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and suppose that has the weak P-property. Define a mapping $f : A \to B$ such that $f(A_0) \subseteq B_0$ and*

$$\frac{1}{2}d^*(x, fx) \leq d(x, y) \Rightarrow d(fx, fy) \leq r[M(x, y) - d(A, B)],$$

where $A, B \subseteq X$, $d^*(x, y) = d(x, y) - d(A, B)$ and $M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy)\}$. Then there exists unique x^* in A such that $d(x^*, fx^*) = d(A, B)$.

Proof. Following Theorem 4.2 by taking $\beta(t) = r$, where $r \in [0, 1)$, then we obtained the desired result. ∎

Corollary 4.8. *Let (X, d) be a complete metric space and $f : X \to X$ be a self-mapping. Assume that there exists $\beta \in \mathcal{F}$ such that*

$$\frac{1}{2}d^*(x, fx) \leq d(x, y) \Rightarrow d(fx, fy) \leq \beta(M(x, y))M(x, y),$$

for all $x, y \in A$, where $d^*(x, y) = d(x, y) - d(A, B)$ and $M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy)\}$. Then f has unique fixed point.

Proof. From Theorem 4.2, we put $A = B = X$, then we obtain desired result. ∎

5. Conclusions

It is the fact that Suzuki contraction is actually the most important extension of the Banach contraction principle and can be proved fixed point theorems by using this type of contraction. The results in this article give a new way to find the best proximity points and fixed points by using α-Geraghty Suzuki contraction in metric spaces. Also justify the uniqueness of best proximity points, fixed points and unify many existing results in the literature of mathematics. In addition, we explain some new contraction and some notions, which are more general results than before.

Competing Interests

The Authors declared that they have no competing interests.

Author's Contribution

All authors contributed significantly in writing this paper. They have also read and approved the final manuscript.

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References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fundam. Math. 3 (1922) 133–181.

[2] R.P. Agarwal, M.A. El-Gebeily, D.O. Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008) 1–8.

[3] M. Geraghty, On contractive mappings, Proc. Am. Math. Soc. 40 (1973) 604–608.

[4] S.H. Cho, J.S. Bae, Common fixed point theorems for mappings satisfying property (E.A) on cone metric spaces, Math. Comput. Model. 53 (2011) 945–951.

[5] E. Hille, R.S. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ., Am. Math. Soc., Providence, R.I., 1957.

[6] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2) (2007) 1468–1476.

[7] M.A. Khamsi, V.Y. Kreinovich, Fixed point theorems for dissipative mappings in complete probabilistic metric spaces, Math. Jpn. 44 (1996) 513–520.

[8] S.K. Yang, J.S. Bae, S.H. Cho, Coincidence and common fixed and periodic point theorems in cone metric spaces, Comput. Math. Appl. 61 (2011) 170–177.

[9] K. Fan, Extensions of two fixed point theorems of F. E. Browder, Mathematische Zeitschrift 112 (1969) 234–240.

[10] S.S. Basha, Best proximity points: global optimal approximate solution, J. Glob. Optim. 49 (2011) 15–21.

[11] M. Abbas, A. Hussain, P. Kumam, A coincidence best proximity point problem in G-metric spaces, Abst. and Appl. Anal. 2015 (2015) Article ID 243753.

[12] A. Akbar, M. Gabeleh, Generalized cyclic contractions in partially ordered metric spaces, Optim. Lett. 6 (2012) 1819–1830.

[13] A. Akbar, M. Gabeleh, Global optimal solutions of non-cyclic mappings in metric spaces, J. Optim. Theory Appl. 153 (2012) 298–305.

[14] S.S. Basha, Extensions of Banach's contraction principle, Numer. Funct. Anal. Optim. 31 (2010) 569–576.

[15] S.S. Basha, Best proximity point theorems generalizing the contraction principle, Nonlinear Anal. 74 (2011) 5844–5850.

[16] S.S. Basha, N. Shahzad, R. Jeyaraj, Best proximity points: approximation and optimization, Optim. Lett. 7 (2013) 145–155.

[17] S.S. Basha, Common best proximity points: global minimization of multi-objective function, J. Glob. Optim. 54 (2012) 367–373.

[18] N. Shahzad, S.S. Basha, R. Jeyaraj, Common best proximity points: global optimal solutions, J. Optim. Theory Appl. 148 (2011) 69–78.

[19] A. Amini-Harandi, H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. 72 (2010) 2238–2242.

[20] J. Caballero, J. Harjani, K. Sadarangani, A best proximity point theorem for Geraghty-contractions, Fixed Point Theory Appl. 2012 (2012) Article no. 231.

[21] J. Zhang, Y. Su, Q. Chang, A note on a best proximity point theorem for Geraghty contractions, Fixed Point Theory Appl. 2013 (2013) Article no. 99.

[22] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces, Fixed point Theory Appl. 2012 (2012) Article no. 93.

[23] W.A. Kirk, Contraction mappings and extensions, In: W.A. Kirk, B. Sims (eds.) Handbook of Metric Fixed Point Theory, Kluwer acedemic publishers, Dordrecht (2001), 1–34.

[24] W.A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (2003) 851–862.

[25] P. Salimi, E. Karapinar, Suzuki-Edelstein type contraction via auxiliary functions, Math. Probl. Eng. 2013 (2013) Article ID 648528.

[26] V.S. Raj, A best proximity point theorem for weakly contractive non-self mappings, Nonlinear Anal. 74 (2011) 4804–4808.

[27] S.H. Cho, J.S. Bae, E. Karapinar, Fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. 2013 (2013) Article no. 329.

[28] B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Am. Math. Soc. 226 (1977) 257–290.

[29] E. Karapinar, P. Kumam, P. Salimi, On α-ψ-Meir-Keeler contractive mappings, Fixed Point Theory Appl. 2013 (2013) Article no. 94.

[30] T. Suzuki, A generalized Banach contraction principle that cheracterizes metric completness, Proc. Amer. Math. Soc. 136 (2008) 1861–1869.

[31] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. 75 (2012) 2154–2165.

[32] P. Chaipunya, W. Sintunavarat, P. Kumam, On $\mathcal{P}$-contractions in ordered metric spaces, Fixed Point Theory Appl. 2012 (2012) Article no. 219.