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A Note on Homotopic Invariance for Endpoints of Multi-Valued Contractive Mappings

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Abstract In this paper, we prove the homotopic invariance for endpoints of multi-valued contractive mappings in Banach spaces. We also give an example to illustrate the main result. At the same time, a question raised in [B. Panyanak, The demiclosed principle for multi-valued nonexpansive mappings in Banach spaces, J. Nonlinear Convex Anal. 17 (2016) 2063–2070] is answered in the negative.

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1. Introduction

Fixed point theory, a branch of analysis, is an important tool for finding solutions of problems in the form of equations or inequalities. One of the fundamental and celebrated results in metric fixed point theory is the Banach contraction principle which states that every contractive mapping on a complete metric space always has a unique fixed point (see [1]). On the other hand, homotopy theory is a branch of topology which aims to classify topological spaces via homotopy equivalences. These equivalences are weaker than homeomorphisms in the sense that the equivalence classes of topological spaces with homotopy equivalences are sometimes bigger than those with homeomorphisms.

A connection between the above two topics was studied by Frigon [2] in 1996. She proved that under some appropriate conditions, the fixed points of contractive mappings in complete metric spaces are invariant under homotopies. Since then, the homotopic invariance for fixed points of such kind of mappings has been developed and many papers have appeared (see, e.g., [3–11]).

In this paper, we prove a multi-valued version of the Banach contraction principle and then apply it to obtain the homotopic invariance for fixed points of multi-valued contractive mappings in Banach spaces.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers.

Let (X,d) be a metric space, $\emptyset \neq K \subseteq X$ and $x \in X$. The *distance* from x to K is defined by

$$\operatorname{dist}(x,K) := \inf\{d(x,y) : y \in K\}.$$

The *radius* of K relative to x is defined by

$$r_x(K) := \sup\{d(x,y) : y \in K\}.$$

The *diameter* of K is defined by

$$\operatorname{diam}(K) := \sup\{d(x,y) : x, y \in K\}.$$

The set K is *bounded* if $\operatorname{diam}(K) < \infty$. It is denoted by $CB(X)$: the family of nonempty closed bounded subsets of X. The *Pompeiu-Hausdorff distance* on $CB(X)$ is defined by

$$\mathcal{H}(A,B) := \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\} \text{ for all } A,B \in CB(X).$$

The following elementary fact was proved by Nadler in 1969.

Lemma 2.1. ([12]) *Let (X,d) be a complete metric space, $A,B \in CB(X)$ and $a \in A$. Then for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a,b) \leq \mathcal{H}(A,B) + \varepsilon$.*

A multi-valued mapping $T : K \to CB(X)$ is said to be *contractive* if there exists a constant $\lambda \in [0,1)$ such that

$$\mathcal{H}(T(x),T(y)) \leq \lambda d(x,y) \text{ for all } x,y \in K.$$

In this case, we call T a $\lambda-$*contractive mapping.* An element x in K is called a *fixed point* of T if $x \in T(x)$. Moreover, if $\{x\} = T(x)$, then x is called an *endpoint* of T. It is denoted by $Fix(T)$: the set of all fixed points of T and by $End(T)$: the set of all endpoints of T. Some elementary facts about fixed points and endpoints for multi-valued mappings were collected as the following lemmas.

Lemma 2.2. *Let $T : K \to CB(X)$ be a mapping. Then the following statements hold.*
(1) *If x is an endpoint of T, then x is a fixed point of T.*
(2) *$x \in Fix(T)$ if and only if $dist(x,T(x)) = 0$.*
(3) *$x \in End(T)$ if and only if $r_x(T(x)) = 0$.*

Lemma 2.3. *Let $T : K \to CB(X)$ be a $\lambda-$contractive mapping for some $\lambda \in [0,1)$. Then the following statements hold.*
(1) *The mapping $\phi : K \to \mathbb{R}$ defined by $\phi(x) := r_x(T(x))$ is continuous.*
(2) *If $End(T) \neq \emptyset$, then $End(T)$ consists of exactly one point.*

Proof. (1) Let $x,y \in K$. For each $n \in \mathbb{N}$, there exist $a_n \in T(x)$ and $b_n \in T(y)$ such that

$$r_x(T(x)) - \frac{1}{n} < d(x,a_n) \text{ and } d(a_n,b_n) < \operatorname{dist}(a_n,T(y)) + \frac{1}{n}.$$

This implies that

$$\begin{aligned} r_x(T(x)) - \frac{1}{n} &< d(x, a_n) \\ &\le d(x,y) + d(y,b_n) + d(b_n,a_n) \\ &\le d(x,y) + r_y(T(y)) + \operatorname{dist}(a_n, T(y)) + \frac{1}{n} \\ &\le d(x,y) + r_y(T(y)) + \mathcal{H}(T(x),T(y)) + \frac{1}{n} \\ &\le (1+\lambda)d(x,y) + r_y(T(y)) + \frac{1}{n}. \end{aligned}$$

Thus

$$r_x(T(x)) - r_y(T(y)) \le (1+\lambda)d(x,y). \tag{2.1}$$

Similarly, we can show that

$$r_y(T(y)) - r_x(T(x)) \le (1+\lambda)d(x,y). \tag{2.2}$$

By (2.1) and (2.2) we get that $|r_x(T(x)) - r_y(T(y))| \le (1+\lambda)d(x,y)$, and hence ϕ is continuous.

(2) Let $x, y \in End(T)$. Then $d(x,y) = \mathcal{H}(\{x\},\{y\}) = \mathcal{H}(T(x),T(y)) \le \lambda d(x,y)$ which implies that $x = y$. ■

Definition 2.4. Let K be a nonempty subset of a Banach space $(X, \|\cdot\|)$ and $T, G : K \to CB(X)$ be two contractive mappings. Then T and G are said to be *homotopic* if there exists a mapping (which is called a *homotopy*) $H : [0,1] \times K \to CB(X)$ such that

(1) $H(0,\cdot) = T(\cdot)$ and $H(1,\cdot) = G(\cdot)$;

(2) $H(t,\cdot)$ is a contractive mapping for each $t \in [0,1]$;

(3) For each $t \in [0,1]$, $H(t,\cdot)$ is fixed point free on the boundary of K;

(4) $H(t,x)$ is *equi-continuous* in $t \in [0,1]$ over $x \in K$, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $t, s \in [0,1]$ with $|t - s| < \delta$ we have $\mathcal{H}(H(t,x), H(s,x)) < \varepsilon$ for all $x \in K$.

Definition 2.5. A mapping $H : [0,1] \times K \to CB(X)$ is said to satisfy *condition (S)* ([11]) if for any sequence $\{t_n\}$ in $[0,1]$ with

$$\inf_{x\in K} r_x(H(t_n,x)) > 0 \quad \text{and} \quad \lim_{n\to\infty} t_n = t_0,$$

it follows that $\inf_{x\in K} r_x(H(t_0,x)) > 0$.

3. Main Results

Recall that a multi-valued mapping $T : K \to CB(X)$ is said to have the *approximate endpoint property* if $\inf\limits_{x\in K} r_x(T(x)) = 0$. This section is begun by proving the existence of endpoints for multi-valued contractive mappings in complete metric spaces. This result can be viewed as an extension of Corollary 2.2 in [13].

Theorem 3.1. *Let K be a nonempty closed subset of a complete metric space (X, d) and $T : K \to CB(X)$ be a $\lambda-$contractive mapping for some $\lambda \in [0,1)$. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

Proof. It is clear that if T has an endpoint, then T has the approximate endpoint property. Conversely, suppose that T has the approximate endpoint property. For each $n \in \mathbb{N}$, we let

$$K_n := \left\{ x \in K : r_x(T(x)) \le \frac{1}{n} \right\}.$$

Then K_n is a nonempty and $K_{n+1} \subseteq K_n$. By Lemma 2.3 (1), K_n is closed in X. Since T is $\lambda-$contractive, for $x, y \in K_n$ we have

$$\begin{aligned} d(x,y) &= \mathcal{H}(\{x\},\{y\}) \\ &\le \mathcal{H}(\{x\},T(x)) + \mathcal{H}(T(x),T(y)) + \mathcal{H}(T(y),\{y\}) \\ &\le r_x(T(x)) + \lambda d(x,y) + r_y(T(y)) \\ &\le \frac{2}{n} + \lambda d(x,y). \end{aligned}$$

Therefore $d(x,y) \le \frac{2}{n(1-\lambda)}$ for all $x, y \in K_n$. This implies that K_n is bounded and $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$. By the Cantor intersection theorem (see e.g., [14]), there exists $z \in K$ such that $\cap_{n\in\mathbb{N}} K_n = \{z\}$ and hence $r_z(T(z)) = 0$. The conclusion follows from Lemmas 2.2 and 2.3. ■

Recall that a Banach space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for each $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in X$ the conditions $\|x\| \le 1$, $\|y\| \le 1$, $\|x-y\| \ge \varepsilon$ imply

$$\frac{1}{2}\|x+y\| \le 1-\delta.$$

A Banach space $(X, \|\cdot\|)$ is said to have the *Opial property* if given whenever $\{x_n\}$ converges weakly to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \text{ for each } y \in X \text{ with } y \ne x.$$

From now on, we will use the notation $x_n \rightharpoonup x$ (resp. $x_n \to x$) for a sequence $\{x_n\}$ converging weakly (resp. converging strongly) to a point x.

Before proving the main theorem, a result is stated which is known as the demiclosed principle.

Lemma 3.2. ([15]) *Suppose X is either a uniformly convex Banach space, or a Banach space with the Opial property. Let K be a nonempty closed convex subset of X, and suppose $T : K \to CB(X)$ is contractive. Then the following implication holds:*

$$\{x_n\} \subseteq K,\ x_n \rightharpoonup x,\ r_{x_n}(T(x_n)) \to 0 \implies x \in End(T).$$

The main theorem is proved as follows:

Theorem 3.3. *Suppose X is either a uniformly convex Banach space, or a reflexive Banach space with the Opial property. Let K be a nonempty closed convex subset of X and $T, G : K \to CB(X)$ be contractive mappings. Suppose $H : [0,1] \times K \to CB(X)$ is a mapping such that*

(1) *$H(0,\cdot) = T(\cdot)$ and $H(1,\cdot) = G(\cdot)$;*

(2) *$H(t,\cdot)$ is a $\lambda-$contractive mapping with $\lambda \in [0,1)$ for each $t \in [0,1]$;*

(3) *H satisfies condition (S);*

(4) $H(t,x)$ *is equi-continuous in* $t \in [0,1]$ *over* $x \in K$.

Then T *has an endpoint in* K *if and only if* G *has an endpoint in* K.

Proof. Suppose that T has an endpoint in K and let

$$V := \{t \in [0,1] : H(t,\cdot) \text{ has an endpoint in } K\}.$$

Then V is nonempty. We can show that V is actually the entire interval $[0,1]$ by verifying that V is both open and closed in $[0,1]$. To see that V is closed, we let $\{t_n\}$ be a sequence in V such that $\lim_{n\to\infty} t_n = t_0$. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $x_n \in End(H(t_n,\cdot))$. By the equi-continuity of H, there exists $\delta > 0$ such that for $t,s \in [0,1]$ with $|t-s| < \delta$, we have

$$\mathcal{H}(H(t,x), H(s,x)) < 1 \text{ for all } x \in K. \tag{3.1}$$

Since $\lim_{n\to\infty} t_n = t_0$, there exists $n_0 \in \mathbb{N}$ such that $|t_n - t_0| < \delta$ for all $n \geq n_0$. By (3.1) we have

$$\sup_{x\in K} \mathcal{H}(H(t_n,x), H(t_0,x)) \leq 1. \tag{3.2}$$

Fix $y \in K$. By Lemma 2.1, for each $n \in \mathbb{N}$ there exists $y_n \in H(t_0,y)$ such that

$$\|x_n - y_n\| \leq \mathcal{H}(H(t_n,x_n), H(t_0,y)) + \frac{1}{n}. \tag{3.3}$$

By (3.2) and (3.3), for $n \geq n_0$ we have

$$\begin{aligned}\|x_n\| &\leq \|x_n - y_n\| + \|y_n\| \\ &\leq \mathcal{H}(H(t_n,x_n), H(t_0,y)) + \frac{1}{n} + \|y_n\| \\ &\leq \mathcal{H}(H(t_n,x_n), H(t_0,x_n)) + \mathcal{H}(H(t_0,x_n), H(t_0,y)) + \frac{1}{n} + \|y_n\| \\ &\leq \sup_{x\in K} \mathcal{H}(H(t_n,x), H(t_0,x)) + \lambda\|x_n - y\| + \frac{1}{n} + \|y_n\| \\ &\leq 2 + \lambda\|x_n\| + \lambda\|y\| + \operatorname{diam}(H(t_0,y)).\end{aligned}$$

This implies that $\|x_n\| \leq \frac{1}{1-\lambda}(2 + \lambda\|y\| + \operatorname{diam}(H(t_0,y)))$ for all $n \geq n_0$. Therefore $\{x_n\}$ is a bounded sequence in K. Since X is reflexive, by passing through a subsequence, we may assume that $x_n \rightharpoonup z \in K$. By the equi-continuity of H, we have

$$r_{x_n}(H(t_0,x_n)) \leq \mathcal{H}\left(H(t_n,x_n), H(t_0,x_n)\right) \longrightarrow 0 \text{ as } n \to \infty.$$

By Lemma 3.2, z is an endpoint of $H(t_0,\cdot)$ which means $t_0 \in V$. Therefore V is closed. Next, we show that V is open in $[0,1]$. Suppose not, then there exists $t_0 \in V$ and a sequence $\{t_n\}$ in $[0,1] - V$ such that $\lim_{n\to\infty} t_n = t_0$. This implies that $r_x(H(t_n,x)) > 0$ for all $n \in \mathbb{N}$ and $x \in K$. We claim that

$$\inf_{x\in K} r_x(H(t_n,x)) > 0 \text{ for all } n \in \mathbb{N}.$$

Otherwise, there exists a sequence $\{x_m\}$ in K such that $\lim_{m\to\infty} r_{x_m}(H(t_n,x_m)) = 0$. This implies that $H(t_n,\cdot)$ has the approximate endpoint property. By Theorem 3.1, $H(t_n,\cdot)$

has an endpoint in K which contradicts to the fact that $t_n \notin V$. So we have the claim. Now, condition (S) implies

$$\inf_{x\in K} r_x(H(t_0,x)) > 0,$$

which in turn implies $t_0 \notin V$ and this is a contradiction. Therefore V is open in $[0,1]$ and hence the proof is complete. ∎

The following example shows that the condition (S) in Theorem 3.3 is necessary. It also gives a negative answer to Question 4.3 of [15].

Example 3.4. Let $X = (\mathbb{R}, |\cdot|)$ and $K = [-1,1]$. Let $T, G : K \to CB(X)$ be defined by $T(x) := \{0\}$ and $G(x) := [-1,1]$ for all $x \in K$. Let $H : [0,1]\times K \to CB(X)$ be defined by

$$H(t,x) := [-|t|,|t|] \text{ for all } t \in [0,1] \text{ and } x \in K.$$

Then the following statements hold.

(1) T and G are contractive mappings with $End(T) = \{0\}$ and $End(G) = \emptyset$.
(2) $H(t,\cdot)$ is a contractive mapping for each $t \in [0,1]$ and $H(0,\cdot) = T(\cdot)$ and $H(1,\cdot) = G(\cdot)$.
(3) For each $t \in [0,1]$, $H(t,\cdot)$ is endpoint free on the boundary of K.
(4) $H(t,x)$ is equi-continuous in $t \in [0,1]$ over $x \in K$.
(5) H does not satisfy condition (S).

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